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ON THE ESTIMATION OF THE VARIOGRAM IN SPATIAL INTERPOLATION METHODS USED IN GROUNDWATERFLOW MODELLING

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ABSTRACT

The solution of the groundwaterflow modelling problem requires the estimation of the piezometric head, considered as a two-dimensional (2-D) random process, at all nodes of a grid from measurements at some points of the spatial domain. This can be achieved using a linear minimum variance unbiased estimation method called "kriging".

Kriging requires that a model be estimated for the variogram of the piezometric head, Var $\{h(x,y)-h(x',y')\}$, as a function of the distance d between the points (x,y) and (x',y') using increments of available measurements. The choice of these increments raises several problems, which are the main object of this paper.

I. MOTIVATION : THE GROUNDWATER FLOW MODELLING PROBLEM

The steady state behavior of the groundwaterflow in an isotropic unconfined aquifer is described by the following two-dimensional flow equation :

$$\frac{\partial}{\partial x}\left(T(x,y)\,\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial h}{\partial y}\right) + q(x,y)=0 \qquad (1)$$

where :
$(x,y) \in \Omega$ are the spatial coordinates in the studied domain Ω
h(x,y) is the piezometric head (≐ "the state")
T(x,y) is the transmissivity (= "the parameter")
q(x,y) is the aquifer recharge (or discharge) rate (= the input).

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The identification problem consists in estimating T(x,y) in the whole spatial domain Ω from available measurements of h(x,y) at certain points in Ω. Various assumptions can be made concerning the recharge q(x,y).

Often q(x,y) can be assumed constant over the whole spatial domain and this constant value can be estimated [1].

The identification problem as described here has been studied by a number of authors and a variety of solutions have been proposed under various assumptions [2]- [6]. In most cases these solutions require a large amount of available piezometric measure points. In [1] various methods have been discussed and compared, and a procedure has been proposed for the case where few measure points are available.

In the proposed procedure T(x,y), h(x,y) and q(x,y) are considered as spatial (2-dimensional) stochastic processes, and the flow equation is a stochastic equation. The method requires a discretization of the spatial domain. The solution is then obtained in two steps. First estimates of the piezometric head are obtained at all nodes of the discretization grid from the available measurements. These estimates are minimum variance unbiased estimates obtained through a procedure known as "kriging" which will be detailed later in this paper. Then estimates of the transmissivities are obtained by minimizing a uniformity criterion on T(x,y) subject to satisfying the state constraints, in which the true piezometric heads are replaced by their estimated values.

This procedure has the advantage that the estimated transmissivities can be given a stochastic interpretation which is consistent with experimental geophysical findings, namely that the logarithm of the transmissivities behave like a 2-D Wiener process. In addition the kriging procedure produces covariance estimates for the estimated piezometric heads. From these error covariance estimates for $\hat{h}(x,y)$ it is possible to derive confidence intervals for the estimated transmissivities T(x,y) (see [7]-[8]).

Kriging requires a model for the variogram Var $\{h(x,y)-h(x',y')\}$ of the piezometric head as a function of the distance d between the points (x,y) and (x',y'). A crucial step of the kriging procedure consists in estimating this variogram from increments of available measurements of h. This is normally done by least squares regression using a "large number" of increments ; however no systematic procedure or theory exists as to how to choose these increments. This choice raises several questions.in particular : what is the effect of statistically dependant increments on the estimated variogram ? how many increments should one chooze ? These questions are the main object of this paper. Partial answers are given and some guidelines are proposed.

Due to space limitation we shall not elaborate on the identification of the transmissivities in the sequel, even though the conference version has largely dealt with this problem. We shall limit ourselves to the presentation of the kriging procedure for the estimation of h, and to the estimation of the variogram that is required for this procedure. We refer the reader to [1]or [8] for the application to groundwaterflow modelling.

II. STOCHASTIC INTERPOLATION IN TWO-DIMENSIONAL SPACE

Stochastic hypotheses for piezometry.

The piezometry $h(x,y)$ is assumed to be a twodimensionnal non stationary stochastic process, which is decomposed as follows :

$$h(x,y) = m(x,y) + z(x,y) \tag{2}$$

$m(x,y)$ is the mean of the process, also called the drift. It is assumed to vary slowly in space and to be of the form

$$m(x,y) = E\{h(x,y)\} = \sum_{l=1}^{L} a_l f_l(x,y) \tag{3}$$

The functions $f_l(x,y)$ are chozen a priori (e.g. they are monomials in x and y of order $\leqslant k$), and the a_l are unknown coefficients. For simplicity of notations we shall from now on designate the location (x_k,y_k) by the single variable u_k.

We now introduce the "variogram" γ_{kj} of the process h ; it is defined as the variance of the increments of h :

$$\gamma_{kj} = \frac{1}{2} \mathrm{Var}\{h(u_k)-h(u_j)\} \ , \ u_k,u_j \in \Omega \tag{4}$$

The reason for introducing the variogram will become clear later, in the context of spatial interpolation.

It will be assumed that the variogram γ_{kj} of the process h depends only on the distance d_{kj} between the points u_k and u_j :

$$\gamma_{kj} = \gamma(d_{kj}) \tag{5}$$

This is an intuitively plausible assumption for many physical processes, of which Wiener processes are a special case.

The actual piezometry is viewed as a realization of a stochastic process that is measured at n data points $h(u_1)$, ..., $h(u_n)$, which do most often not coincide with the grid nodes.

Kriging.

Within the framework of the above hypotheses one can compute unbiased linear minimum variance estimates of the piezometric head at the grid nodes without knowing or even computing the coefficients a_l of the mean $m(x,y)$ (see (3)). Such an optimal interpolation method, called "kriging", is a twodimensional generalization of the Wiener filter and has been studied and developed by geostatisticians |9| -|11|.

Let u_o be the coordinates of an arbitrary grid node. Then it is desired to obtain a linear estimate

$$\hat{h}(u_o) = \sum_{k=1}^{n} \lambda_k h(u_k), \tag{6}$$

where u_1, ..., u_n are the measure points, such that

i) $E\{\hat{h}(u_o)-h(u_o)\} = 0$ (7a)

ii) $\sigma_o^2 \triangleq \mathrm{Var}\{\hat{h}(u_o)-h(u_o)\}$ is minimized (7b)

With $\lambda_o = -1$, one can write

$$\hat{h}(u_o) - h(u_o) = \sum_{k=0}^{n} \lambda_k h(u_k) \tag{8}$$

The solution of the kriging problem is as follows.

Since the a_l's are unknown, (7a) leads to the following L universality conditions on λ_k :

$$\sum_{k=0}^{n} \lambda_k f_l(u_k) = 0, \ l=1,...., L \tag{9}$$

Furthermore σ_o^2 can be rewritten as

$$\sigma_o^2 = -\sum_{k=0}^{n} \sum_{j=0}^{n} \lambda_k \lambda_j \gamma_{kj} \tag{10}$$

Minimizing (10) subject to (9) gives :

$$-2 \sum_{j=0}^{n} \lambda_j \gamma_{kj} + \sum_{l=1}^{L} \mu_l f_l(u_k) = 0, \ k=1,...,n \tag{11a}$$

$$\sum_{j=0}^{n} \lambda_j f_l(u_j) = 0 \ , \qquad l=1,...,L \tag{11b}$$

This so-called "kriging system" is a set of n+L linear equations in the n+L unknowns $\lambda_j(j=1,...,n)$ and $\mu_l(l=1,...L)$. μ_l are Lagrange coefficients.

Let $\hat{\lambda}_j$ and $\hat{\mu}_l$ be the solutions. Then the optimal estimate is

$$\hat{h}(u_o) = \sum_{k=1}^{n} \hat{\lambda}_k h(u_k) \tag{12}$$

The variance of the estimation error is

$$\hat{\sigma}_o^2 = \sum_{k=0}^{n} \hat{\lambda}_k \gamma_{ok} - \sum_{l=1}^{L} \hat{\mu}_l f_l(u_o) \tag{13}$$

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