Frequency Domain and State Space Methods for Linear Systems
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WHEN IS THE SOLUTION OF THE RICCATI EQUATION STABILIZING AT EVERY INSTANT?

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We consider the Kalman filter associated with a continuous-time or discrete-time state-space model with constant parameters. This Kalman filter has a time-varying closed-loop state transition matrix, which is directly obtained from the time-varying solution of a Riccati differential or difference equation (RDE). We consider the problem of selecting an initial covariance matrix P_0 for the RDE to ensure that the closed loop filter is exponentially asymptotically stable at every instant as a time-invariant filter. The results have application in observer design.

INTRODUCTION

This paper is concerned with the stabilizing properties of the solutions P(t) of the Riccati difference or differential equation (RDE) of optimal filtering. We consider either the discrete-time Kalman filter

$$\hat{x}(t+1) = [F-K(t)H]\hat{x}(t) + K(t)y(t) \tag{1.1a}$$

$$K(t) = FP(T)H^T[HP(t)H^T+I]^{-1} \tag{1.1b}$$

and the associated discrete-time RDE:

$$P(t+1) = FP(t)F^T - FP(t)H^T[HP(t)H^T+I]^{-1}HP(t)F^T + LL^T \tag{1.1c}$$

or the continuous-time filter

$$\dot{\hat{x}}(t) = [F-K(t)H]\hat{x}(t) + K(t)y(t) \tag{1.2a}$$

$$K(t) = P(t)H^T \tag{1.2b}$$

and the associated continuous-time RDE:

$$\dot{P}(t) = P(t)F^T + FP(t) - P(t)H^THP(t) + LL^T \tag{1.2c}$$

These Kalman filters are associated with the following signal models.

*The order of the authors' names was decided by a random walk.

In discrete-time:

$$x(t+1) = Fx(t) + w(t) \quad (1.3a)$$

$$y(t) = Hx(t) + v(t)$$

where x and y have dimension n and p respectively, and where w(t) and v(s) are uncorrelated with

$$E(w(t)w^T(s)] = LL^T\delta_{ts} \quad (1.4a)$$

$$E[v(t)v^T(s)] = I\delta_{ts} \quad (1.4b)$$

In continuous-time:

$$\dot{x}(t) = Fx(t) + w(t) \quad (1.5a)$$

$$y(t) = Hx(t) + v(t) \quad (1.5b)$$

with the same conditions on x,y,w and v where δ_{ts} is replaced by $\delta(t-s)$.

We then ask the question: Under what conditions on the underlying signal model (i.e. H,F,L) and for what initial conditions P(∘) of the RDE are the closed-loop filtering matrices {F-K(t)H} exponentially asymptotically stable (e.a.s.) matrices for all t≥0? We say that a matrix A is e.a.s. if

- $|\lambda_i(A)| < 1$, i=1,...,n, in discrete-time (1.6a)
- $Re\lambda_i(A) < 0$, i=1,...,n, in continuous-time (1.6b)

Surprisingly, this problem does not seem to have been studied before. Besides its theoretical interest, the major practical motivation for studying this problem is as follows. The conditions under which the Kalman filter is asymptotically stable as a time-varying filter are well established: see e.g. [1]. However, here we want to study the exponential stability of the time-invariant filter

$$\hat{x}(s+1) = [F-K(t)H]\hat{x}(s) + K(t)y(s) \quad (1.7)$$

in discrete-time, or

$$\dot{\hat{x}}(s) = [F-K(t)H]\hat{x}(s) + K(t)y(s) \quad (1.8)$$

in continuous-time, where K(t) is given, respectively, by (1.1b,c) and (1.2b,c), for some arbitrary but <u>fixed</u> t.

There are conditions under which the Kalman filter is asymptotically stable as a time-varying filter and converges, but the limiting steady-state filter need not be asymptotically stable. This is the case when the pair [F,L] has uncontrollable modes on the unit circle (in discrete-time) or on the jω-axis (in continuous-time): see [2], [3]. The results of this paper then allow the use of the Kalman filter as an optimal time-varying filter from an initial start-up time until a switching-off or freezing time, from which the Kalman gain update is stopped and the filter is run as a time-invariant exponentially asymptotically stable observer. This particular problem arises naturally in short-time Fourier analysis [4], where L=0 and F has eigenvalues equally spaced around the unit circle.

In this paper, sets of sufficient conditions on H,F,L and P(∘) will be presented under which the matrices {F-K(t)H} are e.a.s. for all t≥0. Our results will be obtained by proving the monotonic non-increasing behaviour of the solution P(t) of the RDEs (1.1c) and (1.2c) under those sets of sufficient conditions. The major results of this paper were presented for the discrete-time problem in [5]. In section 2 we shall briefly recall the major discrete-time results without proof. The continuous-time results will be presented and proved in section 3: the proofs of the continuous-time results closely parallel those of [5] although some simplifications occur in continuous-time. The stability results on {F-K(t)H} are obtained by transforming the RDE into a Fake Algebraic Riccati Equation (FARE) and by using existing results on the Algebraic Riccati Equation (ARE) established in [2]-[3].

For the purposes of this paper, the most relevant previous contributions on the RDE and the ARE are those of Willems [6], Kucera [7], Martensson [8], Caines and Mayne [9], Payne and Silverman [10], Chan, Goodwin and Sin [2] and de Souza, Gevers and Goodwin [3].

2. THE DISCRETE-TIME RESULTS

We consider the discrete-time Kalman filter (1.1). The closed-loop state transition matrix of the Kalman filter is

$$\bar{F}(t) = F - K(t)H = F - FP(t)H^T[HP(t)H^T+I]^{-1}H \quad (2.1)$$

To the RDE (1.1c) there is associated an algebraic Riccati equation (ARE):

$$P = FPF^T - FPH^T(HPH^T+I)^{-1}HPF^T + LL^T \quad (2.2)$$

We shall assume throughout that H has full rank. For each P(t), solution of the RDE (1.1c), we define a symmetric matrix Q(t) as follows:

$$Q(t) \triangleq P(t) - FP(t)F^T + FP(t)H^T[HP(t)H^T+I]^{-1}HP(t)F^T \quad (2.3)$$

Note that Q(t) can also be written as:

$$Q(t) = LL^T + P(t) - P(t+1) \quad (2.4)$$

For fixed t, equation (2.3) can be viewed as an ARE, provided Q(t) can be shown to be nonnegative definite: (2.3) will therefore be called a Fake Algebraic Riccati Equation (FARE). The main trick in our proofs is the introduction of the FARE (2.3) and the use of stability results for the ARE recently obtained in [2] and [3]. The major task then is to obtain conditions on H,F,L and P(∘) which guarantee that Q(t) is nonnegative definite. As is clear from (2.4), this property of Q(t) is closely linked to the monotonically nonincreasing behaviour of P(t). The introduction of the FARE has given rise to a whole set of new techniques called Fake Algebraic Riccati Techniques (FART): see [11].

It is well-known (see e.g. [9]) that if P(∘)=0, then the sequence P(t) increases monotonically, in the sense that P(t+1) ≥ P(t) (i.e. P(t+1)-P(t) is nonnegative definite). However, this monotonic behaviour breaks down if P(∘) is an arbitrary nonnegative definite matrix. In deriving conditions that ensured the monotonic decreasing behaviour of the discrete-time RDE, we were led to prove the following lemma, which we believe is of independent interest; it is inspired by a result of Nishimura [12].

<u>Lemma 1</u>. Consider two RDEs (1.1c) with the same F and H matrices but possibly different $Q=LL^T$ matrices, $\hat{Q}$ and $\bar{Q}$, and possibly different initial conditions. Let the solutions to these RDEs be written

$$\hat{P}(t+1) = f[\hat{P}(t),\hat{Q}], \quad \bar{P}(t+1) = f[\bar{P}(t),\bar{Q}] \quad (2.5)$$

where the functional form of f is identical and is given by (1.1c) with LL^T

replaced by Q. Then

$$\hat{P}(t+1) \geqq \bar{P}(t+1) \text{ if } \hat{P}(t) \geqq P(t) \text{ and } \hat{Q} \geqq Q \qquad (2.6)$$

The proof relies on the following expression which can be obtained from [12] after lengthy manipulations (for simplicity P(t) will be written P_t):

$$\hat{P}_{t+1} - \bar{P}_{t+1} = F[I-\bar{P}_t H^T(H\bar{P}_t H^T+I)^{-1}H]g(\hat{P}_t-\bar{P}_t)$$
$$\cdot\, [I-\bar{P}_t H^T(H\bar{P}_t H^T+I)^{-1}H]^T F^T + \hat{Q} - \bar{Q} \qquad (2.7a)$$

where

$$g(\hat{P}_t-\bar{P}_t) = (\hat{P}_t-\bar{P}_t)H^T[H(\hat{P}_t-\bar{P}_t)H^T + H(\hat{P}_t-\bar{P}_t)H^T(H\bar{P}_t H^T+I)^{-1}H(\hat{P}_t-\bar{P}_t)H^T]^{-1}H(\hat{P}_t-\bar{P}_t)$$
$$+ \{I - (\hat{P}_t-\bar{P}_t)H^T[H(\hat{P}_t-\bar{P}_t)H^T]^{-1}H\}(\hat{P}_t-\bar{P}_t)$$
$$\cdot\, \{I - (\hat{P}_t-\bar{P}_t)H^T[H(\hat{P}_t-\bar{P}_t)H^T]^{-1}H\}^T \qquad (2.7b)$$

No similar continuous-time result is needed to prove the continuous-time results, as will appear from section 3.

Our main results on the discrete-time Riccati equation are as follows.

Theorem 1 [5]. Consider the RDE (1.1c) with initial condition P(∘). Define Q(∘) from (2.3). If

(1) [H,F] is detectable
(2) [F,L] is stabilizable
(3) $P(\circ) \geqq 0$ is such that $Q(\circ) \geqq LL^T$

then the solution sequence P(t) of (1.1c) is stabilizing for all t≧0, i.e. $|\lambda_k[\bar{F}(t)]| < 1$ for all t≧0 and for k=1,2,...,n with $\bar{F}(t)$ defined by (2.1) and λ_k denoting the eigenvalues.

Two comments are in order

(1) First note that the set of P(∘) satisfying condition (3) is clearly non-empty. Indeed, take any $Q(\circ) \geqq LL^T$; then it is easy to show that $[F,Q^{\frac{1}{2}}(\circ)]$ is a stabilizable pair because [F,L] is stabilizable. Conditions (1) and (2) then ensure that the ARE (2.3), with t=0, has a unique nonnegative definite solution P(∘) (see e.g. [3]).

(2) The conditions of Theorem 1 guarantee that the sequence {P(t)} is monotonically nonincreasing and convergent to a steady-state P^+. By condition (2), P^+ is actually stabilizing. As we argued earlier, we envisage the stability results on P(t) to be of particular use when the steady-state P^+ is not stabilizing, due to the pair [F,L]} not being stabilizable. We would therefore want to replace condition (2) of Theorem 1 by a weaker condition which would only involve the initial condition. Our aim is to replace the stabilizability condition on [F,L] by a condition on $[F,Q(\circ)^{\frac{1}{2}}]$. this is achieved in the next Theorem.

Theorem 2 [5]. Consider the RDE (1.1c). Define Q(∘) as in (2.3) and assume that

(1) [H,F] is detectable
(2) $[F,Q(\circ)^{\frac{1}{2}}]$ is stabilizable
(3) P(∘)≧0 is such that $Q(\circ) \geqq LL^T$.

Then the solution sequence {P(t)} with initial condition P(∘) is stabilizing for each t≧0.

3. CONTINUOUS TIME RESULTS

We now consider the continuous-time Kalman filter (1.2). The results presented are directly analogous to the discrete-time results presented in the previous section. Furthermore, in this section, we include the proofs of these continuous-time results. Most of the proofs are directly analogous to the proofs of the discrete-time results given in [5].

Associated with the RDE (1.2c) is an ARE

$$PF^T + FP - PH^THP + LL^T = 0 \qquad (3.1)$$

A solution P^+ to this ARE is said to be a strong solution if all eigenvalues λ of $F - P^+H^TH$ satisfy $Re(\lambda) \leq 0$. This definition was first introduced in [2], in a discrete-time context.

Proposition 1. If the RDE (1.2c) is such that [H,F] is detectable then, for any initial condition P(∘)≧0, there exists a solution to (1.2c) defined on [0,∞). Furthermore, if the initial condition P(∘) is such that $P(\circ) \geqq P^+$ then $\lim_{t\to\infty} P(t) = P^+$ where P^+ is the unique strong solution to the ARE (3.1).

Proof. The existence of solutions to the RDE (1.2c) follows from remark M in [13]. Furthermore, the fact that the ARE (3.1) has a unique strong solution P^+ and that $\lim_{t\to\infty} P(t) = P^+$ follows from Poubelle et al. [14]. A discrete-time version of this result has been proved in [3].

We now introduce the Fake Algebraic Riccati Equation for the continuous-time case. If P(t) is the solution to RDE (1.2c), then for each P(t), we define an associated symmetric matrix Q(t) according to

$$Q(t) \triangleq P(t)H^THP(t) - P(t)F^T - FP(t). \qquad (3.2)$$

This is the Fake Algebraic Riccati Equation corresponding to the RDE (1.2c). Comparing RDE (1.2c) to ARE (3.2), we obtain

$$Q(t) = LL^T - \dot{P}(t). \qquad (3.3)$$

The following lemma concerns the monotonicity of the solutions to the RDE (1.2c).

Lemma 2. Consider the RDE (1.2c) with [H,F] detectable and let Q(t) be defined as in (3.2). If the initial condition P(∘)≧0 is such that $Q(\circ) \geqq LL^T$ then $P(t_2) \leqq P(t_1)$ for all times $t_2 \geqq t_1 \geqq 0$ and hence $\dot{P}(t) \leqq 0$ for all t≧0.

Proof. By Proposition 1 and the assumptions of the lemma, the RDE (1.2c) has a solution P(t) for all t≧0. We now introduce the notation

$$\tilde{F} = F - P(\circ)H^TH;$$

$$\tilde{P}(t) = P(t) - P(\circ).$$

It follows that $\tilde{P}(t)$ satisfies the RDE

$$\dot{\tilde{P}}(t) = \tilde{P}(t)\tilde{F}^T + \tilde{F}\tilde{P}(t) - \tilde{P}(t)H^TH\tilde{P}(t) + LL^T - Q(\circ) \qquad (3.4)$$

with initial condition $\tilde{P}(\circ)=0$. We will use a duality argument to interpret $\tilde{P}(t)$ in terms of the solution to a certain optimal control problem.

Indeed, using Theorem 21.1 of [15], it follows that given any vector x_1 and $t_1>0$ then

$$x_1^T\tilde{P}(t_1)x_1 = \min_{u(\cdot)} \int_0^{t_1} [x(t)^T(LL^T-Q(0))x(t) + u(t)^Tu(t)]dt$$

subject to

$$\dot{x}(t) = -\tilde{F}^Tx(t) + H^Tu(t); \quad x(t_1) = x_1.$$

As in [15], this optimal control problem has a solution $\tilde{x}(\cdot)$, $\tilde{u}(\cdot)$. We let $t_2 \geq t_1$ be given and let $\Delta = t_2-t_1$. A control law $\hat{u}(\cdot)$ is now defined according to

$$\hat{u}(t) \triangleq 0 \text{ for } t \in [0,\Delta]$$

$$\hat{u}(t) \triangleq \tilde{u}(t-\Delta) \text{ for } t \in [\Delta,t_2]$$

Let $\hat{x}(t)$ be the corresponding solution to the state equation

$$\dot{x}(t) = -\tilde{F}^Tx(t) + H^Tu(t); \quad x(t_2) = x_1.$$

Using the fact that this system is time-invariant, it follows that $\hat{x}(t) = \tilde{x}(t-\Delta)$ for all t. Therefore

$$x_1^T\tilde{P}(t_1)x_1 = \int_0^{t_1} [\tilde{x}(t)^T(LL^T-Q(0))\tilde{x}(t) + \tilde{u}(t)^T\tilde{u}(t)]dt$$

$$= \int_\Delta^{t_2} [\tilde{x}(t-\Delta)^T(LL^T-Q(0))\tilde{x}(t-\Delta) + \tilde{u}(t-\Delta)^T\tilde{u}(t-\Delta)]dt$$

$$= \int_\Delta^{t_2} [\hat{x}(t)^T(LL^T-Q(0))\hat{x}(t) + \hat{u}(t)^T\hat{u}(t)]dt.$$

However, $LL^T-Q(0) \leq 0$ and $\hat{u}(t) = 0$ for $t \in [0,\Delta]$. Hence

$$\int_\Delta^{t_2} [\hat{x}(t)^T(LL^T-Q(0))\hat{x}(t) + \hat{u}(t)^T\hat{u}(t)]dt$$

$$\geq \int_0^{t_2} [\hat{x}(t)^T(LL^T-Q(0))\hat{x}(t) + \hat{u}(t)^T\hat{u}(t)]dt$$

$$\geq x_1^T\tilde{P}(t_2)x_1$$

using the optimal control interpretation of $\tilde{P}(t_2)$. Thus, we have established that for arbitrary x_1

$$x_1^T\tilde{P}(t_1)x_1 \geq x_1^T\tilde{P}(t_2)x_1.$$

That is $\tilde{P}(t_1) \geq \tilde{P}(t_2)$ and hence $P(t_1) \geq P(t_2)$. The fact that $\dot{P}(t) \leq 0$ now follows directly from the definition of the $\tilde{P}(t)$. □

Using this lemma, we obtain the following theorem, which is the continuous-time version of Theorem 1 in [5].

Theorem 3. Consider the RDE (1.2c). Define $Q(0)$ as in (3.2) and assume that

(1) $[H,F]$ is detectable
(2) $[F,L]$ is stabilizable
(3) $P(0) \geq 0$ is such that $Q(0) \geq LL^T$,

then the solution $P(t)$ of the RDE (1.2c) is stabilizing for all $t \geq 0$, i.e. all eigenvalues λ of the matrix $F - P(t)H^TH$ satisfy $Re(\lambda) < 0$.

Proof: Since $P(0)$ and $Q(0)$ are related by the FARE (3.2), it follows from Theorem 1 of [16] that $Q(0) \geq LL^T$ implies $P(0) \geq P^+$, where P^+ is the unique

strong solution of (3.1). Using Proposition 1, it follows that the RDE (1.2c) has a solution $P(t) \to P^+$. Futhermore, Lemma 2 implies that $\dot{P}(t) \geq 0$ for all $t \geq 0$. Thus, equation (3.3) implies that given any $t \geq 0$

$$Q(t) \geq LL^T \tag{3.5}$$

Using this fact and Condition (2) of the theorem statement, it is straightforward to establish that $[F,Q(t)^{\frac{1}{2}}]$ is stabilizable. However, $P(t)$ satisfies the FARE

$$FP(t) + P(t)F^T - P(t)H^THP(t) + Q(t) = 0$$

where $[H,F]$ is detectable and $[F,Q(t)^{\frac{1}{2}}]$ is stabilizable. Hence using Theorem 4.11 of [17], it follows that the matrix $F - P(t)H^TH$ is a stability matrix. □

As in the discrete-time case, it would be desirable to weaken the requirement that $[F,L]$ is stabilizable. This is achieved in the following theorem.

Theorem 4. Consider the RDE (1.2c). Define $Q(0)$ as in (3.2) and assume that

(1) $[H,F]$ is detectable
(2) $[F,Q(0)^{\frac{1}{2}}]$ is stabilizable
(3) $P(0) \geq 0$ is such that $Q(0) \geq LL^T$.

Then the solution $P(t)$ with initial condition $P(0)$ is stabilizing for each $t \geq 0$.

In order to prove this theorem, we first establish a number of preliminary lemmas.

Lemma 3. Let $P(t)$ be the solution to the RDE (1.2c) with initial condition $P(0) \geq 0$, then $\dot{P}(t) \geq 0$ for all $t \geq 0$. Furthermore, if x_1 is an eigenvector of F^T such that $x_1^*\dot{P}(t_1)x_1 = 0$ for some $t_1 \geq 0$ then $x_1^*\dot{P}(0)x_1 = 0$ (where x_1^* denotes the complex conjugate transpose of the vector x_1).

Proof. This proof relies on a duality argument in which $\dot{P}(t)$ is related to a certain optimal control problem. We observe that it is straightforward to extend Theorem 21.1 of [15] to include the case where both $x(t)$ and $u(t)$ are complex vectors. Hence, given any $t_1 \geq 0$ and any vector x_1,

$$x_1^*\dot{P}(t_1)x_1 = \min_{u(\cdot)} \int_0^{t_1} [x(t)^*Qx(t) + u(t)^*u(t)]dt + x(0)^*\dot{P}(0)x(0) \tag{3.6}$$

subject to

$$\dot{x}(t) = -F^Tx(t) + H^Tu(t); \quad x(t_1) = x_1.$$

Since all terms on the right hand side of (3.6) are nonnegative, it follows that $x_1^*\dot{P}(t_1)x_1 \geq 0$. Furthermore, if x_1 is an eigenvector of F^T such that $x_1^*\dot{P}(t_1)x_1 = 0$, we obtain

$$\tilde{x}(0)^*\dot{P}(0)\tilde{x}(0) = 0, \tag{3.7}$$

$$\tilde{x}(t)^*Q\tilde{x}(t) \equiv 0 \text{ on } [0,t_1]$$

and

$$\tilde{u}(t)^*\tilde{u}(t) \equiv 0 \text{ on } [0,t_1],$$

where $\tilde{x}(\cdot)$, $\tilde{u}(\cdot)$ solves the given optimal control problem. Therefore

$$\dot{\tilde{x}}(t) = -F^T\tilde{x}(t); \quad \tilde{x}(t_1) = x_1.$$

Using the fact that x_1 is an eigenvector of F^T with eigenvalue λ, it follows that

$$\tilde{x}(\circ) = e^{+\lambda t_1} x_1.$$

Substituting this into (3.7) we obtain

$$x_1^* P(\circ) x_1 = 0. \quad \square$$

Lemma 4. Let $P(t) \geq 0$ be a solution to RDE (1.2c) such that $\dot{P}(t) \leq 0$ for all t and $HP(t_1)x = 0$ for some $t_1 \geq 0$ and some vector x. Then $HP(t)x = 0$ for all $t \geq t_1$.

Proof. For each t, we can write

$$P(t) = V_1(t)^T V_1(t) + V_2(t)^T V_2(t)$$

where $V_1(t)^T$ is in the range space of H^T and $V_2(t)^T$ is in the null space of H. Similarly, write $x = x_1 + x_2$. Then $HP(t_1)x = 0$ implies that $V_1(t_1)x_1 = 0$. Furthermore, given any $t \geq t_1$, the fact that $P(t) \leq P(t_1)$ implies that $V_1(t_1)^T V_1(t_1) \geq V_1(t)^T V_1(t)$. Therefore $V_1(t)x_1 = 0$ and hence $HP(t)x = 0$. $\square$

Proof of Theorem 4. Consider the RDE (1.2c) with initial condition $P(\circ)$. Using Proposition 1 and Lemma 3, it follows that there exists a solution P(t) such that $P(t) \geq 0$ for all $t \geq 0$. Furthermore, using Lemma 2, $\dot{P}(t) \leq 0$ for all $t \geq 0$. Now let $t_1 \geq 0$ be given and let x be an eigenvector of $\bar{F}^T \triangleq F^T - H^T HP(t_1)$ with eigenvalue λ. We must show that $Re(\lambda) < 0$.

The matrix $P(t_1)$ satisfies the ARE (3.2) and hence

$$\bar{F}P(t_1) + P(t_1)\bar{F}^T + P(t_1)H^T HP(t_1) + Q(t_1) = 0.$$

Therefore

$$(\lambda+\lambda^*)x^*P(t_1)x + x^*P(t_1)H^T HP(t_1)x + x^*Q(t_1)x = 0. \tag{3.8}$$

We note that $\dot{P}(t_1) \leq 0$ and hence (3.3) implies $Q(t_1) \geq 0$.

We now consider two cases:

Case 1. $x^*P(t_1)x \neq 0$.

In this case, (3.8) implies $\lambda+\lambda^* \leq 0$, i.e. $Re(\lambda) \leq 0$. We will show $Re(\lambda) \neq 0$ by contradiction. Suppose $\lambda+\lambda^* = 0$. Using (3.8) we conclude that $x^*Q(t_1)x = 0$ and $HP(t_1)x = 0$. Hence, using Lemma 4, it follows that $HP(t)x = 0$ for all $t \geq t_1$. Now using [16] and Proposition 1, it follows that $P(t) \to P^+$ and hence $HP^+x = 0$. We also observe that $(F^T - H^T HP(t_1))x = \lambda x$ and hence $F^T x = \lambda x$. Furthermore, substituting $\dot{P}(t_1) \leq 0$ and $x^*Q(t_1)x = 0$ into (3.3) it follows that $x^*LL^T x = 0$. Now, returning to RDE (1.2c), it follows that for all $t \geq t_1$,

$$\begin{aligned} x^*\dot{P}(t)x &= x^*P(t)F^T x + x^*FP(t)x - x^*P(t)H^T HP(t)x + x^*LL^T x \\ &= (\lambda+\lambda^*)x^*P(t)x \\ &= 0. \end{aligned}$$

Hence, $x^*P(t)x = x^*P(t_1)x$ for all $t \geq t_1$ and therefore $x^*P^+x = x^*P(t_1)x \neq 0$.

Now

$$FP^+ + P^+F^T - P^+HH^TP^+ + LL^T = 0.$$

Therefore $FP^+x + P^+F^Tx = 0$ and hence, if we let $z = P^+x$ then

$$Fz = -\lambda P^+x = -\lambda z.$$

Furthermore, $z = P^+x \neq 0$ and therefore z is an eigenvector of F with eigenvalue $-\lambda$. Also, note that $Hz = 0$. The fact that $Re(\lambda) = 0$ now contradicts the detectability of the pair [H,F].

Case 2. $x^*P(t_1)x = 0$.

In this case, it follows from (3.8) that $HP(t_1)x = 0$ and $x^*Q(t_1)x = 0$. Now, we have $(F^T - HH^TP(t_1))x = \lambda x$ and hence $F^Tx = \lambda x$. That is, x is an eigenvector of F^T. Furthermore, it follows from Lemma 3 that $P(\circ)x = 0$. We now conclude from ARE (3.2) that

$$\begin{aligned} x^*Q(\circ)x &= x^*P(\circ)H^T HP(\circ)x - x^*P(\circ)F^T x - x^*FP(\circ)x \\ &= 0 \end{aligned}$$

That is, x is an eigenvector of F^T (with eigenvalue λ) such that $x^*Q(\circ)x = 0$. The detectability of $[Q(\circ)^{\frac{1}{2}}, F^T]$ now implies that $Re(\lambda) < 0$. $\square$

4. CONCLUSION

Our results provide a rather easy procedure to initialize a Riccati equation in such a way that the RDE can be stopped at any time and the corresponding Kalman gain used as the constant gain of an observer, while guaranteeing the exponential stability of that observer. While our conditions are only sufficient, it has proved remarkably difficult to obtain necessary and sufficient conditions. In [5] we have given a number of counterexamples in discrete-time, which show just how badly the Riccati equation solution can behave if one of the conditions of our main theorems is violated.

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Frequency Domain and State Space Methods for Linear Systems
C.I. Byrnes and A. Lindquist (Editors)

RICCATI AND LYAPUNOV SYSTEMS

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By application of the annihilating polynomial method, the resolution problem of a Riccati system, $A_i+B_iX+XC_i+XD_iX=0$, $i=1,2$, is reduced to a linear system of the type $A+XB=0$, $C+DX=0$.

INTRODUCTION

The resolution problem of an algebraic quadratic equation of Riccati type

$$A_1 + B_1 X + X C_1 + X D_1 X = 0 \tag{1.1}$$

or the resolution problem of a linear equation of Lyapunov type

$$A_1 + B_1 X + X C_1 = 0 \tag{1.2}$$

appears, in the context of system theory, in linear-quadratic optimal control problems, filtering problems, stability, ... , [1].

In this paper we study the more general problem of Riccati systems, that is, systems of the type

$$\left.\begin{aligned} A_1 + B_1 X + X C_1 + X D_1 X = 0 \\ A_2 + B_2 X + X C_2 + X D_2 X = 0 \end{aligned}\right\} \tag{1.3}$$

as well as the Lyapunov systems

$$\left.\begin{aligned} A_1 + B_1 X + X C_1 = 0 \\ A_2 + B_2 X + X C_2 = 0 \end{aligned}\right\} \tag{1.4}$$

A motivation for the study of such systems appears in the analysis of perturbation problems. So, for instance, notice that a solution of (1.3) is a solution of (1.1) and of the perturbed Riccati equation

$$(A_1 + A_2) + (B_1 + B_2)X + X(C_1 + C_2) + X(D_1 + D_2)X = 0$$

In the systems and matrices bibliography a great deal of information on the resolution methods of matrix algebraic equations is shown, [2]. The annihilating polynomial method applied by Jameson [3], and Jones [4], in the resolution problems of equations (1.2) and (1.1), respectively, is extended to the resolution problems of systems (1.3) and (1.4).