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THE RICCATI EQUATION : MONOTONICITY, CONVERGENCE AND STABILIZABILITY PROPERTIES.

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ABSTRACT

We present a summary of recent results obtained with various coauthors on the solution of the Riccati differential or difference equations. These include necessary and sufficient conditions for monotonicity of the solution, convergence properties of the solution and sufficient conditions for stability of the closed loop state transition matrix function.

1. INTRODUCTION

We present a compendium of recent results on monotonicity, convergence and stabilizability properties of the solutions of the Riccati Difference and Differential Equation (RDE) of optimal filtering. These results have been obtained by the author with a number of different colleagues : R.R. Bitmead, C.E. de Souza, G.C. Goodwin, R.J. Kaye, I.R. Petersen and M.A. Poubelle : see [1] - [5]. They are obtained under conditions that are weaker than the classical ones. In particular, our new necessary and sufficient conditions for monotonicity of the solution of the RDE require neither detectability nor stabilizability of the underlying system, whereas the results on convergence and stability of the closed loop system require detectability only.

We study the following Riccati equation of optimal filtering :

- the discrete-time (DT) RDE :

$$P(t+1) = FP(t)F^T - FP(t)H^T[HP(t)H^T+R]^{-1} HP(t)F^T + Q \qquad (1)$$
$$P(0) = P_0;$$

- the continuous-time(CT) RDE :

$$\dot{P}(t) = FP(t) + P(t)F^T - P(t)H^TR^{-1}HP(t) + Q \qquad (2)$$
$$P(0) = P_0$$

with H, F, Q, R constant, $R > 0$ and $Q = Q^T$.

Associated with these two RDE's are a discrete-time and a continuous-time Algebraic Riccati Equation (ARE) :

$$P = FPF^T - FPH^T[HPH^T + R]^{-1}HPF^T + Q \qquad (3)$$

$$0 = FP + PF^T - PH^TR^{-1}HP + Q \qquad (4)$$

Also associated with these RDE's are the discrete-time and continuous-time Fake Algebraic Riccati Equations (FARE) :

$$\bar{Q}(t) = P(t) - FP(t)F^T + FP(t)H^T[HP(t)H^T + R]^{-1} HP(t)F^T \qquad (5)$$

$$\bar{Q}(t) = P(t)\, H^TR^{-1}HP(t) - FP(t) - P(t)F^T \qquad (6)$$

These FARE's are actually definitions for $Q(t)$.

The Fake Algebraic Riccati Techniques were first introduced in [1] and have proved very useful in establishing monotonicity and stabilizability results for the RDE. We now successively present our monotonicity, convergence and stabilizability results for the RDE.

2. MONOTONICITY RESULTS

In order to prove our major monotonicity result, we first establish two preliminary lemmas, which are of independent interest.

Lemma 1

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Consider two RDE's (either DT(1) or CT(2)). With the same F, H, R but possibly different Q matrices, $\hat{Q}$ and Q^*, and possibly different initial conditions, $\hat{P}(t_0)$ and $P^*(t_0)$. Denote the corresponding solutions $P(t)$ and $P^*(t)$ respectively.

Then $\hat{P}(t_0) \geq P^*(t_0)$ and $\hat{Q} \geq Q^*$ implies $\hat{P}(t) \geq P^*(t)$ for all $t \geq t_0$.

to $\hat{P}(t) - P^*(t)$

The proof of lemma 1 was given in discrete-time in [1]-[2]. It uses a rather complicated formula relating $\hat{P}(t+1)-P^*(t)$ and $\hat{Q}-Q^*$; this formula was derived after lengthy manipulations from related formulae obtained by Nishimura [6]. In continuous-time, a much simpler argument, initially derived in [5], can be used.

Proof of Lemma 1 (Continous-time)

Denote $\tilde{P}(t) = \hat{P}(t)-P^*(t)$. Then

$$\dot{\tilde{P}}(t) = A(t)\tilde{P}(t) + \tilde{P}(t)A^T(t) + W(t) \tag{7}$$

where

$$A(t) = F - \hat{P}(t)H^TR^{-1}H \tag{8}$$

$$W(t) = \tilde{P}(t)H^TR^{-1}H\tilde{P}(t) + \hat{Q}-Q^* \tag{9}$$

Equation (7) is a time-varying Lyapunov equation.

Let $\Phi(t,\tau)$ be the fundamental matrix associated with $A(t)$; then the solution of (7) is

$$\tilde{P}(t) = \int_0^t \Phi(t,\tau)W(\tau)\Phi^T(t,\tau)d\tau + \Phi(t,0)\tilde{P}(0)\Phi^T(t,0) \tag{10}$$

The result follows immediately.

Lemma 2 (DT version) :

Consider the RDE (1). If for some t_0, $P(t_0) \geq P(t_0+1)$ (resp. $P(t_0) \leq P(t_0+1)$) then $P(t_0+k) \geq P(t_0+k+1)$ (resp. $P(t_0+k) \leq P(t_0+k+1)$) for all $k \geq 0$.

Proof : From Lemma 1, by considering $\hat{P}(t_0) = P(t_0)$, $P^*(t_0) = P(t_0+1)$ and $\hat{Q} = Q^* = Q$ (and vice-versa).

Lemma 2 (CT version)

Consider the RDE (2). If for some t_0, $\dot{P}(t_0) \leq 0$ (resp. $\dot{P}(t_0) \geq 0$) then $\dot{P}(t) \leq 0$ (resp. $\dot{P}(t) \geq 0$) for all $t \geq t_0$.

Proof : Differentiating the RDE (2) yields

$$\ddot{P}(t) = \dot{P}(t)A^T(t) + A(t)\dot{P}(t) \tag{11}$$

where

$$A(t) = F - P(t)H^TR^{-1}H \tag{12}$$

Thus $\dot{P}(t)$ satisfies the Lyapunov equation (11) whose solution is

$$\dot{P}(t) = \Phi(t,t_0)\dot{P}(t_0)\Phi^T(t,t_0) \tag{13}$$

where $\Phi(t,\tau)$ is the fundamental matrix associated with $A(t)$. The result follows from (13).

We can now establish our main monotonicity result.

Theorem 1 :

The solution $\{P(t)\}$ of the RDE (1) or (2) is monotonically non-increasing (resp. monotonically nondecreasing) if and only if $\overline{Q}(0) \geq Q$ (resp. $\overline{Q}(0) \leq Q$).

Proof : We prove the nonincreasing result.

a) Discrete-time : It follows from (1) and (5) that

$$\overline{Q}(t) = Q + P(t) - P(t+1) \tag{14}$$

Setting $t = t_0 = 0$, the result follows from Lemma 2.

b) Continuous-time : It follows similarly from (2) and (6) that

$$\overline{Q}(t) = Q - \dot{P}(t)$$

The result follows again from Lemma 2.

Comment 1

Note that the statement is identical for the continuous-time and the discrete-time RDE.

Comment 2

The result does not require detectability of [H,F] or stabilizability of $[F,Q^{1/2}]$.

3. CONVERGENCE RESULTS

We first recall the following definition [7].

Definition 1

A real nonnegative definite solution P^+ of the DT (resp. CT) ARE is called strong if the corresponding closed-loop state-transition matrix $F^+ = F - FP^+H^T(HP^+H^T+R)^{-1}H$ (resp. $F^+ = F-P^+H^TR^{-1}H$) has all its eigenvalues inside or on the unit circle (resp. in the closed left half plane). It is called stabilizing if the eigenvalues are strictly inside the unit circle (resp. in the open left half plane).

The following convergence result combines elements from Theorems 1 and 3 of [3] and Theorem 3.2 and 4.2 of [4].

Theorem 2

Consider the RDE (1) in DT, or (2) in CT, with [H,F] detectable and the associated~~s~~ ARE's. Then

1) the ARE (3) in DT, or (4) in CT, has a unique strong solution P^+;

2) if $P_0 \geqslant P^+$, then $\lim_{t \to \infty} P(t) = P^+$;

3) if in addition $[F, Q^{1/2}]$ has no unreachable mode on $|z| = 1$ (resp. on the jw-axis), then P^+ is stabilizing.

Proof : the proofs are rather lengthy and are given in [3] and [4].

4. STABILIZABILITY RESULTS

We now present results for the following problem : under what conditions (on H, F, Q, R, P_0) are the closed loop state transition matrices F(t) exponentially asymptotically stable for all $t \geqslant 0$, i.e.

- in discrete-time

A
$$|\lambda_k(\cancel{F}(t))| < 1 \ , \ k = 1, \ldots, n, \quad \text{for all } t \geqslant 0, \text{ with} \tag{16}$$
$$A(t) = F - FP(t)H^T[HP(t)H^T + R]^{-1} H$$

- in continuous-time

A
$$\operatorname{Re} \lambda_k(\cancel{F}(t)) < 0, \ k = 1, \ldots, n, \text{ for all } t \geqslant 0, \text{ with} \tag{17}$$
$$A(t) = F - P(t)H^TR^{-1}H$$

The reason for looking at this problem is that conditions (16) (resp. (17)) will guarantee the asymptotic stability of the linear time-invariant "frozen" closed loop system :

$$x(\tau+1) = A(t)\, x(\tau), \qquad \tau = 0,1,2,\ldots \tag{18}$$

respectively

$$\dot{x}(\tau) = A(t)\, x(\tau) \quad , \qquad \tau \geqslant 0 \tag{19}$$

for any fixed $t \in (0,\infty)$ (hence the name "frozen"). This problem has applications in signal processing where the observer gain of a filter can be chosen as the gain of a Kalman filter frozen after a few iterations (see e.g. [8]). It also has applications in adaptive LQG control where one may want to apply finite horizon LQG control laws. The finite horizon is a design parameter and a meaningful question is to ask whether stability of the closed loop system for some finite horizon will guarantee stability for all larger horizons.

The main tool for establishing asymptotic stability of the closed loop state-transition matrices $F(t)$ will be the FARE's. Indeed, if $\overline{Q}(t) \geqslant 0$, then the FARE's (5) and (6) become legitimate ARE's for $P(t)$. We can then use Theorem 2 to establish the following result.

Theorem 3 :

Assume that :

1) $[H,F]$ is detectable
2) $\overline{Q}(t) \geqslant 0$ and $[F,\overline{Q}(t)^{1/2}]$ has no unreachable mode on $|z| = 1$ (resp. on the $j\omega$-axis).

Then $A(t)$ is asymptotically stable.

Proof : If $\overline{Q}(t) \geqslant 0$, then $P(t)$ satisfies the FARE (5) in DT, or (6) in CT. The result then follows from parts 1) and 3) of Theorem 2, applied to theses FARE's.

Theorem 3 tells us that, for each t for which condition 2 on $\overline{Q}(t)$ is satisfied, the closed-loop transition matrix $A(t)$ is asymptotically stable. A more interesting question is to find conditions on H,F,R,Q and P_0 that will produce a sequence $\overline{Q}(t)$ that satisfies condition 2) of Theorem 3. We have the following result.

Theorem 4

Consider the RDE (1) in DT, or (2) in CT, with the following assumptions

1) $[H,F]$ is detectable

2) $[F,\bar{Q}(0)^{1/2}]$ is stabilizable

3) P_0 is such that $\bar{Q}(0) \geq Q$

Then the solution $P(t)$ of the RDE (1) (resp.(2)) is stabilizing for each $t \geq 0$, i.e. the conditions (16) (resp. (17)) are satisfied.

The proof of Theorem 4 is rather complicated and will not be given here. A discrete version can be found in [1], a continuous version in [2]. Both proofs use the FARE technique, but the CT version uses a dual optimal control argument which was not used in the DT version.

Comment 3

Condition 3) of Theorem 4 implies in particular that $P_0 \geq P^+$, where P^+ is the solution of the ARE. The conditions of the Theorem then insure that the sequence (or function) $P(t)$ is monotonically nonincreasing and convergent to P^+. In finite horizon optimal control problems, P_0 (which is the weighting on the final state in the cost function) is typically smaller than P^+. In such case, another result is needed to guarantee that if $A(t)$ is asymptotically stable for some $t = t_0$, it remains asymptotically stable for all $t \geq t_0$. The following continuous-time result is proved in [5].

Theorem 5

Assume that

1) $[H,F]$ is detectable

2) $P(t_0) = P_0$ is such that

a/ $\bar{Q}(t_0) \geq 0$ and $[F,\bar{Q}^{1/2}(0)]$ has no unreachable modes on the $j\omega$-axis b/ $[Q-\bar{Q}(t_0)]\, A^T(t_0) + A(t_0)[Q-\bar{Q}(t_0)] \leq 0$ (20)

Then $A(t)$ is asymptotically stable for all $t \geq t_0$.

Proof : Differentiating (11) yields

$$\dddot{P}(t) = A(t)\ddot{P}(t) + \ddot{P}(t)A^T(t) - 2\dot{P}(t)H^TR^{-1}H\dot{P}(t) \quad (21)$$

That is, $\ddot{P}(t)$ satisfies a Lyapunov equation. Its solution is

$$\ddot{P}(t) = -2\int_{t_0}^{t} \Phi(t,\tau)\dot{P}(\tau)H^TR^{-1}H\dot{P}(\tau)\Phi^T(t,\tau)d\tau + \Phi(t,t_0)\ddot{P}(t_0)\Phi^T(t,t_0) \quad (22)$$

By (11) and (15) the quantity on the left hand side of (20) is $\ddot{P}(t_0)$. It then follows from (20) and (22) that $\ddot{P}(t) \leq 0 \quad \forall\, t \geq t_0$. Therefore $\dot{P}(t) \leq \dot{P}(t_0)$ and $\bar{Q}(t) \geq \bar{Q}(t_0) \quad \forall\, t \geq t_0$.

The result then follows from Theorem 3.

A discrete-time verions of Theorem 5, which would be helpful in solving the discrete-time finite horizon LQG problem, is not yet available. It would require a DT analogue of (21).

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