PERSISTENCE OF EXCITATION CRITERIA

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ABSTRACT:

We present conditions under which the persistency of excitation of one regressor vector implies the persistency of another regressor derived from the first via a linear, dynamical transformation.

1. INTRODUCTION

Persistency of excitation and sufficient richness have a longstanding history, both in identification [1] and adaptive control theory [2,3].

A wealth of results concerning the generation of persistently exciting regressor vectors in the context of linear time invariant systems is available. Here, we consider mainly time varying systems, as motivated by [4]. The results are presented in continuous time, but unlike most other developments including [5], can be transliterated into the discrete time situation without difficulty. To the authors' knowledge the only other instance in the literature where nonstationary systems are considered in connection with persistency of excitation is [5]. In comparison with [5], we believe that our approach yields a far simpler proof, for a more general result.

2. DEFINITIONS AND NOTATIONS

We define the concept of persistently exciting vector functions and introduce some notation.

Definition 2.1

A bounded locally square integrable vector function ϕ: $R^+ \to R^n$ is said to be persistently exciting if there exist positive constants s_0, T_0 and α such that

$$\frac{1}{T_0}\int_s^{s+T_0} \phi(t)\phi^T(t)dt \geq \alpha I > 0 \quad \forall\ s \geq s_0 > 0 \qquad (2.1)$$

□

Throughout the paper, we shall assume that the signal vector u(t) is bounded and Riemann integrable. In addition we shall use the following notation:
PE: persistently exciting (or persistence of excitation); L_∞: the space of bounded functions defined on $(-\infty,\infty)$; D: $Du \equiv d/dt(u)$; D_2: $D_2uv^T \equiv uDv^T$ and for $A(s) = A_ns^n + A_{n-1}s^{n-1} + \ldots + A_0$, $A(D_2)uv^T \equiv A_nuD^nv^T + \ldots + A_0uv^T$; $||x||$: any vector norm, or its corresponding induced matrix norm; $||x||_\infty$: the L_∞ norm of the vector function x(t): $||x||_\infty = \sup_{t\epsilon R}||x(t)||$, or the corresponding induced matrix norm.

3. SOME SWAPPING LEMMAS

In analyzing adaptive identification and/or control algorithms it is important that one can infer the persistency of excitation of one vector valued function from the persistency of excitation of another vector valued function, which is linked to the first by a linear, dynamical transformation. In this section we set up some tools allowing us to do so. Consider the linear systems:

$$\dot{x}_i(t) = F_ix_i(t) + G_iu(t) \qquad (3.1)$$

$$y_i(t) = H_ix_i(t) + J_iu(t) \qquad (3.2)$$

where $u:R^+\to R^m$, $x_i:R^+\to R^{n_i}$ $y_i:R^+\to R^{p_i}$, and F_i, G_i, H_i, J_i are (real) matrices of appropriate dimensions. Define:

$$p^{(i)}(s) = \det(sI-F_i) \qquad (3.3)$$

$$Q^{(i)}(s) = H_i(sI-F_i)^{adj}G_i + J_i\det(sI-F_i) \qquad (3.4)$$

The first Lemma links the inner product of y_i and a suitable test function ψ with the inner product of u and ψ.

Lemma 3.1: For any row vector test function $\psi: R^+\to R^r$, n_i times continuously differentiable on R^+ we have that for any $t_0 < t_1 \epsilon R^+$

$$\int_{t_0}^{t_1} p^{(i)}(-D_2)y_i(t)\psi(t)dt$$
$$= \int_{t_0}^{t_1} Q^{(i)}(-D_2)u(t)\psi(t)dt + g^{(i)}(t_0,t_1) \qquad (3.5)$$

where $g^{(i)}(t_0,t_1)$ is given by:

$$g^{(i)}(t_0,t_1) = H_i(-D_2-F_i)^{adj}\ x_i(t)\psi(t)\Big|_{t_0}^{t_1} \qquad (3.6)$$

Proof: Starting from the left hand side of (3.5), using integration by parts and the system equations (3.1)-(3.4), and Lemma 3.1 yields the desired result. □

The expression (3.5) relates y_i to u since, roughly Qbadxmey% $p^{(i)}(D)y_i(t) = Q^{(i)}(D)u(t)$, without imposing restrictive assumptions on the differentiability of u and y_i. The following result goes a step further:

Lemma 3.2: Consider the systems (3.1) - (3.4) and suppose there exist polynomial matrices $T^{(1)}(s)$, $T^{(2)}(s)$ such that

$$T^{(1)}(s)\ Q^{(1)}(s) = T^{(2)}(s)Q^{(2)}(s) \qquad (3.7)$$

Then for any row vector test function $\Psi : R^+ \to R^r$, sufficiently continuously differentiable on R^+ ($\Psi \epsilon C^r$, $r = \max(n_2+\deg T^{(2)}, n_1+\deg T^{(1)})$ we have that

$$\int_{t_0}^{t_1} T^{(1)}(-D_2)p^{(1)}(-D_2)y_1(t)\Psi(t)dt$$
$$= \int_{t_0}^{t_1} T^{(2)}(-D_2)p^{(2)}(-D_2)y_2(t)\Psi(t)dt + g^{(1,2)}(t_0,t_1) \qquad (3.8))$$

where $g^{(1,2)}$ is given by:

$$g^{(1,2)}(t_0,t_1) = -\ T^{(1)}(-D_2)H_1(-D_2-F_1)^{adj}x_1(t)\Psi(t)\Big|_{t_0}^{t_1}$$

$$+ T^{(2)}(-D_2)H_2(-D_2-F_2)\mathrm{adj}\, x_2(t)\Psi(t)\Big|_{t_0}^{t_1} \qquad (3.9)$$

□

Proof: The proof follows from a repeated application of Lemma 3.2. □

4. TIME-INVARIANT SYSTEMS

In this section we state that if the output of a time-invariant MIMO system is persistently exciting, the output of a "related" time-invariant MIMO system is also persistently exciting.

Theorem 4.1: Consider two MIMO systems (defined as in (3.1)-(3.4); i=1,2) with the following assumptions:
A.1: $u(t) \in L_\infty$
A.2: $\mathrm{Re}\lambda_j(F_i) < 0$ $i=1,2$; $j=1,\dots,n_i$ and $n_1 \geq n_2$
A.3: there exist constants $t_1>0$, $\alpha_1>0$, $\beta_1>0$, $T_1>0$ such that

$$\beta_1 I \geq \frac{1}{T}\int_t^{t+T} y_1(\tau)y_1^T(\tau)d\tau \geq \alpha_1 I \qquad \forall T\geq T_1,\ t\geq t_1$$

A.4: there exists a constant matrix $R \in R^{p_2 x P_1}$, of full row rank such that

$$Q^{(2)}(s) = RQ^{(1)}(s) \qquad (4.2)$$

Then there exist constants $\alpha_2>0$, $\beta_2>0$ and $T_2\geq T_1$ such that

$$\beta_2 I \geq \frac{1}{T_2}\int_t^{t+T_2} y_2(\tau)y_2^T(\tau)d\tau \geq \alpha_2 I \qquad t\geq t_1 \qquad (4.3)$$

□

Proof: For any $t\geq t_1$ and $T\geq T_1$, define $\phi(\tau)$ on $(t,t+T)$ as the solution of

$$p^{(1)}(-D)\phi(\tau) = y_1(\tau),$$
$$\phi(t+T) = \phi^{(1)}(t+T) = \dots = \phi^{(n_1-1)}(t+T) = 0 \qquad (4.4)$$

and $\psi(\tau)$ as

$$\psi(\tau) = p^{(2)}(-D)\phi(\tau) \qquad \tau \in (t,t+T) \qquad (4.5)$$

(ψ is well defined because $n_1\geq n_2$). The Cauchy-Schwartz inequality yields:

$$\int_t^{t+T} y_2(\tau)y_2^T(\tau)d\tau \int_t^{t+T} \psi^T(\tau)\psi(\tau)d\tau$$
$$\geq \left(\int_t^{t+T} y_2(\tau)\psi^T(\tau)d\tau\right)\left(\int_t^{t+T} y_2(\tau)\psi^T(\tau)d\tau\right)^T \qquad (4.6)$$

By Lemma 3.2

$$\int_t^{t+T} y_2(\tau)\psi^T(\tau)d\tau = \int_t^{t+T} Ry_1(\tau)y_1^T(\tau)d\tau + g^{(1,2)}(t,t+T) \qquad (4.7)$$

It follows by A.1 and A.2 that there exist K_1, K_2 such that

$$||\psi(t)||_\infty^2 \leq K_1 \text{ and } ||g^{(1,2)}(t,t+T)||_\infty \leq K_2 \qquad (4.8)$$

Therefore, for $T\geq T_1$, using the full rank property of R:

$$\frac{1}{T}\int_t^{t+T} y_2(\tau)y_2^T(\tau)d\tau \geq \frac{1}{K_1T^2}\left[\alpha_1^2T^2||R||^2 - 2\beta_1T||R||K_2\right] \qquad (4.9)$$

Taking

$$T_2 = r\frac{2\beta_1K_2}{\alpha_1^2||R||} \qquad r > 1 \qquad (4.10)$$

yields (4.3) with $\beta_2 = ||y_2||_\infty$ and

$$\alpha_2 = \frac{r-1}{r}\,||R||^2 \cdot \alpha_1 \cdot \left(\frac{\alpha_1}{K_1}\right) \qquad (4.11)$$

□

Theorem 4.1 can also be written (mixing time and Laplace transform notations):

$$y_i(t) = \frac{1}{p^{(i)}(s)}\,Q^{(i)}(s)u(t) \qquad i=1,2 \qquad (4.13)$$

The result then says that if $p^{(1)}(s)$ and $p^{(2)}(s)$ are Hurwitz, with $\deg(p^{(1)}) \geq \deg(p^{(2)})$ and if $Q^{(2)}(s) = RQ^{(1)}(s)$, then the persistence of excitation of $y_1(t)$ implies the persistence of excitation of $y_2(t)$.

This result can be used to obtain a natural definition for the persistency of excitation of a vector valued input signal $u(t)$, in terms of a basis vector $\phi(t)$ derived from $u(t)$, and which can serve as a basis for all vectors $y(t)$ generated by a MIMO system of given structure.

5. TIME VARYING SYSTEMS

In this section we derive persistence of excitation conditions for time-varying systems whose parameter variations are sufficiently slow.

Theorem 5.1: Consider the time-varying system

$$\dot{x}(t) = F(t)x(t) + G(t)u(t)$$
$$y(t) = H(t)x(t) + J(t)u(t) \qquad (5.1)$$

with $x\in R^n$, $u\in R^m$, $y\in R^p$, and with the following assumptions:
A.1: $F(t)$, $G(t)$, $H(t)$, $J(t)$ are bounded regulated matrix functions of t
A.2: there exist $K_1>0$ and $\alpha_1>0$ such that

$$||\phi(t,t_0)|| \leq K_1 e^{-\alpha_1(t-t_0)}$$

for all t,t_0, where $\phi(t,t_0)$ is the state-transition matrix of $\dot{z}(t) = F(t)z(t)$.
A.3: there exists $\delta>0$ such that for all $t,T>0$ and for all $s,\tau \in (t,t+T)$:

$$\frac{1}{T}\,||F(s) - F(\tau)|| < \delta,\ \frac{1}{T}\,||G(s) - G(\tau)|| < \delta,$$
$$\frac{1}{T}\,||H(s) - H(\tau)|| < \delta \ ,\ \frac{1}{T}\,||J(s) - J(\tau)|| < \delta \qquad (5.2)$$

A.4: there exists $\alpha_2>0$, and $\forall T>0$ and t_0, there exists $\sigma\in(t_0,t_0+T)$ such that
i) $\mathrm{Re}\lambda_i(F(\sigma)) \leq -\alpha_2$ $\quad i=1,\dots,n$
ii) The output $\bar{y}(t)$ of the system

$$\dot{\bar{x}}(t) = F(\sigma)\bar{x}(t) + G(\sigma)u(t) \quad \bar{x}(t_0)=x(t_0)$$
$$\bar{y}(t) = H(\sigma)\bar{x}(t) + J(\sigma)u(t) \qquad (5.3)$$

is PE, i.e. there exist $\alpha_3>0$, and $t_-\geq 0$ such that

$$\frac{1}{T}\int_{t_0}^{t_0+T}\bar{y}(\tau)\bar{y}^T(\tau)d\tau \geq \alpha_3 I \qquad \forall t_0\geq t_- > 0 \qquad (5.4)$$

Then $y(t)$ is PE for δ sufficiently small, i.e. there exist $\alpha_4>0$ and $\delta_1>0$ such that, if $\delta<\delta_1$, then

$$\frac{1}{T}\int_{t_0}^{t_0+T} y(\tau)y^T(\tau)d\tau \geq \alpha_4 I \qquad \forall t_0\geq t_- > 0 \qquad (5.5)$$

Proof: Consider an arbitrary $t_0\geq t_-$ and $T>0$, and a $\sigma\in(t_0,t_0+T)$ satisfying A.4. Denote $e(t) = x(t)-\bar{x}(t)$, $t\in(t_0,t_0+T)$. It follows by A.2, A.3, A.4 that

$$\sup_{t_0\leq t\leq t_0+T} ||e(t)|| \leq K_2\delta T||u||_\infty \qquad (5.6)$$

for some finite $K_2>0$. Therefore, using A.1, A.3 and (5.6)

$$\sup_{t_0\le t\le t_0+T} ||y(t) - \bar{y}(t)|| \le K_3\delta T||u||_\infty \qquad (5.7)$$

for some finite $K_3>0$. Now $\forall t_0 \ge t_-$

$$\frac{1}{T}\int_{t_0}^{t_0+T} y(\tau)y^T(\tau)d\tau \ge \frac{1}{2T}\int_{t_0}^{t_0+T_-} \bar{y}(\tau)\bar{y}(\tau)^T d\tau$$

$$- \frac{1}{T}\int_{t_0}^{t_0+T} [y(\tau) - \bar{y}(\tau)][y(\tau) - \bar{y}(\tau)]^T d\tau$$

$$\ge \left(\frac{\alpha_3}{2} - K_3^2\delta^2T^2||u||_\infty^2\right)I \qquad (5.8)$$

Let $\delta_1 = (\alpha_3/2)^{\frac{1}{2}}(K_3T||u||_\infty)^{-1}$. Then for $\delta < \delta_1$, there exists $\alpha_4>0$ such that (5.5) holds. □

Our main result for time-varying SISO systems shows that if the input is sufficiently rich of prescribed order and if the time variation of the parameters is slow enough, then a regressor vector containing filters of the input and output is persistently exciting provided a uniform output reachability condition is satisfied.

Theorem 5.2: Consider the time-varying SISO system (5.1) ($m = p = 1$) and the regression vector

$$\phi(t) = \frac{1}{p(s)} Q(s) \begin{vmatrix} y(t) \\ u(t) \end{vmatrix} \qquad (5.9)$$

where $p(s)$ is a polynomial and $Q(s)$ is a rx2 polynomial matrix, with the following assumptions:

A.1, A.2 and A.3: as in Theorem 6.1

A.4: $\deg Q(s) \triangleq q < n$, where $\deg Q(s) \triangleq \max_{i,j} \deg q_{ij}(s)$

A.5: $p(s)$ is Hurwitz and $\deg p(s) \ge \deg Q(s)$

A.6: With $M \in R^{rx2(q+1)}$ defined by

$$Q(s) = M[I_2 \; sI_2 \; \dots \; s^q I_2]^T$$

we have $\lambda_{min}(MM^T) = \alpha_2 > 0$.

A.7: $u(t)$ is bounded and sufficiently rich of order $n+q+1$, i.e. there exist $t_1, \alpha_3>0$, $\alpha_4>0$ and $T_1>0$ such that

$$\alpha_3 I \le \frac{1}{T}\int_t^{t+T} \psi(\tau)\psi^T(\tau)d\tau \le \alpha_4 I \quad \forall T\ge T_1, t\ge t_1 \qquad (5.10)$$

where

$$\psi(t) = \frac{1}{(s+\gamma)^{n+q}} [1 \; s \; \dots \; s^{n+q}]^T u(t) \qquad (5.11)$$

A.8: There exist $\alpha_5>0$, $\alpha_6>0$, $K_2>0$ and $\forall T>0$ and t_0, there exists $\sigma \in (t_0, t_0+T)$ such that

i) $\mathrm{Re}\,\lambda_i(F(\sigma)) < -\alpha_5$

ii) $H(\sigma)[sI-F(\sigma)]^{-1} G(s) + J(s) = b_\sigma(s)/a_\tau(s)$

Denote the Sylvester matrix of the polynomials b_σ and a_σ by S_σ, and assume that $\lambda_{min}(S_\sigma S_\sigma^T) \ge \alpha_6$ for all σ, some $\alpha_6>0$.

Then $\phi(t)$ is PE for sufficiently small δ, i.e. there exist $\alpha_7>0$, $T_2\ge T_1$, such that if $\delta < \delta_1$ then

$$\frac{1}{T_2}\int_t^{t+T_2} \phi(\tau)\phi^T(\tau)d\tau \ge \alpha_7 I \quad \forall t\ge t_1 \qquad (5.12)$$

□

Proof:

Step 1: Let σ satisfy A.8 and consider the output $\bar{y}(t)$ generated by (5.3) for that fixed σ. Define

$$\bar{\phi}(t) = \frac{1}{p(s)} Q(s) \begin{vmatrix} \bar{y}(t) \\ u(t) \end{vmatrix}$$

Using the proof of Theorem 5.1 and A.5, it is easy to show that

$$\sup_{t_0\le t\le t_0+T} ||\phi(t) - \bar{\phi}(t)|| \le C_1\delta T||u||_\infty \qquad (5.14)$$

where C_1 is a function of the system (5.1) and the filter $Q(s)/p(s)$.

Step 2: Now notice that $\bar{\phi}(t)$ can be written as

$$\bar{\phi}(t) = \frac{1}{p(s)a_\sigma(s)} MS_\sigma[1 \; s \; \dots \; s^{n+q}]^T u(t) \qquad (5.15)$$

$$= \frac{(s+\gamma)^{n+q}}{p(s)a_\sigma(s)} U_\sigma \Psi(t) \quad \text{with } U_\sigma \triangleq MS_\sigma \qquad (5.16)$$

With the uniform output reachability conditions A.6 and A.8.ii) we prove the uniform reachability of (5.16). It then follows from Theorem 4.1 that there exist $\alpha_8>0$ and $T_2\ge T_1$, such that

$$\frac{1}{T_2}\int_t^{t+T_2-} \bar{\phi}(\tau)\bar{\phi}^T(\tau)d\tau \ge \alpha_8 I \qquad (5.17)$$

Step 3: The result then follows from (5.14) and (5.17), using the same argument as in Theorem 5.1. □

7. CONCLUSIONS

The main contribution has been to present conditions under which the persistency of excitation of one regression vector implies that of another related regression vector, for both time-invariant and time-varying stable systems.

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