Controller validation for Stability and Performance based on an uncertainty region designed from an identified model†

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Abstract

This paper focuses on the validation (for stability and for performance) of a controller that has been designed from an unbiased model of the true system, identified either in open-loop or in closed-loop using a prediction error framework. A controller is said to be *validated for stability* if it stabilizes all models defined by an ellipsoidal parametric uncertainty set containing the true system with some prescribed probability. Such uncertainty set is computed from the covariance matrix of the parameters of the identified model. Our first contribution is to design the general LFT framework for robustness stability analysis linking the controller to be validated with such parametric uncertainty region resulting from prediction error identification (open-loop and indirect closed-loop identification). This leads us to a necessary and sufficient condition for the robust stabilization of all plants in this nonstandard uncertainty region. Our second contribution is to show that we can compute the worst case performance of a given controller over all systems in such uncertainty set described by ellipsoidal regions in parameter space, by recasting the problem as an LMI-based optimization problem,

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for which the exact solution can be computed. A controller is then said to be validated for performance if the worst case performance is better than some threshold value.

1 Introduction

This paper is part of our continuing investigation of identification for control, as well as controller design and controller validation based on identified models and their uncertainty regions [13, 14, 15, 3]. Here we consider the case where a nominal model G_{mod} has been identified, together with an uncertainty set $\mathcal{D}$ to which the true system G_0 is known to belong with some prescribed probability. This uncertainty set $\mathcal{D}$ is defined as a set of parametrized rational transfer functions whose parameter vector lies in an ellipsoidal confidence region. This is clearly a nonstandard uncertainty set in robust control analysis and design. We then focus on the validation of a controller C, designed from G_{mod}, both for robust stability and for robust performance. We present a validation procedure for stability which ensures that the controller C stabilizes all systems in this nonstandard uncertainty set $\mathcal{D}$. We also present a validation procedure for performance in which we compute the worst case performance over all closed loop systems made up of the controller C and all plants in $\mathcal{D}$. Our two procedures (stability and performance validation) are based on the particular structure of the ellipsoidal parametric confidence regions delivered by prediction error identification (see e.g. [21]). Our major contribution is to show that this particular uncertainty description, which results directly from classical prediction error identification, allows one to use classical robust stability results (see e.g. [7, 10, 30, 18]) for the validation for stability, and to develop an LMI-based optimization problem that computes the worst case performance level. In addition, we also show that, for this particular uncertainty description, exact values of the stability radius and of the worst case performance can be computed leading to necessary and sufficient condition statements, both for the robust stability and for the robust performance problem.

Uncertainty region. Prediction error identification theory (see e.g. [21]) delivers an estimated model G_{mod} for the true plant G_0 and provides us with tools for the estimation of an uncertainty region. If the parametric structure is sufficiently complex to represent the true system, then G_{mod} is asymptotically unbiased and the uncertainty is described by the covariance matrix of the identified model G_{mod}. This covariance matrix allows one to construct a parametric uncertainty region U containing the parameters of the true system G_0 at a certain probability level that we can fix at, say, 95 %. The uncertainty region U in the parameter space defines an equivalent uncertainty region $\mathcal{D}$ in the space of transfer functions. We show that this uncertainty region $\mathcal{D}$ can be obtained for both open-loop identification and indirect closed-loop identification.

Controller validation for stability. Robust stability theory developed in e.g. [10, 7, 30, 24, 18] provides necessary and sufficient conditions for the stabilization, by some given controller C, of all plants in an uncertainty region, provided this uncertainty region is defined in the general LFT (linear fractional transformation) framework for robust stability analysis. Our contribution in the proposed stability validation procedure is to show that one can rewrite the closed-loop connection of the controller C and all plants in the uncertainty region $\mathcal{D}$ obtained from both types of identification (open-loop and indirect closed-loop identification) into a particular LFT that takes into account the parametric description of $\mathcal{D}$ (i.e. the uncertainty part of the obtained LFT is a real vector) and whose (real) stability radius is exactly computable, using the result presented in [18, 26]. The proposed approach has the complementary advantage of being easily extensible to robust control design using the result in [27]. Indeed, [27] gives a convex parametrization of all controllers stabilizing the plants defined by rank one LFT's (that is the type of LFT's we here obtain). The main advantage of the convex parametrization of [27] is that several robust synthesis problems can then be stated in terms of convex or quasi-convex optimization.

In an earlier paper [3], the problem of the robust stability of all plants in the $\mathcal{D}$ domain of parametrized transfer functions has already been addressed. The solution presented in [3] was to embed the uncertainty region $\mathcal{D}$ into a larger coprime factor uncertainty region, leading therefore only to a sufficient robust stability condition. The advantage in the present approach is that the obtained robust stability condition is necessary and sufficient. This is a consequence of the fact that our new stability results apply directly to the parametrized set $\mathcal{D}$ resulting from the identification step, thereby avoiding the conservativeness resulting from the overbounding of $\mathcal{D}$ by a coprime factor uncertainty set.

In the case of open-loop identification of an ARMA structure, the structure of $\mathcal{D}$ has a simpler expression. In [26], it is shown that this simpler structure can be expressed as an LFT. In this paper, we give a general formulation of this LFT for all model sets and for both open-loop and indirect closed-loop identification using a general expression of the uncertainty region $\mathcal{D}$ obtained with both types of prediction error identifications. In the case of open-loop identification and ARX structure, a similar approach to ours and to that presented in [26] can also be found in [20].

Other authors have tackled the robust stability problem in the presence of parametric uncertainties from another point of view (see e.g. [2, 1] and references therein). In this literature, the stability of an uncertain polynomial is analyzed. For control purposes, the analyzed polynomial is the denominator of the closed-loop transfer function. In [2], the authors present a procedure that gives, for a given controller, the largest ellipsoid in the space of the system parameters for which the stabilization of the closed-loop transfer function denominator is guaranteed. Their approach uses Euclidean space geometry to project the parameters of the open-loop system into those of the common denominator of the closed-loop transfer functions and conversely. The main advantage of our procedure is to use the general framework of the robustness theory which allows one to use all standard robust theory tools: e.g., as said earlier, the results of [27] allows robust control design using convex or quasi-convex optimization. This is not possible with the approach proposed in [2] where only a local minimum of the proposed controller design criterion can be found.

Our approach also differs significantly from the approach used in traditional *set membership identification* ([23] and references therein). In the set membership literature, a hard bound assumption is made on the noise and a known upper bound is required on the impulse response of the true system, leading to the identification of an uncertainty set around a nominal model. In [17], a method to identify an additive uncertainty region with a stochastic noise assumption is presented, but a known prior bound on the true system impulse response is again required. Furthermore, the approach presented in [17] is restricted to linearly parametrized models, such as FIR models. In our approach, rational transfer functions with denominator uncertainty can be used. In addition, no prior assumptions are required on the magnitude of the noise and of the impulse response. Our uncertainty regions are derived from the data using classical prediction error identification. The only important restriction in this paper is that we assume that the system is in the model set and that the uncertainty sets are therefore entirely defined by covariance errors on the parameters. This restriction can be relaxed using the stochastic embedding approach of [16] to construct uncertainty regions that then take into account both bias and variance errors in the estimated transfer functions. This will be the subject of a future paper.

Controller validation for performance. Our procedure for controller validation for performance is based on the computation of the worst case performance of a closed-loop made up of the to-be-validated controller and a system in the uncertainty region $\mathcal{D}$. The performance for a closed loop (see [30, 9, 29]) is often defined via the modulus of the frequency response of the four different closed-loop transfer functions. The worst case performance, for each of these four transfer functions, will be defined by the maximum of their modulus computed over all plants in the uncertainty region $\mathcal{D}$. These maxima over all plants in $\mathcal{D}$, for a given controller C, define four templates, which are used as performance indicators. A number of standard performance indicators (such as perturbation rejection rate, resonance peak, ...) can be derived from these four indicators. Our contribution is to show that the computation of the worst case performance can be formulated as an LMI-based optimization problem. In fact, we give a general LMI-based optimization problem which allows computation of the worst case performance for the four different closed-loop transfer functions by an appropriate choice of the weights in the general LMI problem. The LMI formulation of the problem uses the fact that the parametric uncertainty appears linearly in the expression of both the numerator and the denominator of the systems in the uncertainty region $\mathcal{D}$ and, as a consequence, also appears linearly in the expression of the different closed-loop transfer functions.

Our approach to compute the worst case performance differs significantly from the usual approach proposed in e.g. [9, 11]. In these papers, the computation of the worst case performance in an uncertainty region described by an LFT involves the computation of a quantity ν. The quantity ν is an extension of the structured singular value μ. Only upper and lower bounds of ν (and μ) are computable in polynomial time. [9] and [11] only give a way to compute the upper bound for the standard structured uncertainties. The case of an uncertainty given by a real vector (such as in the uncertainty region $\mathcal{D}$) is not tackled. Our approach therefore presents two advantages with respect to that in [9, 11]. First, the worst case performance can be computed for all types of uncertainties (and not only the standard ones) that appear linearly both in the numerator and the denominator of the systems in the uncertainty region. Second, it avoids the conservatism that results from using upper (and lower) bounds.

In [22], the worst case performance is also computed for a parametric uncertainty region, but in that paper the criterion that measures the performance is an LQ time criterion and not the modulus of the frequency responses of the closed-loop transfer functions.

Paper outline. In Section 2, we briefly review how open-loop and indirect closed-loop identification lead to ellipsoidal parametric uncertainty regions U, which define equivalent uncertainty regions $\mathcal{D}$ in the space of transfer functions. In Section 3, we show how the closed-loop connections of the systems in the uncertainty region $\mathcal{D}$ and the "to-be-validated controller" can be expressed in the general LFT framework for robust stability analysis. A necessary and sufficient condition for the robust stabilization of all plants in $\mathcal{D}$ is then derived from classical robust stability theory. In Section 4, the concept of worst case performance level is introduced and the LMI-based optimization problem developed for its computation is given. The procedures for validation for stability and for performance are illustrated by an example in Section 5. Finally, some conclusions are given in the last section.

2 Identification and parametric uncertainty region

In this section, we briefly recall the uncertainty regions delivered by classical prediction error identification, assuming that unbiased model structures are used [21]. We here consider both open-loop and indirect closed-loop identification. We assume that the *open-loop* true system is linear and time-invariant, with a rational input-output transfer function G_0:

$$y = G_0 u + v$$

where v is additive noise.

2.1 Open-loop identification

In the case of open-loop identification, we consider a uniformly stable[1] model set $\mathcal{M}_{OL}$ with the following structure:

$$\mathcal{M}_{OL} = \left\{ G(\theta) \mid G(\theta) = \frac{b_1 z^{-1} + ... + b_m z^{-m}}{1 + a_1 z^{-1} + ... + a_n z^{-n}} = \frac{Z_2 \theta}{1 + Z_1 \theta} \right\} \tag{1}$$

with

- $\theta^T = [a_1 \; ... \; a_n \; b_1 \; ... \; b_m] \in \mathbf{R}^{q \times 1}, \; q \triangleq n + m$
- $Z_1(z) = [z^{-1} \; z^{-2} \; ... \; z^{-n} \; 0 \; ... \; 0] \in \mathbf{C}^{1 \times q}$
- $Z_2(z) = [0 \; ... \; 0 \; z^{-1} \; z^{-2} \; ... \; z^{-m}] \in \mathbf{C}^{1 \times q}$

[1] A model set is said to be uniformly stable if the one-step ahead predictors linked to the systems in the model set, their gradients and the gradients of their gradients are all stable [21].

The noise model is assumed to be independently parametrized.

We make the important assumption that $G_0 \in \mathcal{M}_{OL}$, and hence

$$G_0 = G(\theta_0) \in \mathcal{M}_{OL} \ for \ some \ \theta_0 \in \mathbf{R}^{q \times 1} \tag{2}$$

A model $G_{mod} = G(\hat{\theta}) \in \mathcal{M}_{OL}$ is then identified from experimental data $[u_{id} \ y_{id}]$, as well as an estimate P_θ of the covariance matrix of $\hat{\theta}$. It is well known that $\hat{\theta}$ is an asymptotically unbiased estimate of θ_0 (since $G_0 \in \mathcal{M}_{OL}$) and that it is normally distributed [21]. The true parameter vector θ_0 lies with probability $\alpha(q, \chi^2_{ol})$ in the ellipsoidal uncertainty region

$$U_{OL} = \{\theta \mid (\theta - \hat{\theta})^T P_\theta^{-1} (\theta - \hat{\theta}) < \chi^2_{ol}\} \tag{3}$$

where $\alpha(q, \chi^2_{ol}) = Pr(\chi^2(q) \leq \chi^2_{ol})$ with $\chi^2(q)$ the chi-square probability distribution with q parameters[2]. This parametric uncertainty region U_{OL} defines a corresponding uncertainty region in the space of transfer functions which we denote $\mathcal{D}_{OL}$:

$$\mathcal{D}_{OL} = \left\{ G(\theta) \mid G(\theta) = \frac{Z_2 \theta}{1 + Z_1 \theta} \ and \ \theta \in U_{OL} \right\} \tag{4}$$

Properties of $\mathcal{D}_{OL}$.

$$G_0 \in \mathcal{D}_{OL} \ with \ probability \ \alpha(q, \chi^2_{ol})$$

We have thus defined an uncertainty region $\mathcal{D}_{OL}$ which contains both the model G_{mod} and the true sytem G_0 with probability $\alpha(q, \chi^2_{ol})$ (e.g. $\alpha = 0.95$).

2.2 Indirect closed-loop identification

Let us now consider a controller K which stabilizes the true system G_0. In indirect closed-loop identification, we collect experimental data on the closed loop composed of the true system G_0 and the stabilizing controller K in order to identify a model of one of the four closed-loop transfer functions describing the loop $[K \ G_0]$. These four "true" closed-loop transfer functions are:

$$T_0^1 = \tfrac{G_0 K}{1+G_0 K} \quad T_0^2 = \tfrac{G_0}{1+G_0 K} \quad T_0^3 = \tfrac{K}{1+G_0 K} \quad T_0^4 = \tfrac{1}{1+G_0 K} \tag{5}$$

The model G_{mod} for G_0 is then computed from the estimate of any one of these four transfer functions by inversion of the mapping (5), using knowledge of the controller K. The selection of one of those transfer functions for identification is linked to the available signals and the structure of the controller K. Indeed, it is proved in [5] that the presence of unstable (or unit-circle) poles or zeros in K imposes restrictions on the subset of these

[2] The use of the chi-square probability distribution with q parameters to define the probability density linked to U_{OL} is in fact an appoximation. Indeed, since P_θ is only an estimate of the true parameter covariance matrix obtained with e.g. N experimental data, the probability of the presence of θ_0 in U_{OL} defined in (3) is $Pr(F(q, N-q) < \chi^2_{ol}/q)$, where $F(q, N-q)$ is the F-distribution [20]. Nevertheless, since N will generally be large, we have that $Pr(F(q, N-q) < \chi^2_{ol}/q) \approx Pr(\chi^2(q) \leq \chi^2_{ol})$. The same remark holds for indirect closed-loop identification.

transfer functions that can be identified.

In the sequel, we show how we can construct an uncertainty region $\mathcal{D}_{CL}$ containing the true system in the case where the third closed-loop transfer function T_0^3 is estimated. An uncertainty region $\mathcal{D}_{CL}$ can be derived similarly for the other cases (see e.g. [15, 3] for the identification of T_0^1).

If the true system G_0 is assumed to have the same generic form as defined in (1)-(2) and the controller K is assumed to have the following generic expression $K = (k_{n,0} + k_{n,1}z^{-1} + ... + k_{n,nn}z^{-nn})/(1 + k_{d,1}z^{-1} + ... + k_{d,nd}z^{-nd})$, it is easy to prove that the first term of the numerator of T_0^3 is the first term of the numerator of the controller K, i.e. $k_{n,0}$, and that the denominator of T_0^3 is monic. For the identification of the closed-loop transfer function T_0^3, we therefore consider a uniformly stable model set $\mathcal{M}_{CL}$ having a monic denominator and a numerator whose first term is known.

$$\mathcal{M}_{CL} = \left\{ T(\xi) \mid T(\xi) = \frac{k_{n,0} + c_1 z^{-1} + ... + c_l z^{-l}}{1 + d_1 z^{-1} + ... + d_p z^{-p}} = \frac{k_{n,0} + Z_3\xi}{1 + Z_4\xi} \right\} \tag{6}$$

with

- $k_{n,0}$ is the first term of the numerator of K
- $\xi^T = [d_1 \; ... \; d_p \; c_1 \; ... \; c_l] \in \mathbf{R}^{f\times 1}, \; f \triangleq l + p$
- $Z_4(z) = [z^{-1} \; z^{-2} \; ... \; z^{-p} \; 0 \; ... \; 0] \in \mathbf{C}^{1\times f}$
- $Z_3(z) = [0 \; ... \; 0 \; z^{-1} \; z^{-2} \; ... \; z^{-l}] \in \mathbf{C}^{1\times f}$

Just for open-loop identification, we again make the important assumption that $T_0^3 \in \mathcal{M}_{CL}$. Therefore

$$T_0^3 = T(\xi_0) \in \mathcal{M}_{CL} \; for \; some \; \xi_0 \in \mathbf{R}^{f\times 1} \tag{7}$$

A model $T_{mod} = T(\hat{\xi}) \in \mathcal{M}_{CL}$ of the closed-loop transfer function T_0^3 can now be identified using experimental data $[r_{id} \; u_{id}]$ collected on the closed loop $[K \; G_0]$, together with an estimate P_ξ of the covariance matrix of $\hat{\xi}$. Just as in the open-loop case, we can define an ellipsoidal parametric uncertainty region U_{CL}:

$$U_{CL} = \{\xi \mid (\xi - \hat{\xi})^T P_\xi^{-1} (\xi - \hat{\xi}) < \chi_{cl}^2\} \tag{8}$$

From this set U_{CL}, we can deduce the set of corresponding open loop plants $G(\xi)$ defined as:

$$\mathcal{D}_{CL} = \left\{ G(\xi) \mid G(\xi) = \frac{1}{T(\xi)} - \frac{1}{K} \; and \; \xi \in U_{CL} \right\} \tag{9}$$

The notation $G(\xi)$ used in (9) denotes the rational transfer function model whose coefficients are uniquely determined from ξ by the mapping

$$G(\xi) = \frac{1}{T(\xi)} - \frac{1}{K}. \tag{10}$$

The nominal open-loop model derived from $T(\hat{\xi})$ is $G_{mod} = G(\hat{\xi})$. It is important to note that, using the expression of $T(\xi)$ in (6), the uncertainty region $\mathcal{D}_{CL}$ can also be rewritten as follows:

$$\mathcal{D}_{CL} = \left\{ G(\xi) \mid G(\xi) = \frac{e(z) + Z_5(z)\xi}{1 + Z_6(z)\xi} \textit{ and } \xi \in U_{CL} \right\} \tag{11}$$

with $e(z) = (1/k_{n,0}) - (1/K)$, a known transfer function with one delay and $Z_5 = (Z_4/k_{n,0}) - (Z_3/(k_{n,0}K))$ and $Z_6 = Z_3/k_{n,0}$.

Properties of U_{CL} and $\mathcal{D}_{CL}$. The probability level linked to the uncertainty regions U_{CL} and $\mathcal{D}_{CL}$ depends on the way the noise model of the closed-loop has been modelled [21, Chapter 9]. If the closed-loop noise model has been independently parametrized, then the following statements hold:

$$\xi_0 \in U_{CL} \textit{ with probability } \alpha(f, \chi^2_{cl})$$

$$G_0 = G(\xi_0) \in \mathcal{D}_{CL} \textit{ with probability } \alpha(f, \chi^2_{cl})$$

If the closed-loop noise model and $T(\xi)$ have common parameters and if the noise model set also contains the true noise model, then, denoting r ($r > f$) the size of the total parameter vector ($T(\xi)$ + noise model), the uncertainty regions U_{CL} and $\mathcal{D}_{CL}$ have the following properties:

$$\xi_0 \in U_{CL} \textit{ with probability } \alpha(r, \chi^2_{cl})$$

$$G_0 = G(\xi_0) \in \mathcal{D}_{CL} \textit{ with probability } \alpha(r, \chi^2_{cl})$$

We have thus defined an uncertainty region $\mathcal{D}_{CL}$ which contains both the model G_{mod} and the true system G_0 with probability $\alpha(f, \chi^2_{cl})$ or $\alpha(r, \chi^2_{cl})$ (e.g. $\alpha = 0.95$).

Similar uncertainty regions $\mathcal{D}_{CL}$ can be deduced from the indirect closed-loop identification of T_0^1, T_0^2 and T_0^4 (see e.g. [15, 3] for the identification of T_0^1). It is to be noted that $e(z) = 0$ in the other three cases.

2.3 General structure of the uncertainty regions obtained from identification tools

In the previous subsections, uncertainty regions $\mathcal{D}_{OL}$ and $\mathcal{D}_{CL}$ containing the true system have been obtained as a result of open-loop identification or indirect closed-loop identification, respectively. In both cases, these uncertainty regions take the form of a set of parametrized open-loop transfer functions where the parameter vector lies in an ellipsoid U_{OL} and U_{CL}, respectively. This fact can be summarized in the following proposition.

Proposition 1 *Consider* $\delta \in \mathbf{R}^{k\times 1}$*, the real parameter vector;* $G(\delta_0)$*, the true open-loop system and* $G(\hat{\delta})$*, the open-loop model obtained either "directly" by open-loop identification or "indirectly" by indirect closed-loop identification. The uncertainty region* $\mathcal{D}$ *containing* $G(\delta_0)$ *at a certain probability level has the following general form.*

$$\mathcal{D} = \left\{ G(\delta) \mid G(\delta) = \frac{e + Z_N \delta}{1 + Z_D \delta} \text{ and } \delta \in U = \{\delta \mid (\delta - \hat{\delta})^T R (\delta - \hat{\delta}) < 1\} \right\} \tag{12}$$

with

- R *is a symmetric positive definite matrix* $\in \mathbf{R}^{k\times k}$*. It is proportional to the inverse of the covariance matrix of* $\hat{\delta}$ *and has been scaled so as to obtain 1 on the right hand side.*
- $Z_N(z)$ *and* $Z_D(z)$ *are known row vectors of size* k*.*
- $e(z)$ *is either a known transfer function with delay one or is equal to 0.*

Proof. This proposition is a direct consequence of expressions (4), (11). □

3 Controller validation for stability or robust stability condition for the uncertainty region $\mathcal{D}$

In the previous section, it has been shown that an uncertainty region $\mathcal{D}$ whose generic structure is given in (12) can be constructed in the space of transfer function models both with open-loop ($\mathcal{D}_{OL}$) and with indirect closed-loop identification ($\mathcal{D}_{CL}$). This uncertainty region $\mathcal{D}$ contains both the true system G_0 and the model $G_{mod} = G(\hat{\delta})$. We now say that a controller C, designed from G_{mod}, is *validated for stability* if it stabilizes all models in this uncertainty region $\mathcal{D}$ (and therefore also the true system G_0).

Robust stability theory provides a necessary and sufficient condition for the stabilization of all plants in an uncertainty region by some given controller [10, 7, 30, 24, 18]. But for the application of robust stability results, it is required that the closed loop connections of this controller to all plants in the uncertainty region be rewritten as a set of loops that connect a known fixed dynamic matrix $M(z)$ to an uncertainty part $\Delta(z)$ of known structure that belongs to a prescribed uncertainty domain.

Our contribution in this section is to show that the uncertainty region $\mathcal{D}$ is amenable to classical robust stability analysis. Indeed, we present a way to describe the set of closed-loop connections of all plants in $\mathcal{D}$ with the "to be validated controller" C as a set of loops $[M_{\mathcal{D}}(z)\ \phi]$ where the uncertainty part ϕ is a real vector. We also also show that the (real) stability radius linked with the set of loops $[M_{\mathcal{D}}(z)\ \phi]$ can be computed exactly and efficiently, using the result presented in [18, 26]. This elegant result is only available for the identification of SISO systems. In the case of MIMO systems, the necessity of the deduced robust stability condition can no more be assured as will be shown in a future paper.

Before we proceed to this, we recall an important result of robust stability analysis [26, 18] in the case when the uncertainty is assumed to be a real vector.

3.1 Robust stability for a real vector uncertainty

We consider here a set of loops $[M(z)\ \beta]$ that obey the following system of equations (see Figure 1).

$$\begin{cases} p = \beta q \\ q = M(z)p \end{cases} \tag{13}$$

In this set of loops, it is assumed that $M(z) \in H_\infty$ is a known *fixed* row vector of size b and that the uncertainty part β is a real vector $\in \mathbf{R}^{b\times 1}$ that varies in the following uncertainty domain: $|\beta|_2 < 1$. $|\beta|_2$ represents the 2-norm of the vector β i.e. $|\beta|_2 = \sqrt{\beta^T\beta}$.

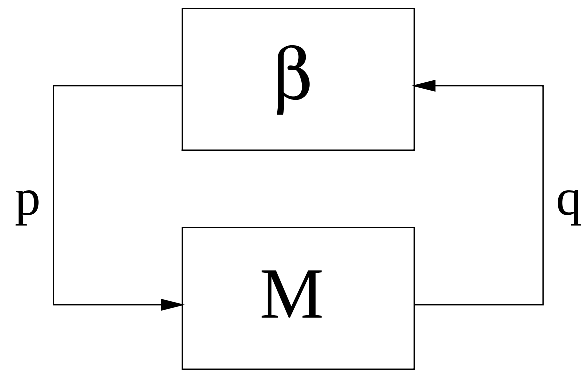

Figure 1: set of loops $[M(z)\ \beta]$

The robust stability theorem linked to the set of loops $[M(z)\ \beta]$ is now summarized in the following proposition.

Proposition 2 *If $M(z) \in H_\infty$ and $\beta \in \mathbf{R}^{b\times 1}$, then the loops $[M(z)\ \beta]$ given in (13) are internally stable for all $\beta \in \mathbf{R}^{b\times 1}$ such that $|\beta|_2 < 1$ if and only if*

$$\max_{\Omega} \mu_\beta(M(e^{j\Omega})) \leq 1 \tag{14}$$

The value $\mu_\beta(M(e^{j\Omega}))$ is called the stability radius of the loop $[M(z)\ \beta]$ at the frequency Ω and is defined below.

Definition 1 (stability radius [26, 18]) *For $M(e^{j\Omega})$ a known complex matrix $\in \mathbf{C}^{1\times b}$ and $\beta \in \mathbf{R}^{b\times 1}$, the stability radius $\mu_\beta(M(e^{j\Omega}))$ is defined as follows if $Im(M(e^{j\Omega})) \neq 0$:*

$$\mu_\beta(M(e^{j\Omega})) = \sqrt{|Re(M)|_2^2 - \frac{(Re(M)Im(M)^T)^2}{|Im(M)|_2^2}} \tag{15}$$

and $\mu_\beta(M(e^{j\Omega})) = |M|_2$, if $Im(M) = 0$.

Remarks. The stability radius is in fact the structured singular value linked to the loop $[M(z)\ \beta]$. Therefore, $\mu_\beta(M(e^{j\Omega}))$ is the inverse of the smallest value of $|\beta|_2$ such that $1 - M(e^{j\Omega})\beta = 0$. In [26], the stability radius at a given frequency is defined for a real uncertainty that is a row vector. The case of a column vector is similar and yields Definition 1. Note also that the stability radius is discontinuous only at the frequencies where M is real [25].

3.2 LFT framework for the uncertainty region $\mathcal{D}$ and a controller C

In order to apply Proposition 2 to check the stabilization of all plants in the uncertainty region $\mathcal{D}$ described in Proposition 1 by some model-based controller C, the first step is to find the particular set of loops $[M(z)\ \beta]$ that correspond to the closed-loop connections of all plants in $\mathcal{D}$ with C. This first step can be achieved using the following theorem.

Theorem 1 (LFT framework for $\mathcal{D}$) *Consider an uncertainty region $\mathcal{D}$ of plant transfer functions given by (12) and a controller $C(z)$ whose numerator and denominator are denoted $X(z)$ and $Y(z)$, respectively ($C(z) = X(z)/Y(z)$). The set of closed-loop connections $[G(\delta)\ C]$ for all $G(\delta) \in \mathcal{D}$ are equivalent to the set of loops $[M_\mathcal{D}\ \phi]$ which obey the following system of equations*

$$\begin{cases} p = \phi q \\ q = M_\mathcal{D}(z) p \end{cases}$$

where the uncertainty part ϕ is a real column vector of size k that varies in the uncertainty domain: $|\phi|_2 < 1$, and where the part $M_\mathcal{D}(z)$ is a row vector of size k defined as :

$$M_\mathcal{D}(z) = \frac{-(Z_D + \frac{X(Z_N - eZ_D)}{Y+eX})T^{-1}}{1 + (Z_D + \frac{X(Z_N - eZ_D)}{Y+eX})\hat{\delta}}, \tag{16}$$

with T a square root of the matrix R defining U in (12) : $R = T^T T$.

Proof. The closed-loop connection of C and a particular plant $G(\delta) = (e + Z_N\delta)/(1 + Z_D\delta)$ in $\mathcal{D}$ (see (12)) is given by

$$\begin{cases} y = \frac{e + Z_N\delta}{1 + Z_D\delta} u \\ u = -Cy \end{cases} \tag{17}$$

Let us rewrite (17) in a convenient way for the LFT formulation:

$$\begin{cases} y = (e + \frac{(Z_N - eZ_D)\delta}{1 + Z_D\delta}) u \\ u = -Cy \end{cases} \tag{18}$$

By introducing two new signals q and p_1, we can restate (18) as

$$\begin{cases} \begin{pmatrix} q \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} -Z_D & 1 \\ Z_N - eZ_D & e \end{pmatrix}}^{H(z)} \begin{pmatrix} p_1 \\ u \end{pmatrix} \\ p_1 = \delta q \\ u = -Cy \end{cases} \tag{19}$$

By doing so, we have isolated the uncertainty vector δ from the known transfer matrix $H(z)$ and the controller $C(z)$, as is shown in Figure 2.

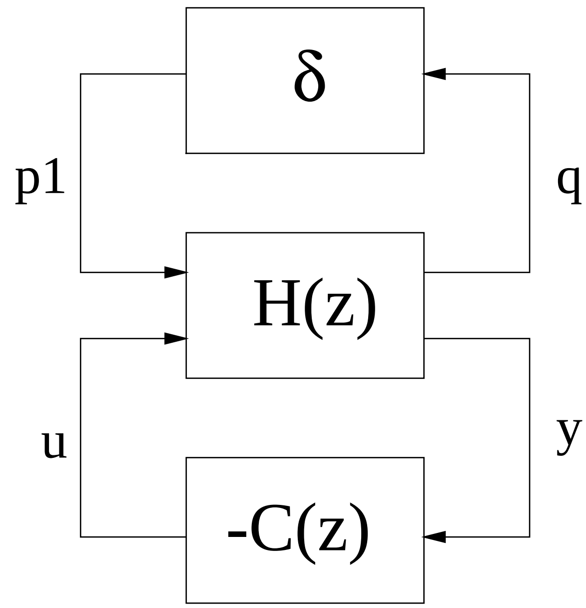

Figure 2: Equivalent loop for $[G(\delta)\ C]$

The variables y and u are now eliminated from (19), yielding the following system of equations representing a loop which is of the type (13) required by Proposition 2.

$$\begin{cases} p_1 = \delta q \\ q = (\overbrace{-Z_D - \frac{C(Z_N - eZ_D)}{1+eC}}^{M_1(z)})\, p_1 \end{cases} \tag{20}$$

The system (20) is equivalent to the closed-loop connection of a particular $G(\delta)$ in $\mathcal{D}$ with the controller C. In order to consider the closed-loop connections for all plants in $\mathcal{D}$, we have to consider all $\delta \in \mathbf{R}^{k\times 1}$ lying in the ellipsoid U given by:

$$U = \{\delta \mid (\delta - \hat{\delta})^T R(\delta - \hat{\delta}) < 1\}. \tag{21}$$

This last expression is the uncertainty domain of the real uncertainty vector δ. This uncertainty domain is not quite standard. Therefore, the set of loops $[M_1(z)\ \delta]$ with $\delta \in U$ can not be immediatly used in this form in Proposition 2. A last step is then to normalize the uncertainty domain using a method presented e.g. in [26, 20] . Using $R = T^T T$, we now define the real vector $\phi \in \mathbf{R}^{k\times 1}$ as follows:

$$\phi \triangleq T(\delta - \hat{\delta}). \tag{22}$$

Using now (21) and (22), we have

$$\delta \in U \Leftrightarrow \phi^T\phi < 1 \Longleftrightarrow |\phi|_2 < 1 \tag{23}$$

ϕ is therefore an uncertainty vector with same structure as δ (i.e. $\phi \in \mathbf{R}^{k\times 1}$) but with an uncertainty domain as required by Proposition 2. The uncertainty vector δ is therefore replaced by ϕ in (20). For this purpose, we first denote $p \triangleq \phi q$. Since $\delta = \hat{\delta} + T^{-1}\phi$, we have

$$\begin{cases} p_1 = \delta q \\ q = M_1(z) p_1 \end{cases} \Leftrightarrow \begin{cases} p = \phi q \\ q = \dfrac{M_1 T^{-1}}{1 - M_1 \hat{\delta}} p = \overbrace{\dfrac{-(Z_D + \frac{X(Z_N - eZ_D)}{Y+eX})T^{-1}}{1 + (Z_D + \frac{X(Z_N - eZ_D)}{Y+eX})\hat{\delta}}}^{M_{\mathcal{D}}(z)} p \end{cases} \tag{24}$$

The set of loops $[M_{\mathcal{D}}(z)\ \phi]$ for $\phi \in \mathbf{R}^{k\times 1}$ and $|\phi|_2 < 1$ is therefore equivalent to the set of closed-loop connections $[G(\delta)\ C]$ for all plants $G(\delta)$ in $\mathcal{D}$. This completes the proof. □

3.3 Robust stability condition for the uncertainty region $\mathcal{D}$

Theorem 1 allows us to "transform" our problem of testing if the controller C stabilizes all the plants in the uncertainty region $\mathcal{D}$ into the *equivalent* problem of testing if the set of loops $[M_{\mathcal{D}}\ \phi]$ are stable for all real vector $\phi \in \mathbf{R}^{k\times 1}$ such that $|\phi|_2 < 1$. This equivalent problem is the one which is treated by Proposition 2. Therefore, using Proposition 2 and Theorem 1, we can now formulate our main stability theorem.

Theorem 2 (robust stability condition) *Consider an uncertainty region $\mathcal{D}$ of plant transfer functions having the general form given in (12) and let C be a controller that stabilizes the nominal model $G(\hat{\delta})$. All the plants in the uncertainty region $\mathcal{D}$ are stabilized by the controller C if and only if*

$$\max_{\Omega} \mu_\phi(M_{\mathcal{D}}(e^{j\Omega})) \leq 1 \tag{25}$$

where the stability radius μ and $M_{\mathcal{D}}(z)$ are defined in Definition 1 and in (16), respectively.

Proof. $M_{\mathcal{D}}(z)$ lies in H_∞ since its denominator is the denominator of the sensitivity function of the closed loop $[G(\hat{\delta})\ C]$ which is stable by assumption. Therefore, this theorem is a direct consequence of Proposition 2 and Theorem 1. □

This theorem gives a necessary and sufficient condition for the stabilization of all plants in $\mathcal{D}$ by a controller that has been designed from the nominal model and that stabilizes it. This necessary and sufficient condition involves the computation at each frequency of the stability radius $\mu_\phi(M_{\mathcal{D}}(e^{j\Omega}))$. This computation is achieved using Definition 1.

4 Controller validation for performance or performance robustness analysis

In Section 3, we have presented a procedure to check whether a controller C designed from the model also stabilizes all plants in the uncertainty region $\mathcal{D}$ which contains the true system at a certain probability level. Modulo this probability level, we have thus given a condition that ensures that the considered controller stabilizes the true system. However, stabilization does not imply good performance with all plants in $\mathcal{D}$ (including the true system). In this section, we show that we can evaluate the worst case performance in the uncertainty region $\mathcal{D}$, i.e. the worst level of performance of a closed loop made up of the connection of the considered controller and a particular plant in $\mathcal{D}$. It is obvious that the worst case performance in $\mathcal{D}$ is a lower bound for the closed-loop performance achieved with the true system. We then say that a controller is validated for performance if the difference between the nominal performance obtained with the model and the worst case performance in $\mathcal{D}$ remains below some threshold.

There is no unique way of defining the performance of a closed-loop system. However most commonly used performance criteria can be derived from some norm of a frequency weighted version of the stability matrix $H(G,C)$ of the closed-loop system $[C\ G]$ made up of G in feedback with the controller C.

Definition 2 (stability matrix) *Given a plant G and a stabilizing controller C, the stability matrix $H(G,C)$ of the closed loop $[C\ G]$ is given by:*

$$H(G,C) = \begin{pmatrix} H_{11}(G,C) & H_{12}(G,C) \\ H_{21}(G,C) & H_{22}(G,C) \end{pmatrix} = \begin{pmatrix} \frac{GC}{1+GC} & \frac{G}{1+GC} \\ \frac{C}{1+GC} & \frac{1}{1+GC} \end{pmatrix}. \tag{26}$$

4.1 The general criterion measuring the worst case performance

The worst case performance criterion over all plants in an uncertainty region $\mathcal{D}$ will be similarly defined as the worst possible norm, over all plants in $\mathcal{D}$, of a frequency weighted version of the stability matrix $H(G(\delta),C)$, where $G(\delta)$ is any plant in $\mathcal{D}$ and C is the "to-be-validated" controller C.

General Criterion. Consider an uncertainty region $\mathcal{D}$ given by (12) and containing all systems $G(\delta) = G(z,\delta)$ with $\delta \in U$. Consider also a controller $C(z)$ validated for stability. The general criterion measuring the worst case performance level is defined at a frequency Ω as follows:

$$J_{WC}(\mathcal{D},C,W_l,W_r,\Omega) = \max_{G(z,\delta)\in\mathcal{D}} \sigma_1\left(\overbrace{\begin{pmatrix} W_{l1} & 0 \\ 0 & W_{l2} \end{pmatrix}}^{W_l} H(G(e^{j\Omega},\delta),C(e^{j\Omega})) \overbrace{\begin{pmatrix} W_{r1} & 0 \\ 0 & W_{r2} \end{pmatrix}}^{W_r} \right). \tag{27}$$

where $W_l(z)$ and $W_r(z)$ are diagonal weights[3] that allow one to define specific worst case performance levels and where $\sigma_1(A)$ denotes the largest singular value of A. Note that J_{WC} is a frequency function : it defines a template.

[3] Assuming a diagonal structure for W_l and W_r is not a loss of generality since the four transfer functions in $H(G,C)$ can all be weighted differently.

4.2 More specific worst case performance levels derived from the general criterion

In [6], the performance of a loop $[C\ G]$ is defined as $\| W_l H(G,C) W_r \|_\infty$. In this framework, the nominal performance of the designed loop is therefore $\| W_l H(G_{mod},C) W_r \|_\infty$ and the worst case performance for an uncertainty region $\mathcal{D}$ is the maximum over all frequencies of the general criterion $J_{WC}(\mathcal{D}, C, W_l, W_r, \Omega)$. In order to validate a controller in this framework, this maximum must be compared with the nominal performance.

A more fundamental way of defining the performance of a closed loop $[C\ G]$ is that proposed in [29]. The performance can be "measured" by the shape of the modulus of the frequency response of the different closed-loop transfer functions (i.e. $H_{11}(G,C)$, $H_{12}(G,C)$, $H_{21}(G,C)$ and $H_{22}(G,C)$ defined in (26)). Let us take the example of the sensitivity function $H_{22}(G,C)$ to motivate this choice. The modulus of the frequency response of $H_{22}(G,C)$ at a particular frequency Ω gives the rejection rate of an output perturbation at frequency Ω. Furthermore, the bandwidth of this frequency response gives an idea of the rejection time for constant disturbance rejection. The importance of the resonance peak is also an indication of the overshoot in constant disturbance rejection.

If the performance is defined as the modulus of the frequency response of one of the transfer functions H_{ij} $(i,j{=}1{,}2)$, the worst case performance in the uncertainty region $\mathcal{D}$ is defined as the largest modulus, over all $G(\delta) \in \mathcal{D}$, of the corresponding closed-loop transfer function H_{ij}. Let us now define this worst case performance related to H_{ij} $(i,j{=}1{,}2)$ more formally.

Definition 3 (The worst case performance for H_{ij}) *Consider an uncertainty region $\mathcal{D}$ given by (12) and containing all systems $G(\delta) = G(z,\delta)$ with $\delta \in U$. Consider also a controller $C(z)$ validated for stability and the closed-loop transfer function H_{ij} $(i,j{=}1{,}2)$ defined in (26). The worst case performance for H_{ij} is the following frequency function :*

$$t_{\mathcal{D}}(\Omega, H_{ij}) = \max_{G(z,\delta)\in\mathcal{D}} \| H_{ij}(e^{j\Omega},\delta) \|, \tag{28}$$

where $H_{ij}(z,\delta) = H_{ij}(G(z,\delta), C(z))$ and $\| H_{ij}(e^{j\Omega},\delta) \|$ denotes the modulus of $H_{ij}(e^{j\Omega},\delta)$.

For instance, if we choose the sensitivity function H_{22}, $t_{\mathcal{D}}(\Omega, H_{22})$ provides the lowest rejection rate of a perodic output disturbance at Ω, the minimal bandwidth and the maximal resonance peak over the set of closed-loop systems composed of the controller C and all plants in $\mathcal{D}$. These worst case values must be compared with the static error, the bandwidth and the resonance peak of the sensitivity function of the designed closed loop $[G_{mod}\ C] = [G(\hat{\delta})\ C]$.

The worst case performance for H_{ij} can be derived from the computation of the general criterion defined in (27). This property is summarized in the following proposition whose proof is trivial.

Proposition 3 *The worst case performance for the closed-loop transfer function H_{ij} i.e. $t_{\mathcal{D}}(\Omega, H_{ij})$ is equal to the general criterion $J_{WC}(\mathcal{D}, C, W_l, W_r, \Omega)$ when the following weights are used.*

$$W_l = \begin{pmatrix} f(i) & 0 \\ 0 & 1 - f(i) \end{pmatrix} \quad W_r = \begin{pmatrix} f(j) & 0 \\ 0 & 1 - f(j) \end{pmatrix} \tag{29}$$

where $f(x) = 1$ *if* $x = 1$ *and* $f(x) = 0$ *if* $x = 2$.

4.3 Computation of the general criterion

The general criterion measuring the worst case performance level has been defined in Section 4.1. In Section 4.2, more specific worst case performance levels have been shown to be derivable from this general criterion by appropriately choosing the diagonal weights W_r and W_l. We now present a procedure for the computation of the general criterion $J_{WC}(\mathcal{D}, C, W_l, W_r, \Omega)$ at a given frequency Ω.

This computation boils down to an optimization problem involving Linear Matrix Inequality (LMI) constraints [4]. An LMI is a matrix inequality of the form $F(\zeta) \triangleq F_0 + \sum_{i=1}^{q} \zeta_i F_i \leq 0$, where $\zeta \in \mathbf{R}^q$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{t\times t}$, $i = 0, \ldots, q$ are given. Several algorithms have been devised for solving these problems, see [28]. The LMI problems can be solved using the freeware code SP [28] and its Matlab/Scilab interface `LMITOOL` [8] or the available commercial Matlab LMI Control Toolbox [12].

For ease of formulating the LMI problem, we rewrite the weighted matrix $H_w(z, \delta) \triangleq W_l H(G(z, \delta), C(z))W_r$ for a plant $G(z, \delta) = (e + Z_N\delta)/(1 + Z_D\delta)$ in the uncertainty region $\mathcal{D}$ and a controller $C(z)$, whose polynomial numerator and denominator are denoted $X(z)$ and $Y(z)$, respectively ($C(z) = X(z)/Y(z)$). Using (26) and the expression of $G(z, \delta)$, the weighted matrix $H_w(z, \delta)$ can be rewritten as follows:

$$H_w(z, \delta) = \frac{1}{Y + eX + (XZ_N + YZ_D)\delta} \begin{pmatrix} W_{l1}X(e + Z_N\delta)W_{r1} & W_{l1}Y(e + Z_N\delta)W_{r2} \\ W_{l2}X(1 + Z_D\delta)W_{r1} & W_{l2}Y(1 + Z_D\delta)W_{r2} \end{pmatrix} \tag{30}$$

It is important to note that $H_w(z, \delta)$ is of rank one and (30) can therefore be written as follows:

$$H_w(z, \delta) = \begin{pmatrix} \dfrac{W_{l1}(e + Z_N\delta)}{Y + eX + Z_1\delta} \\ \dfrac{W_{l2}(1 + Z_D\delta)}{Y + eX + Z_1\delta} \end{pmatrix} \begin{pmatrix} XW_{r1} & YW_{r2} \end{pmatrix} \tag{31}$$

with $Z_1 = XZ_N + YZ_D$.

The following theorem now gives a procedure for the computation of the criterion $J_{WC}(\mathcal{D}, C, W_l, W_r, \Omega)$ at the frequency Ω.

Theorem 3 *Consider an uncertainty region $\mathcal{D}$ defined in (12) and the weighted version of the stability matrix $H_w(z, \delta)$ defined in (30) or (31). The general criterion $J_{WC}(\mathcal{D}, C, W_l, W_r, \Omega)$ defined in (27) is equal to $\sqrt{\gamma_{opt}}$, where γ_{opt} is the optimal value of γ for the following standard convex optimization problem involving LMI constraints evaluated at the frequency Ω:*

$$\begin{array}{ll} \textit{minimize} & \gamma \\ \textit{over} & \gamma,\ \tau \\ \textit{subject to} & \tau \geq 0 \textit{ and} \end{array}$$

$$\begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} R & -R\hat{\delta} \\ (-R\hat{\delta})^T & \hat{\delta}^T R\hat{\delta} - 1 \end{pmatrix} < 0 \tag{32}$$

where

- $a_{11} = (Z_N^* W_{l1}^* W_{l1} Z_N + Z_D^* W_{l2}^* W_{l2} Z_D) - \gamma(Q Z_1^* Z_1)$
- $a_{12} = Z_N^* W_{l1}^* W_{l1} e + W_{l2}^* W_{l2} Z_D^* - \gamma(Q Z_1^*(Y + eX))$
- $a_{22} = e^* W_{l1}^* W_{l1} e + W_{l2}^* W_{l2} - \gamma(Q(Y + eX)^*(Y + eX))$
- $Q = 1/(X^* W_{r1}^* W_{r1} X + Y^* W_{r2}^* W_{r2} Y)$

Proof. Proving this theorem is equivalent to proving that the solution γ_{opt} of the LMI problem (32), evaluated at Ω, is such that:

$$\sqrt{\gamma_{opt}} = \max_{\delta \in U} \sigma_1(H_w(e^{j\Omega}, \delta)) \Longleftrightarrow \gamma_{opt} = \max_{\delta \in U} \lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta))$$

where $U = \{\delta \mid (\delta - \hat{\delta})^T R(\delta - \hat{\delta}) < 1\}$ and $\sigma_1(A)$ and $\lambda_1(A)$ denotes the largest singular value and the largest eigenvalue of A, respectively [4].

An equivalent and convenient way of restating the problem of computing $\max_{\delta \in U} \lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta))$ is as follows:

$$\textit{minimize } \gamma \textit{ such that } \lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta)) - \gamma \leq 0 \qquad \forall \delta \in U.$$

Since $H_w(e^{j\Omega}, \delta)$ has rank one, we have:

$$\lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta)) - \gamma \leq 0 \Longleftrightarrow$$

$$\begin{pmatrix} \dfrac{W_{l1}(e + Z_N\delta)}{Y + eX + Z_1\delta} \\ \dfrac{W_{l2}(1 + Z_D\delta)}{Y + eX + Z_1\delta} \end{pmatrix}^* \begin{pmatrix} \dfrac{W_{l1}(e + Z_N\delta)}{Y + eX + Z_1\delta} \\ \dfrac{W_{l2}(1 + Z_D\delta)}{Y + eX + Z_1\delta} \end{pmatrix} (X^* W_{r1}^* W_{r1} X + Y^* W_{r2}^* W_{r2} Y) - \gamma \leq 0 \Longleftrightarrow$$

$$\begin{pmatrix} \dfrac{W_{l1}(e + Z_N\delta)}{Y + eX + Z_1\delta} \\ \dfrac{W_{l2}(1 + Z_D\delta)}{Y + eX + Z_1\delta} \\ 1 \end{pmatrix}^* \begin{pmatrix} I_2 & 0 \\ 0 & -\gamma Q \end{pmatrix} \begin{pmatrix} \dfrac{W_{l1}(e + Z_N\delta)}{Y + eX + Z_1\delta} \\ \dfrac{W_{l2}(1 + Z_D\delta)}{Y + eX + Z_1\delta} \\ 1 \end{pmatrix} \leq 0 \tag{33}$$

[4] Note that, since A is Hermitian here, its eigenvalues are all nonnegative.

where $Q = 1/(X^* W_{r1}^* W_{r1} X + Y^* W_{r2}^* W_{r2} Y)$.

By pre-multiplying (33) by $(Y + eX + Z_1\delta)^*$ and post-multiplying the same expression by $(Y + eX + Z_1\delta)$, we obtain:

$$\begin{pmatrix} W_{l1}(e + Z_N\delta) \\ W_{l2}(1 + Z_D\delta) \\ Y + eX + Z_1\delta \end{pmatrix}^* \begin{pmatrix} I_2 & 0 \\ 0 & -\gamma Q \end{pmatrix} \begin{pmatrix} W_{l1}(e + Z_N\delta) \\ W_{l2}(1 + Z_D\delta) \\ Y + eX + Z_1\delta \end{pmatrix} \leq 0 \tag{34}$$

which is equivalent to the following constraint on δ with variable γ

$$\begin{pmatrix} \delta \\ 1 \end{pmatrix}^* \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} \leq 0 \tag{35}$$

where

$$\begin{aligned} a_{11} &= (Z_N^* W_{l1}^* W_{l1} Z_N + Z_D^* W_{l2}^* W_{l2} Z_D) - \gamma(Q Z_1^* Z_1) \\ a_{12} &= Z_N^* W_{l1}^* W_{l1} e + W_{l2}^* W_{l2} Z_D^* - \gamma(Q Z_1^* (Y + eX)) \\ a_{22} &= e^* W_{l1}^* W_{l1} e + W_{l2}^* W_{l2} - \gamma(Q (Y + eX)^* (Y + eX)) \end{aligned}$$

Since δ is real, it can be shown that (35) is equivalent with

$$\overbrace{\begin{pmatrix} \delta \\ 1 \end{pmatrix}^T \begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix}}^{\alpha(\delta)} \leq 0 \tag{36}$$

This last expression is equivalent to stating that $\lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta)) - \gamma \leq 0$ for a particular δ in U. However, this must be true for all $\delta \in U$. Therefore (36) must be true for all δ such that

$$\overbrace{\begin{pmatrix} \delta \\ 1 \end{pmatrix}^T \begin{pmatrix} R & -R\hat{\delta} \\ (-R\hat{\delta})^T & \hat{\delta}^T R \hat{\delta} - 1 \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix}}^{\rho(\delta)} < 0 \tag{37}$$

which is equivalent to the statement "$\delta \in U$".

Let us now recapitulate. Computing $\max_{\delta \in U} \lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta))$ is equivalent to finding the smallest γ such that $\alpha(\delta) \leq 0$ for all δ for which $\rho(\delta) \leq 0$. By the $\mathcal{S}$ procedure [19, 4], this problem is equivalent to finding the smallest γ and a positive scalar τ such that $\alpha(\delta) - \tau\rho(\delta) \leq 0$, for all $\delta \in \mathbf{R}^{k \times 1}$, which is precisely (32).

To complete this proof, note that since $\lambda_1(H_w(e^{j\Omega}, \delta)^* H_w(e^{j\Omega}, \delta)) = \sigma_1^2(H_w(e^{j\Omega}, \delta))$, the value $\max_{\delta \in U} \sigma_1(H_w(e^{j\Omega}, \delta))$ at Ω is equal to $\sqrt{\gamma_{opt}}$, where γ_{opt} is the optimal value of γ. $\square$

5 Example

To illustrate our results, we present an example of controller validation for a model identified in closed-loop. We identify an unbiased model $T(\hat{\xi})$ of the true closed-loop transfer function

T_0^1 defined in (5). The corresponding open-loop model $G(\hat{\xi})$ is then used to design a controller C. This controller is then validated for stability using the procedure of Section 3, and for performance using the procedure of Section 4.

Identification step. Let us consider the following true system G_0 with an Output Error structure:

$$y = \overbrace{\frac{0.1047z^{-1} + 0.0872z^{-2}}{1 - 1.5578z^{-1} + 0.5769z^{-2}}}^{G_0} u + e$$

where e is a unit-variance white noise. The sampling time is 0.05 second.

We perform a closed-loop identification of an unbiased $T(\hat{\xi})$ by collecting 1000 reference and output data on the true system in closed loop with an output-feedback controller $u = 3(r-y)$. This controller stabilizes G_0. We choose an ARMAX model structure for the closed loop model since it is the structure of the actual closed loop $[K\ G_0]$. This identification yields:

$$y(\hat{\xi}) = \overbrace{\frac{0.3179z^{-1} + 0.2783z^{-2}}{1 - 1.2129z^{-1} + 0.8251z^{-2}}}^{T(\hat{\xi})} r + \frac{1 - 1.4695z^{-1} + 0.4986z^{-2}}{1 - 1.2129z^{-1} + 0.8251z^{-2}} e$$

The estimated covariance matrix of $\hat{\xi} = [-1.2129\ 0.8251\ 0.3179\ 0.2783]^T$ is

$$P_{\xi} = 10^{-3} \begin{pmatrix} 0.2353 & -0.1250 & 0.0205 & 0.0947 \\ -0.1250 & 0.1639 & -0.0723 & 0.1053 \\ 0.0205 & -0.0723 & 0.8458 & -0.8815 \\ 0.0947 & 0.1053 & -0.8815 & 1.0917 \end{pmatrix}$$

Although $\hat{\xi}$ has only four parameters, our ARMAX model has a total of 6 independent parameters. Therefore, the size χ^2_{cl} of the uncertainty region U_{CL} (see (8)) containing the parameters of the true closed-loop transfer function with probability 95 % is equal to 12.6, because $Pr(\chi^2(6) \leq 12.6) = 0.95$.

The model $G(\hat{\xi})$ corresponding to $T(\hat{\xi})$ is equal to

$$G_{mod} = G(\hat{\xi}) = \frac{1}{K} \times \frac{T(\hat{\xi})}{1 - T(\hat{\xi})} = \frac{0.1060z^{-1} + 0.0928z^{-2}}{1 - 1.5308z^{-1} + 0.5467z^{-2}}$$

Control design. From the model G_{mod}, we have designed a controller with a phase advance :

$$C(z) = \frac{1.8464 - 1.3647z^{-1}}{1 - 0.4545z^{-1}}.$$

With this controller, the designed closed-loop $[G_{mod}\ C]$ has a stability margin of 57 degrees and a gain margin of 10dB. The cut-off frequency Ω_c is equal to 0.5 which corresponds to a *real* frequency $\omega_c = 11 \; rad/s$.

Before applying this controller $C(z)$ to the true system, we verify whether it achieves satisfactory behaviour with all plants in the uncertainty region $\mathcal{D}_{CL}$ (and therefore also with the true system G_0). The uncertainty region $\mathcal{D}_{CL}$ is here defined as follows.

$$\mathcal{D}_{CL} = \left\{ G(\xi) \mid G(\xi) = \frac{1}{K} \times \frac{T(\xi)}{1 - T(\xi)} \; and \; \xi \in U_{CL} = \{\xi \mid (\xi - \hat{\xi})^T P_{\xi}^{-1} (\xi - \hat{\xi}) < 12.6\} \right\}$$

Validation of C for stability. Using the procedure presented in Section 3, we check whether C stabilizes all plants in $\mathcal{D}_{CL}$. For this purpose, we construct the row vector $M_{\mathcal{D}_{CL}}(z)$ defined in Theorem 1 and we compute the corresponding stability radius $\mu_\phi(M_{\mathcal{D}_{CL}}(e^{j\Omega}))$ at all frequencies. According to Definition 1, we know that $\mu_\phi(M_{\mathcal{D}_{CL}}(e^{j\Omega}))$ has a different expression at the frequencies where $M_{\mathcal{D}_{CL}}(e^{j\Omega})$ is real. It occurs here at $\Omega = 0$ and $\Omega = \pi$. The stability radii at these two frequencies are:

$$\mu_\phi(M_{\mathcal{D}_{CL}}(e^{j0})) = 0.0962 \; and \; \mu_\phi(M_{\mathcal{D}_{CL}}(e^{j\pi})) = 0.0340$$

The stability radii at the other frequencies (i.e. in $(0 \; \pi)$) are plotted in Figure 3.

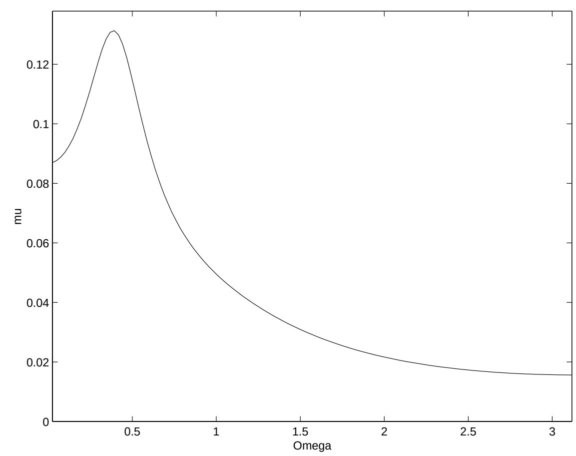

Figure 3: $\mu_\phi(M_{\mathcal{D}_{CL}}(e^{j\Omega}))$ in $(0 \; \pi)$

The maximum over all frequencies in $[0 \; \pi]$ is 0.1313. Since this maximum is smaller than 1, we conclude that $C(z)$ stabilizes all plants in $\mathcal{D}_{CL}$ and therefore also the true system G_0. In other words, the controller $C(z)$ is validated for stability.

Validation of C for performance. In order to verify that C gives satisfactory performance with all plants in $\mathcal{D}_{CL}$, we compute the worst case performance level $t_{\mathcal{D}_{CL}}(\Omega, H_{22})$ for the sensitivity function H_{22} at each frequency. This can be done by computing $J_{WC}(\mathcal{D}_{CL}, C, W_l, W_r, \Omega)$ using Theorem 3 with the particular weights $W_l = W_r = diag(0, 1)$. The worst case modulus of all sensitivity functions over $\mathcal{D}_{CL}$ is represented in Figure 4.

In this figure, the worst case performance level $t_{\mathcal{D}_{CL}}(\Omega, H_{22})$ is compared with the sensitivity functions of the designed closed loop $[G_{mod}\ C]$ and of the achieved closed loop $[G_0\ C]$. From $t_{\mathcal{D}_{CL}}(\Omega, H_{22})$, we can find that the worst case static error ($=t_{\mathcal{D}_{CL}}(0, H_{22})$) resulting from a constant disturbance of unit amplitude is equal to 0.1692, whereas this static error is 0.0834 in the designed closed-loop. The achieved static error is 0.1017. Using $t_{\mathcal{D}_{CL}}(\Omega, H_{22})$, we can also see that the bandwidth of $\Omega_c = 0.5$ in the designed closed-loop is preserved for all closed loops with a plant in $\mathcal{D}_{CL}$ since $t_{\mathcal{D}_{CL}}(\Omega, H_{22})$ is equal to 1 at $\Omega_c \simeq 0.5$. The difference between the resonance peak of the designed sensitivity function (i.e. $\max_\Omega \parallel H_{22}(G_{mod}, C) \parallel = 1.6184$) and the worst case reasonance peak achieved by a plant in $\mathcal{D}_{CL}$ (i.e. $\max_\Omega t_{\mathcal{D}_{CL}}(\Omega, H_{22}) = 1.7075$) also remains small. Note that the actually achieved resonance peak (i.e. $\max_\Omega \parallel H_{22}(G_0, C) \parallel$) is equal to 1.6229.

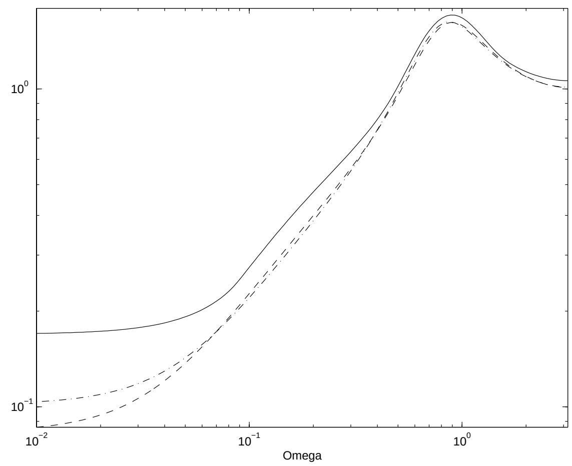

Figure 4: $t_{\mathcal{D}_{CL}}(\Omega, H_{22})$ (solid) and modulus of the designed sensitivity function $H_{22}(G_{mod}, C)$ (dashed) and actually achieved sensitivity function $H_{22}(G_0, C)$ (dashdot)

We may therefore conclude that the controller C is validated for performance since the difference between the nominal and worst case performance level remains very small at every frequency. With such stability and performance analysis results, one would confidently apply the controller to the real system, assuming that the nominal performance is judged to be satisfactory.

6 Conclusions

We have developed the tools for the robust stability and robust performance analysis of a controller designed from a nominal model, when the uncertainty set $\mathcal{D}$ containing the true system is described via ellipsoidal perturbations around the parameter vector of the nominal model. Such ellipsoidal parameter perturbations arise when the nominal model is the result of a prediction error identification procedure using an unbiased model structure. The more difficult case where biased model structures are used will be treated in a subsequent paper, using a stochastic embedding approach.

Our solution to the validation for stability problem is in the form of a necessary and sufficient condition for the stabilization of all models in this parametric uncertainty set $\mathcal{D}$ by a given controller C. Our solution to the validation for performance problem takes the form of the exact computation of the worst case performance of the controller C in closed loop with all models in the uncertainty set $\mathcal{D}$.

The main technical novelties of our paper are threefold:

- to show that the robust stability analysis for the uncertainty set $\mathcal{D}$, which is clearly nonstandard in mainstream robust control analysis, can be recast in an LFT framework with special structure for both open-loop and indirect closed-loop identification;
- to show that the worst case performance analysis for this nonstandard uncertainty set $\mathcal{D}$ can be recast as an LMI optimization problem with special structure using the fact that the uncertainty appears linearly in both the numerator and the denominator of the systems in the uncertainty region $\mathcal{D}$;
- to show that the special structures of these two nonstandard problems allow one to compute an exact solution to these two problems, thus leading us to necessary and sufficient conditions for robust stability and robust performance, respectively.

Our procedures for validation for stability and for performance have been illustrated by an example.

References

[1] B.R. Barmish. *New Tools for Robustness of Linear Systems.* MacMillan, 1994.

[2] R.M. Biernacki, H. Hwang, and S.P. Bhattacharyya. Robust stability with structured real parameter perturbations. *IEEE Transactions on Automatic Control*, 32(6):495–506, 1987.

[3] X. Bombois, M. Gevers, and G. Scorletti. Controller validation based on an identified model. accepted for presentation at CDC 1999.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*, volume 15 of *Studies in Appl. Math.* SIAM, Philadelphia, June 1994.

[5] B. Codrons, B.D.O. Anderson, and M. Gevers. Closed-loop identification with an unstable or nonminimum phase controller. submitted to SYSID 2000.

[6] R.A. de Callafon and P.M.J. Van den Hof. Suboptimal feedback control by a scheme of iterative identification and control design. *Mathematical Modelling of Systems*, 3(1):77–101, 1997.

[7] J.C. Doyle. Analysis of feedback systems with structured uncertainties. *IEE Proc.*, 129-D(6):242–250, November 1982.

[8] L. El Ghaoui, R. Nikoukhah, and F. Delebecque. `LMITOOL`*: A front-end for LMI optimization, users's guide*, February 1995. Avalaible via anonymous ftp to `ftp.ensta.fr`, under `/pub/elghaoui/lmitool`.

[9] M. K. H. Fan and A. L. Tits. A measure of worst-case H_∞ performance and of largest acceptable uncertainty. *Syst. Control Letters*, 18:409–421, 1992.

[10] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, 36(1):25–38, January 1991.

[11] G. Ferreres and V. Fromion. Computation of the robustness margin with the skewed μ-tool. *Syst. Control Letters*, 32:193–202, 1997.

[12] P. Gahinet, A. Nemirovsky, A. L. Laub, and M. Chilali. *LMI Control Toolbox*. The Mathworks Inc., 1995.

[13] M. Gevers. Towards a joint design of identification and control? *Essays on Control : Perspectives in the Theory and its Applications, Birkhauser, New York*, pages 111–151, 1993.

[14] M. Gevers, B. D.O. Anderson, and B. Codrons. Issues in modeling for control. In *Proc. American Control Conference*, pages 1615–1619, Philadelphia, USA, 1998.

[15] M. Gevers, B. Codrons, and F. De Bruyne. Model validation in closed-loop. In *Proc. American Control Conference,*, pages 326–330, San Diego, California,1999.

[16] G.C. Goodwin, M. Gevers, and B. Ninness. Quantifying the error in estimated transfer functions with application to model order selection. *IEEE Trans. Automatic Control*, 37:913–928, 1992.

[17] R.G. Hakvoort. *System Identification for Robust Process Control - PhD Thesis*. Delft University of Technology, Delft, The Netherlands, 1994.

[18] D. Hinrichsen and A.J. Pritchard. New robustness results for linear systems under real perturbations. In *Proc. Conference on Decision and Control,*, pages 1375–1378, Austin, Texas,1988.

[19] V.A. Jakubovič. The $\mathcal{S}$-procedure in nonlinear control theory. *Vestnik Leningrad Univ. (russian) Vestnik Leningrad Univ. Math. (amer.)*, 4 (amer.)(1 (russian)), 1971 (russian) 1977 (amer.).

[20] R.L. Kosut and B.D.O. Anderson. Least-squares parameter set estimation for robust control design. In *Proc. American Control Conference*, pages 3002–3006, Baltimore,Maryland,1994.

[21] L. Ljung. *System Identification: Theory for the User (second edition)*. Prentice-Hall, Englewood Cliffs, NJ, 1999.

[22] C. Marsh and H. Wei. Robustness bounds for systems with parametric uncertainty. *Automatica*, 32(10):1447–1453, 1996.

[23] M. Milanese. Learning models from data: the set membership approach. In *Proc. American Control Conference*, pages 178–182, Philadelphia, USA, 1998.

[24] A. Packard and J.C. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[25] L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young, and J.C. Doyle. A formula for computation of the real stability radius. *Automatica*, 31(6):879–890, 1995.

[26] A. Rantzer. Convex robustness specifications for real parametric uncertainty in linear systems. In *Proc. American Control Conference*, pages 583–585, 1992.

[27] A. Rantzer and A. Megretski. A convex parametrization of robustly stabilizing controllers. *IEEE Trans. Aut. Control*, 39(9):1802–1808, September 1994.

[28] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.

[29] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Aut. Control*, AC-26(2):301–320, April 1981.

[30] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, New Jersey, 1995.