Least costly identification experiment for control. A solution based on a high-order model approximation

X. Bombois, G. Scorletti, P. Van den Hof and M. Gevers

Abstract— This paper presents a new approach for identification experiment design where the objective is to design the least costly experimental conditions such that the controller designed with the identified model stabilizes and achieves a prescribed level of H_∞ performance with the unknown true system G_0.

I. INTRODUCTION

In the industrial practice, a controller for a real-life system G_0 is generally designed using a model $\hat{G}$ of G_0 identified using data collected on the true system. When designing the identification experiment, the control engineer generally faces the problem of making a trade-off between its desire of obtaining as much information as possible about the true system by using a very long identification experiment and a very powerful input signal, and the economical constraint asking her/him to reduce as much as possible the costs of the identification by keeping this identification short and by exciting G_0 with a low power signal. In this paper, we propose an elegant solution to that problem of trade-off by determining the least costly identification experiment for control. The least costly identification experiment is here defined as the experiment on G_0 (with a fixed data length) where the spectrum $\Phi_u(\omega)$ of the input signal $u(t)$ is the one for which the total power P_u of $u(t)$ (i.e. $P_u = (1/2\pi) \int \Phi_u(\omega) d\omega$) is minimized under the constraint that the identification still delivers a model $\hat{G}$ sufficiently close to G_0 for the controller $\hat{C}$ designed with $\hat{G}$ to stabilize and to achieve sufficient performance with G_0. In this paper, the desired performance on G_0 will be expressed by magnitude bounds on one (or several) closed-loop transfer functions of $[\hat{C}\ G_0]$ (H_∞ performance constraints). Note that, to remain brief, we will focus on the design of the input signal spectrum $\Phi_u(\omega)$, given a fixed data length, since this variable $\Phi_u(\omega)$ is generally the variable considered in experiment design. However, our results can be easily extended to the design of the shortest identification experiment for control, given a fixed $\Phi_u(\omega)$.

Our approach of the experiment design problem has thus as objective to minimize the total power P_u of the input signal under a performance constraint on the loop $[\hat{C}\ G_0]$. This approach is thus somehow the dual of the approach previously developed in the broad literature on experiment design (see [9][Chapters 12 and 13] and e.g. [4], [6], [5], [7]). Indeed, in these papers, a measure of the performance degradation between the loops $[\hat{C}\ G_0]$ and $[\hat{C}\ \hat{G}]$ is minimized under a constraint on the power of $u(t)$. In other words, the difference between our approach and the approach previously taken in the literature is that, in the latter, the authors seek the best performance with $\hat{C}$ for a given cost of the identification while, in our approach, we seek the minimal cost for the identification to obtain the desired performance with $\hat{C}$. Another novelty of our approach is that, as opposed to the performance characterization generally used until now in the literature and which was more related to control design methods such as minimum variance control and model reference control (see e.g [4], [6]), the performance objective is here formulated in the modern H_∞-control paradigm.

In order to find a solution to our new experiment design problem, we will use the same assumptions as in the approach of the literature: the modeling error is assumed to be due to the variance error only (no bias error) and will be approximated with an expression accurate for a high-order model and a large number of data. Under these assumptions, the spectrum $\Phi_u(\omega)$ that solves our experiment design problem is determined in two steps. We first determine how far the true system G_0 may be from the identified model $\hat{G}$ for the $\hat{G}$-based controller $\hat{C}$ to be still a satisfying controller for G_0. Once this largest admissible uncertainty around $\hat{G}$ has been determined, we design the spectrum $\Phi_u(\omega)$ of the identification input signal with the minimal total power P_u such that, at a self-chosen probability level, the obtained modeling error $G_0 - \hat{G}$ is smaller than this largest admissible uncertainty around $\hat{G}$.

Overview. After this introduction, we will present in Section II the framework we will consider in this paper and formally state the experiment design problem we will solve. In Section III, we determine the largest admissible uncertainty around $\hat{G}$ for control; this quantity is then used in Section IV to give a new formulation of the experiment design problem and to solve it. In Sections V and VI, we finish this contribution by an example and some brief conclusions.

X. Bombois and P. Van den Hof are with Delft Center for Systems and Control, Mekelweg 2, 2628 CD Delft, The Netherlands, x.j.a.bombois@dcsc.tudelft.nl and p.m.j.vandenhof@dcsc.tudelft.nl

G. Scorletti is with LAP ISMRA, Caen, France, gerard.scorletti@greyc.ismra.fr

M. Gevers is with CESAME, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, gevers@csam.ucl.ac.be

II. CONSIDERED PROBLEM

We will assume that the stable true system $G_0(z)$ can be parametrized using an unknown parameter vector $\theta_0 \in \mathbf{R}^k$ i.e. $G_0(z) = G(z, \theta_0)$, and that the input-output relation of this true system is given by $y(t) = G(z, \theta_0)u(t) + v(t)$ where the additive noise $v(t)$ is the realization of a zero-mean stochastic process. As said in the introduction, we want to design the least costly identification experiment on G_0 which yields a model sufficiently close to G_0 for the controller designed with this model to stabilize and achieve sufficient performance with G_0. For this purpose let us make the following assumptions.

A. Assumptions on the identification expreiment

We will consider that the identification experiment on the true system is performed in open-loop in the Prediction Error (PE) Identification framework [9]. We further assume that this identification experiment will be performed using a full-order model structure $\mathcal{M} = \{ G(z, \theta) \}$ and with a fixed amount N of collected data. In the design of such identification experiment, we have consequently only one degree of freedom: the power spectrum $\Phi_u(\omega)$ of the input signal $u(t)$ that we will apply to the true system to identify it.

Such an identification experiment delivers a parameter vector $\hat{\theta}_N \in \mathbf{R}^k$ which defines the identified model $G(z, \hat{\theta}_N)$ [9]. Due to the stochastic property of the noise $v(t)$, the identified parameter vector $\hat{\theta}_N$ is (asymptotically) normally distributed with mean θ_0 and a covariance matrix $P_\theta \in \mathbf{R}^{k \times k}$ which can be estimated with the data, i.e $\hat{\theta}_N \sim \mathcal{N}(\theta_0, P_\theta)$. This means that the error $\hat{\theta}_N - \theta_0$ between identified and true parameter vectors is a random variable, but that, with probability 0.95, the true parameter vector θ_0 lies in the following ellipsoid U_Φ centered at $\hat{\theta}_N$ and defined by the covariance matrix P_θ:

$$U_\Phi = \left\{ \theta \mid (\theta - \hat{\theta}_N)^T P_\theta^{-1} (\theta - \hat{\theta}_N) < \chi \right\} \qquad (1)$$

with χ such that $Pr(\chi^2(k) < \chi) = 0.95$. The ellipsoid U_Φ (which gives an upper bound on the error $\hat{\theta}_N - \theta_0$) is a function of the chosen input spectrum $\Phi_u(\omega)$. Indeed, the covariance matrix P_θ is a (complicated) function of $\Phi_u(\omega)$ [9], and using (1), it is easy to see that the smaller P_θ, the smaller the ellipsoid U_Φ (and thus the smaller the error $\hat{\theta}_N - \theta_0$).

The error between identified and true parameter vectors being a random variable, so is the modeling error $G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)$. Using the property that $Pr(\theta_0 \in U_\Phi) = 0.95$, we can nevertheless state the following: the probability that the modeling error $|G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)|$ is smaller than

$$r_\Phi(\omega) \overset{\Delta}{=} \sup_{\theta \in U_\Phi} \left| G(e^{j\omega}, \theta) - G(e^{j\omega}, \hat{\theta}_N) \right| \qquad (2)$$

for all ω is equal to 95%. It can be proven that $r_\Phi(\omega)$ can be exactly computed by solving a LMI-based optimization problem at each ω (see the appendix).

Since the ellipsoid U_Φ is a function of the chosen input spectrum $\Phi_u(\omega)$, the upper bound $r_\Phi(\omega)$ on the modeling error is also a function of $\Phi_u(\omega)$. A more straightforward relation between the bound $r_\Phi(\omega)$ on the modeling error at each frequency and the input spectrum $\Phi_u(\omega)$ can be deduced if we assume that both the number of data N and the McMillan degree n of $G(z, \theta)$ are large ($N \to \infty$ and $n \to \infty$). Indeed, under this assumption, we have the following result:

$$E\left(((G(e^{j\omega}, \hat{\theta}_N) - G_0(e^{j\omega}))^*(G(e^{j\omega}, \hat{\theta}_N) - G_0(e^{j\omega})) \right)$$

$$\approx \frac{n}{N} \frac{\Phi_v(\omega)}{\Phi_u(\omega)}$$

where $\Phi_v(\omega)$ is the spectrum of the noise $v(t)$. This expression shows that, at each ω, the expected value of the square of the modulus of the modeling error is approximately inversely proportional to $\Phi_u(\omega)$. Based on this approximation, we can deduce that, at each ω, the square of the quantity $r_\Phi(\omega)$ defined in (2) must be also approximately inversely proportional to the input spectrum $\Phi_u(\omega)$ when N and n are large, and thus that

$$\frac{r_{\Phi,1}^2(\omega)}{r_{\Phi,2}^2(\omega)} \approx \frac{\Phi_{u,2}(\omega)}{\Phi_{u,1}(\omega)} \qquad (3)$$

where $r_{\Phi,1}(\omega)$ and $r_{\Phi,2}(\omega)$ are the upper bound on the modeling error corresponding to two identification experiments performed in the same full-order model structure on the same true system and with the same number of data N, but with input spectrum $\Phi_{u,1}(\omega)$ and $\Phi_{u,2}(\omega)$, respectively.

B. Assumptions on the control design

As said previously, we aim at designing a "satisfying" controller $\hat{C}(z)$ for the unknown true system G_0 using a model $\hat{G} = G(z, \hat{\theta}_N)$ of G_0 identified in the PE framework. We will state that a controller $\hat{C}$ is a satisfying controller for G_0 if $\hat{C}$ stabilizes G_0 and achieves a certain performance level with G_0. This required performance level will be here formulated under the form of constraints on one or more H_∞ cost functions: $J(G_0(z), \hat{C}(z), W(z)) < 1$ where $W(z)$ is some given performance filter. In this paper, for simplicity, we will consider only one H_∞ performance constraint given by:

$$J(G_0, \hat{C}, W) = \left\| \frac{W}{1 + G_0 \hat{C}} \right\|_\infty < 1. \qquad (4)$$

The controller $\hat{C}$ for G_0 will be designed with an identified model $\hat{G}$ of G_0 and using a fixed H_∞ control design method (i.e. H_∞ control design with fixed weights)

[11]. This control design must of course at least ensure that $\hat{C}$ stabilizes $\hat{G}$ and that $J(\hat{G}(z), \hat{C}(z), W(z)) \leq \gamma < 1$ where γ is a fixed number smaller than one. In other words:

$$\hat{G}(z) \;\rightarrow\; \hat{C}(z) \text{ such that} [\hat{C}\;\hat{G}] \text{ stable and} \atop J(\hat{G}(z), \hat{C}(z), W(z)) \leq \gamma < 1. \tag{5}$$

Since the identified model $\hat{G} = G(z, \hat{\theta}_N)$ is, in practice, never equal to G_0, the controller $\hat{C}(z)$ designed with $G(z, \hat{\theta}_N)$ is not guaranteed to stabilize G_0 and to satisfy (4). In this paper, we will nevertheless show that it is possible to design the identification experiment delivering $\hat{G}$ in such a way that, modulo a self-chosen probability level, the controller $\hat{C}$ designed with the identified model $\hat{G}$ will stabilize the unknown true system G_0 and satisfy (4). Moreover, we will determine the least costly identification experiment having this property.

C. The least costly experiment design

It is obvious that, for identification experiments with a fixed data length N such as presented in Section II-A, the costs of the identification are minimized if the total power P_u of the input signal is minimized. Consequently, the experiment design problem in our framework can be formulated as follows:

Experiment design problem: *For an identification experiment performed on G_0 with N data, determine the power spectrum $\Phi_u(\omega)$ of the input signal $u(t)$ which minimizes the total power P_u of u(t) i.e. $P_u \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega$ under the constraint that $\Phi_u(\omega) > 0 \; \forall \omega$ and the constraint that, with a confidence level of 95 %, the controller $\hat{C}$ designed from the identified model $\hat{G}$ using (5) is guaranteed to stabilize G_0 and to achieve the desired level of performance (4) with G_0.*

III. ADMISSIBLE UNCERTAINTY AROUND THE IDENTIFIED MODEL

In order to determine the spectrum solving the experiment design problem presented above, a logical first step is to determine how close the true system G_0 has to be from $\hat{G}$ for (4) to hold and for $\hat{C}$ to stabilize G_0. For this purpose, we use the following proposition.

Proposition 3.1: Consider a stable model $\hat{G}$ and the performance criterion (4). Then, the controller $\hat{C}$ designed from $\hat{G}$ using (5) stabilizes and achieves $J(G, \hat{C}, W) < 1$ with all stable plants G such that $|G(e^{j\omega}) - \hat{G}(e^{j\omega})| \leq r(\omega)$ if and only if

$$r(\omega) \leq r_{adm}(\omega, \hat{G}) \quad \forall \omega \tag{6}$$

where $r_{adm}(\omega, \hat{G})$ is called the largest admissible uncertainty around $\hat{G}$ and is defined by:

$$r_{adm}(\omega, \hat{G}) \triangleq \frac{\left|1 + \hat{G}(e^{j\omega})\hat{C}(e^{j\omega})\right| - \left|W(e^{j\omega})\right|}{\left|\hat{C}(e^{j\omega})\right|}. \tag{7}$$

Proof. We will first prove that, at each ω, $r_{adm}(w, \hat{G})$ given in (7) is the largest value of $r(\omega)$ for which $J(G, \hat{C}, W) < 1$ with all stable plants G such that $|G(e^{j\omega}) - \hat{G}(e^{j\omega})| \leq r(\omega)$. For this purpose, note that $J(G, \hat{C}, W) < 1$ holds if and only if, at each ω,

$$\left|1 + \hat{G}(e^{j\omega})\hat{C}(e^{j\omega}) + \Delta(e^{j\omega})\hat{C}(e^{j\omega})\right| > \left|W(e^{j\omega})\right|$$

where $\Delta(z) \triangleq G(z) - \hat{G}(z)$. Consequently, the largest value $r_{adm}(\omega, \hat{G})$ for which $J(G, \hat{C}, W) < 1$ holds for all $\Delta(e^{j\omega})$ such that $\left|\Delta(e^{j\omega})\right| \leq r_{adm}(\omega, \hat{G})$, obeys the following:

$$\left|1 + \hat{G}(e^{j\omega})\hat{C}(e^{j\omega})\right| - r_{adm}(\omega, \hat{G})\left|\hat{C}(e^{j\omega})\right| = \left|W(e^{j\omega})\right|$$

which delivers (7). The remaining of the proof consists then of verifying that the loops $[\hat{C}\;G]$ are also stable for all these G. For this purpose, notice that, at each ω,

$$\left|G(e^{j\omega}) - \hat{G}(e^{j\omega})\right| \leq r_{adm}(\omega, \hat{G}) < \frac{\left|1 + \hat{G}(e^{j\omega})\hat{C}(e^{j\omega})\right|}{\left|\hat{C}(e^{j\omega})\right|}$$

The stability property is therefore a direct consequence of the small gain theorem [11][page 145]. ∎

Note that the largest admissible uncertainty region $r_{adm}(\omega, \hat{G})$ around $\hat{G}$ is a function of the considered model $\hat{G}$, of the controller $\hat{C}$ designed from $\hat{G}$ with (5) and of the chosen performance weight $W(z)$.

IV. SOLUTION OF THE EXPERIMENT DESIGN PROBLEM

The results presented in the previous sections will allow us to reformulate the experiment design problem considered in this paper in a more compact way. Indeed, on the one hand, according to Proposition 3.1, the frequency function $r_{adm}(\omega, \hat{G})$ determines the maximal distance $|G(e^{j\omega}) - \hat{G}(e^{j\omega})|$ such that the $\hat{G}$-based controller $\hat{C}$ still stabilizes and achieves sufficient performance with G. On the other hand, the frequency function $r_\Phi(\omega)$ given in (2) determines an upper bound (with confidence level 0.95) for the modeling error $|G_0(e^{j\omega}) - \hat{G}(e^{j\omega})|$ obtained after an identification experiment using an input signal with spectrum $\Phi_u(\omega)$. Using these two notions, the identification experiment problem presented at the end of Section II-C can therefore now be rewritten as follows:

Experiment design problem. *For an identification experiment performed on G_0 with N data, determine the power spectrum $\Phi_u(\omega)$ of the input signal $u(t)$ which minimizes the power P_u of u(t) under the constraint that $\Phi_u(\omega) > 0 \; \forall \omega$ and the constraint that*

$$r_\Phi(\omega) \leq r_{adm}(\omega, \hat{G}) \quad \forall \omega \tag{8}$$

where $r_\Phi(\omega)$ is defined in (2) and $r_{adm}(\omega, \hat{G})$ is the largest admissible uncertainty around the particular model $\hat{G}$ identified in this experiment.

As usual in the experiment design theory, the experiment design presented just above is not solvable without prior information about the system we want to identify and the model we will identify. This will be evidenced in the sequel. This prior information will be gained using an initial and exploratory identification experiment. Let us make the following assumptions about this initial identification experiment.

Assumption 4.1: An initial identification experiment performed on G_0 with N data and an input spectrum $\Phi_{u,init}(\omega)$ has delivered a model $\hat{G}_{init}$. We further assume that this identification experiment was such that:

$$\exists \omega \quad \text{for which} \quad r_{\Phi,init}(\omega) > r_{adm}(\omega, \hat{G}_{init}). \quad (9)$$

where $r_{\Phi,init}(\omega)$ is the upper bound of the modeling error obtained in this initial experiment (see (2)) and $r_{adm}(\omega, \hat{G}_{init})$ is the largest admissible uncertainty around the identified $\hat{G}_{init}$. The frequency function $r_{\Phi,init}(\omega)$ can be computed using LMI optimization as stated earlier and, according to (7), $r_{adm}(\omega, \hat{G}_{init})$ can be computed as follows:

$$r_{adm}(\omega, \hat{G}_{init}) = \frac{\left|1 + \hat{G}_{init}(e^{j\omega})\hat{C}_{init}(e^{j\omega})\right| - \left|W(e^{j\omega})\right|}{\left|\hat{C}_{init}(e^{j\omega})\right|} \quad (10)$$

with $\hat{C}_{init}$, the controller designed from $\hat{G}_{init}$ with (5).

Note that, according to Proposition 3.1, assumption (9) implies that $\hat{C}_{init}$ is not guaranteed to be a satisfying controller for G_0. Because of this assumption (9), the initial identification experiment is thus assumed to have been performed with an input signal whose power is inferior to the power we will have to choose when designing the least costly identification experiment for control (i.e. the solution of the considered experiment design problem). The initial identification experiment is thus assumed to be a "cheaper" identification experiment than the one we want to design.

We now show that this initial identification experiment is indeed necessary to deduce a solution for the experiment design problem. A first use of this initial experiment is that the quantity $r_{adm}(\omega, \hat{G}_{init})$ defined in (10) can be considered as an approximation of $r_{adm}(\omega, \hat{G})$. This quantity $r_{adm}(\omega, \hat{G})$ is necessary to the solution of the experiment design problem as shown in (8) and can not be computed a-priori since the model $\hat{G}$ is unknown at the very moment we design the identification experiment. The experiment design problem would therefore be unfeasible if we could not replace the unknown $r_{adm}(\omega, \hat{G})$ by an known approximation i.e. $r_{adm}(\omega, \hat{G}_{init})$.

The experiment design problem would also not be solvable if we would not possess a direct relation between the

quantity $r_{\Phi}(\omega)$ in (8) and the design variable $\Phi_u(\omega)$. The results of the initial identification experiment combined with the approximation (3) provides us with such an (approximated) relation. Indeed, based on (3) and Assumptions 4.1, we have that

$$r_{\Phi}^2(\omega) \approx r_{\Phi,init}^2(\omega)\frac{\Phi_{u,init}(\omega)}{\Phi_u(\omega)} \quad (11)$$

Using the estimate $r_{adm}(\omega, \hat{G}_{init})$ of $r_{adm}(\omega, \hat{G})$ and the (approximated) relation (11) between $r_{\Phi}(\omega)$ and $\Phi_u(\omega)$, the experiment design problem stated in the beginning of this section and leading to the least costly identification experiment for control can be solved.

Proposition 4.1: Assume than an initial identification experiment has been performed (see Assumptions 4.1). Then, the spectrum $\Phi_u(\omega)$, solution of the experiment design problem presented above in this section, can be approximated at each ω by:

$$\Phi_u(\omega) \approx \Phi_{u,init}(\omega)\frac{r_{\Phi,init}^2(\omega)}{r_{adm}^2(\omega, \hat{G}_{init})} \quad (12)$$

where $r_{adm}(\omega, \hat{G}_{init})$ and $r_{\Phi,init}(\omega)$ are defined in Assumptions 4.1.

Proof. By replacing $r_{\Phi}(\omega)$ by its approximation (11) and by replacing $r_{adm}(\omega, \hat{G})$ by its estimate $r_{adm}(\omega, \hat{G}_{init})$, the condition $r_{\Phi}(\omega) \leq r_{adm}(\omega, \hat{G}) \; \forall \omega$ becomes:

$$r_{\Phi,init}^2(\omega)\frac{\Phi_{u,init}(\omega)}{\Phi_u(\omega)} \leq r_{adm}^2(\omega, \hat{G}_{init}) \quad \forall \omega \quad (13)$$

This last expression directly yields (12) as the solution of the experiment design problem. ∎

Let us summarize. We wanted to determine the least powerful input signal for an identification experiment such that (8) holds (and guaranteeing consequently that the model-based controller stabilizes and achieves the required performance level). For this purpose, an initial experiment is necessary. This identification experiment will typically be performed with a (very) low power input signal and therefore be such that the obtained upper bound on the modeling error is too large to guarantee with a confidence level of 0.95 that $\hat{C}_{init}$ is a satisfying controller for G_0 (see Assumptions 4.1). However the information delivered by this initial and "unsuccessful" identification experiment enables us to determine the solution of our experiment design problem as shown in the previous proposition. Indeed, the previous proposition gives us the least costly experimental condition (i.e. the spectrum $\Phi_u(\omega)$ corresponding to the minimal power of $u(t)$) for an identification experiment delivering a model $\hat{G}$ for which the corresponding controller $\hat{C}$ will well stabilize G_0 and satisfy (4).

The expression (12) for the power spectrum $\Phi_u(\omega)$ of $u(t)$ that solves our experiment design problem can be explained as follows. Expression (12) shows that the spectrum $\Phi_u(\omega)$ must be chosen larger than $\Phi_{u,init}(\omega)$ at those frequencies where the modeling error $r_{\Phi,init}(\omega)$ is larger than the admissible uncertainty and $\Phi_u(\omega)$ must be chosen smaller than $\Phi_{u,init}(\omega)$ at those frequencies where the modeling error $r_{\Phi,init}(\omega)$ is smaller than the admissible uncertainty. In order to choose the factor $\Phi_u(\omega)/\Phi_{u,init}(\omega)$ with which $\Phi_u(\omega)$ has to be increased or decreased we use the fact that:

$$\frac{\Phi_u(\omega)}{\Phi_{u,init}(\omega)} \approx \frac{r_{\Phi,init}^2(\omega)}{r_\Phi^2(\omega)}$$

to precisely determine the spectrum $\Phi_u(\omega)$ for which, based on the high-order model approximation (11), $r_\Phi(\omega) = r_{adm}(\omega, \hat{G}_{init})$ ($\approx r_{adm}(\omega, \hat{G})$) at each frequency.

V. ILLUSTRATION: FLEXIBLE TRANSMISSION SYSTEM

We consider as *true system* the half-load flexible transmission system having the following input-output relation: $y(t) = (B_0(z)/A_0(z))u(t) + (1/A_0(z))e(t)$ with $B_0(z) = z^{-3}(0.10276 + 0.18123z^{-1})$, $A_0(z) = 1 - 1.99185z^{-1} + 2.20265z^{-2} - 1.84083z^{-3} + 0.89413z^{-4}$ and $e(t)$ is the realization of a white noise signal of variance $\sigma_e^2 = 0.5$. This true system is a slightly modified version of the plant used as a benchmark in a special issue of the European Journal of Control [8]. In this example, we want to design the input spectrum $\Phi_u(\omega)$ leading to the smallest power P_u for an identification experiment with $N = 500$ data under the constraint that, with a confidence level of 0.95, the controller $\hat{C}$ designed from the identified model $\hat{G}$ with the control design method presented in the sequel is a satisfying controller for G_0. A controller $\hat{C}$ is stated satisfying if $\hat{C}$ stabilizes G_0 and achieves the performance specification (4) with $W(z) = (0.5165 - 0.4632z^{-1})/(1 - 0.999455z^{-1})$. The controller $\hat{C}$ will be designed from an identified model $\hat{G}$ using the 4-block H_∞ control design method of [3] where one of the constraints is $\left\|W/(1 + \hat{G}\hat{C})\right\|_\infty < 1$ with the same $W(z)$ as above.

Initial identification experiment. In order to solve our problem, we need to perform an initial identification experiment. We have decided to perform a very cheap initial identification experiment: the input signal $u(t)$ is chosen as a white noise signal of variance $\sigma_{u,init}^2 = 0.1$ i.e. $\Phi_{u,init}(\omega) = 0.1$ $\forall \omega$ (which has to be compared with $\sigma_e^2 = 0.5$). It yields an identified model $\hat{G}_{init}$. Using the information obtained in this initial identification experiment, we can not guarantee, with a confidence level of 0.95, that the controller $\hat{C}_{init}$ designed with $\hat{G}_{init}$ using the method of [3] stabilizes G_0 and satisfies (4). Indeed, if we compute the bound $r_{\Phi,init}(\omega)$ on the obtained modeling error and the

frequency function $r_{adm}(\omega, \hat{G}_{init})$ using (10), we see that $r_{\Phi,init}(\omega) > r_{adm}(\omega, \hat{G}_{init})$ in the frequency range [0 0.8] as evidenced in Figure 1. Since $\hat{G}_{init}$ and $\hat{C}_{init}$ are not satisfactory enough, we need to design a new experiment design which will deliver enough information about G_0 to design a satisfying controller for G_0. According to Proposition 4.1, the cheapest experiment design having this property is the one with an input signal $\Phi_u(\omega)$ having the spectrum given by (12). This spectrum $\Phi_u(\omega)$ is represented in Figure 2. In this figure, we see that the spectrum $\Phi_u(\omega)$ is larger than $\Phi_{u,init}(\omega)$ in the frequency range [0 0.8] where $r_{\Phi,init}(\omega)$ was too large with respect to the admissible uncertainty $r_{adm}(\omega, \hat{G}_{init})$.
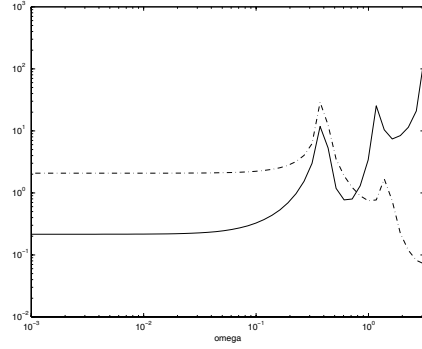

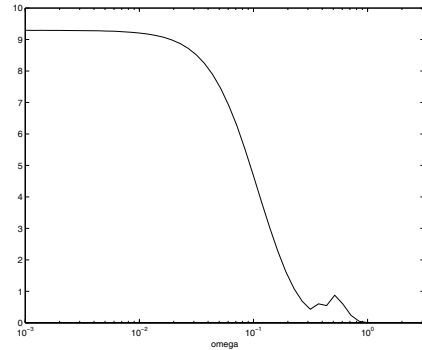Fig. 1. $r_{\Phi,init}(\omega)$ (dashdot) and $r_{adm}(\omega, \hat{G}_{init})$ (solid)

Fig. 2. the spectrum $\Phi_u(\omega)$ given by (12)

The least costly experiment design for robust control. Figure 2 determines the spectrum $\Phi_u(\omega)$ of the input signal $u(t)$ corresponding to the least costly identification experiment. Normally, we should design an input signal having this particular spectrum $\Phi_u(\omega)$. However, here, for simplicity, we will design the least costly identification experiment where the input signal is constrained to be a white noise signal[1] and, thus, we will design the input signal

[1]Such a choice has also the advantage that, in this case, the high-order model approximation (11) is also "accurate" for lower order model (see [10]).

as a white noise with a variance σ_u^2 equal to $\max_\omega \Phi_u(\omega) = 9.3$. Note that σ_u^2 is now 93 times larger than in the initial identification experiment. This identification experiment delivers a model $\hat{G}$ and we design the candidate controller $\hat{C}$ using the pre-specified control design method. We then compute the upper bound $r_\Phi(\omega)$ on the modeling error corresponding to this new identification experiment as well as the largest admissible uncertainty $r_{adm}(\omega, \hat{G})$ around the model $\hat{G}$ and we represent these two frequency functions in Figure 3 where we can see that the condition $r_{\Phi,init}(\omega) \leq r_{adm}(\omega, \hat{G}_{init}) \; \forall \omega$ is well satisfied. We can therefore guarantee with a confidence level of 0.95 that the controller $\hat{C}$ designed with $\hat{G}$ using the method of [3] will stabilize G_0 and achieves sufficient performance with G_0 i.e. $J(G_0, \hat{C}, W) < 1$. This is confirmed in this particular identification experiment by Figure 4.
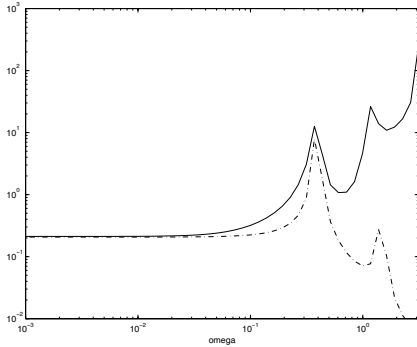

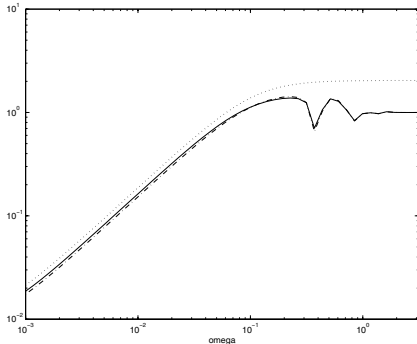Fig. 3. $r_\Phi(\omega)$ (dashdot) and $r_{adm}(\omega, \hat{G})$ (solid)

Fig. 4. $|1 + \hat{G}(e^{j\omega})\hat{C}(e^{j\omega})|^{-1}$ (dashdot), $|1 + G_0(e^{j\omega})\hat{C}(e^{j\omega})|^{-1}$ (solid) and $\left| W(e^{j\omega}) \right|^{-1}$ (dotted)

It is to be noted in Figure 3 that, in low frequencies, we have $r_\Phi(\omega) \approx r_{adm}(\omega, \hat{G})$ which is what we expected since for those frequencies the actual input spectrum $\Phi_u(\omega) = 9.3 \; \forall \omega$ used in this identification experiment is equal to the spectrum determined by (12) (see Figure 2). The spectrum given by (12) is, by construction and provided the approximation (11) is accurate, the spectrum such that the upper bound $r_\Phi(\omega)$ on the modeling error

obtained in the corresponding identification experiment is precisely equal to the largest admissible uncertainty $r_{adm}(\omega, \hat{G}_{init}) \approx r_{adm}(\omega, \hat{G})$.

VI. Conclusions

We have presented in this paper a new approach to determine the input spectrum of an identification experiment when the objective is to design a controller with the identified model achieving sufficient H_∞ performance with G_0. Since the submission of this paper, we have been working on a way to avoid the high-order-model approximation used in this paper. These new results are presented in the contribution [2].

References

[1] X. Bombois, M. Gevers, and G. Scorletti. A measure of robust stability for a set of parametrized transfer functions. *IEEE Transactions on Automatic Control*, 45(11):2141–2145, 2000.

[2] X. Bombois, G. Scorletti, P. Van den Hof, M. Gevers, and R. Hildebrand. Least costly identification experiment for control. Submitted to MTNS'04, Louvain, Belgium, July 2004.

[3] G. Ferreres and V. Fromion. H_∞ control for a flexible transmission system. In *CD-ROM Proc. European Control Conference*, Brussels, Belgium, 1997.

[4] M. Gevers and L. Ljung. Optimal experiment design with respect to the intended model applications. *Automatica*, 22:543–554, 1986.

[5] R. Hildebrand and M. Gevers. Identification for control: optimal input design with respect to a worst-case ν-gap cost function. *SIAM Journal on Control and Optimization*, 41(5):1586–1608, 2003.

[6] H. Hjalmarsson, M. Gevers, and F. De Bruyne. For model-based control design, closed-loop identification gives better performance. *Automatica*, 32(12):1659–1673, 1996.

[7] H. Hjalmarsson and H. Jansson. Using a sufficient condition to analyze the interplay between identification and control. In *CD-ROM Proc. IFAC Symposium on System Identification*, paper WeA01-06, Rotterdam, The Netherlands, 2003.

[8] I.D. Landau, D. Rey, A. Karimi, A. Voda, and A. Franco. A flexible transmission system as a benchmark for robust digital control. *European Journal of Control*, 1(2):77–96, 1995.

[9] L. Ljung. *System Identification: Theory for the User, 2nd Edition*. Prentice-Hall, Englewood Cliffs, NJ, 1999.

[10] B. Ninness and H. Hjalmarsson. The analysis of variance error. Part I: Accurate quantification. Technical Report EE2002020, Dept. of Electrical and Computer Engineering, University of Newcastle, Australia, 2002.

[11] K. Zhou and J. Doyle. *Essentials of Robust Control*. Prentice Hall, Upper Saddle River, New Jersey, 1998.

Appendix

Consider the ellipsoid U_ϕ as given in (1) and the quantity $r_\Phi(\omega)$ at ω defined in (2). Following the same reasoning as in [1], $r_\Phi(\omega) = \sqrt{\gamma_{opt}}$, where γ_{opt} is then the optimal value of γ in the following standard convex optimization problem involving LMI constraints evaluated at ω:

$$\begin{aligned} minimize \quad & \gamma \\ over \quad & \gamma, \tau \\ subject\ to \quad & \tau \geq 0 \ and \end{aligned}$$

$$\begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} R & -R\hat{\theta}_N \\ (-R\hat{\theta}_N)^T & \hat{\theta}_N^T R \hat{\theta}_N - 1 \end{pmatrix} < 0 \tag{14}$$

where $R = P_\theta^{-1}/\chi$, $a_{11} = Z_N^* Z_N - Z_N^* x Z_D - Z_D^* x^* Z_N + Z_D^* x^* x Z_D - \gamma Z_D^* Z_D$, $a_{12} = -Z_N^* x + Z_D^* x x^* - \gamma Z_D^*$, $a_{22} = x x^* - \gamma$, $G(e^{j\omega}, \theta) \triangleq (Z_N\theta)/(1 + Z_D\theta)$ and $x = G(e^{j\omega}, \hat{\theta}_N)$.