Controller validation based on an identified model†

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Abstract

This paper focuses on the validation of a controller that has been designed from an unbiased model of the true system, identified either in open-loop or in closed-loop using a prediction error framework. A controller is said to be validated if it stabilizes all models in a parametric uncertainty set containing the parameters of the true system with some prescribed probability. This uncertainty set is deduced from the covariance matrix of the parameters of the identified model. Our contribution is to embed this set in the smallest possible overbounding coprime factor uncertainty set. This then allows us to use the results of mainstream robust control theory such as the Vinnicombe gap between plants and its related stability theorems.

1 Introduction

This paper is part of our continuing investigation of identification for control, as well as controller design and controller validation based on identified models [6, 7, 8]. Here we focus on controller validation on the basis of an identified model. We present a procedure to ensure that a controller C, designed from an identified model, also stabilizes all systems in an uncertainty region where the true system G_0 is known to lie with a prescribed probability. Our procedure is based on the computation of uncertainty sets as required by mainstream robust control theory (see e.g. [18, 13, 16, 19]) on the basis of ellipsoidal parametric confidence regions as delivered by prediction error identification theory (see e.g. [11]).

Robust control theory provides theorems about the stabilization of the true system but with the assumption that the true system G_0 lies in particular uncertainty regions such as the

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so-called additive, (inverse) multiplicative or coprime factor uncertainties. The prediction error identification theory developed in [11] gives us tools such as the covariance matrix of the parameters of the identified model that allow one to construct a parametric uncertainty region U containing the parameters of the true system G_0 at a certain probability level that we can fix at, say, 95 %. Such an uncertainty region takes the form of an ellipsoid in the parameter space. However, such parametric uncertainty region U is not amenable to the application of robust control theory. In order to apply the standard robust control results, we show how to embed our parametric uncertainty region U into one of the standard uncertainty regions, namely coprime factor uncertainty sets. Such sets present several advantages. First, a coprime factor uncertainty gives a satisfactory representation of the uncertainty both in high frequencies and in low frequencies (see [17, section 2.3.5]). Second, a coprime factor uncertainty set can contain plants with different numbers of unstable poles. Third, such sets can also be defined, via the Vinnicombe gap metric between two plants, as the set of plants whose Vinnicombe distance to some nominal plant is less than a given number. A fourth reason is that some powerful stability and controller design results have been developed for this kind of uncertainty set (see [13, 16, 17]).

In an earlier paper [1] we have already developed a method for the embedding of an uncertainty region in a coprime factor uncertainty set. This uncertainty region was composed of ellipsoids at each frequency in the Nyquist plane. It was obtained from the parametric ellipsoidal uncertainty region U defined by the covariance matrix through a first order approximation described in, e.g. [12, 8]. The two step procedure from the parametric uncertainty region U to an overbounding coprime factor uncertainty set described in [1] introduces both an error (by the first order approximation) and a conservatism (via the intermediate uncertainty region in the Nyquist plane). The progress in the present paper is that we can now embed the parametric uncertainty region U directly into the smallest possible coprime factor uncertainty set by formulating this embedding as a convex optimization problem involving Linear Matrix Inequality (LMI) constraints [2].

Our embedding approach differs significantly from the approach used in *set membership identification* ([14] and references therein). In the set membership literature, a hard bound assumption is made on the noise and a known upper bound is required on the pulse response of the true system, leading to the identification of an uncertainty set around a nominal model. In [9], a method to identify an additive uncertainty region with a stochastic noise assumption is presented, but a known prior bound on the true system pulse response is again required. Furthermore, the approach presented in [9] is restricted to linearly parametrized models, such as FIR models. In our embedding approach, such assumptions are not required; in particular, it applies to rational transfer function models.

In [17, Chapter 7], a parametric uncertainty region is also embedded in a coprime factor uncertainty set. However, in this chapter, the parameters are assumed to vary in fixed intervals, which contain the parameters of the true system. Nothing is said about the way these intervals are obtained from data collected on the system, and, furthermore, the covariance between the parameters are not taken into account as opposed to the method proposed in the present paper.

Paper outline. In Section 2, we briefly review how open-loop and indirect closed-loop identification lead to ellipsoidal parametric uncertainty regions U, which define equivalent uncertainty regions $\mathcal{D}$ in the space of transfer functions. In Section 3, the coprime factor uncertainty that is used in order to embed our parametric uncertainty region U is presented. In this same section, we explain how we can compute the size of this embedding set using convex optimization problems involving LMI constraints. In Section 4, different stability theorems related to the coprime factor uncertainties described in Section 3 are presented. They are illustrated by an example in Section 5. Finally, in the last section, some conclusions are given.

2 Identification and parametric uncertainty region

In this section, we briefly review the uncertainty regions delivered by classical prediction error identification, assuming that unbiased models are used [11]. For the sake of later use, we present separately open-loop and indirect closed-loop identification. We assume that the *open-loop* true system is linear and time-invariant, with a rational input-output transfer function G_0:

$$y = G_0 u + v$$

where v is additive noise.

2.1 Open-loop identification

In the case of open-loop identification, we consider a uniformly stable model set $\mathcal{M}_{OL}$ with the following structure:

$$\mathcal{M}_{OL} = \{G(\theta) \mid G(\theta) = \frac{b_1 z^{-1} + ... + b_m z^{-m}}{1 + a_1 z^{-1} + ... + a_n z^{-n}} \; and \; \theta^T = [a_1 \, ... \, a_n \; b_1 \, ... \, b_m] \in \mathbf{R}^k, k \triangleq n+m\} \tag{1}$$

and an independently parametrized noise model.

We make the important assumption that $G_0 \in \mathcal{M}_{OL}$, and hence

$$G_0 = G(\theta_0) \in \mathcal{M}_{OL} \; for \; some \; \theta_0 \in \mathbf{R}^k \tag{2}$$

A model $G_{mod} = G(\hat{\theta}) \in \mathcal{M}_{OL}$ is then identified from experimental data $[u_{id} \; y_{id}]$, as well as an estimate of the covariance matrix P_θ of $\hat{\theta}$. It is well known that $\hat{\theta}$ is asymptotically unbiased (since $G_0 \in \mathcal{M}_{OL}$) and normally distributed [11]. The true parameter θ_0 lies with probability $\alpha(k, \chi^2_{ol})$ in the ellipsoidal uncertainty region

$$U_{OL} = \{\theta \mid (\theta - \hat{\theta})^T P_\theta^{-1} (\theta - \hat{\theta}) < \chi^2_{ol}\} \tag{3}$$

where $\alpha(k, \chi^2_{ol}) = Pr(\chi^2(k) \leq \chi^2_{ol})$ with $\chi^2(k)$ the chi-square probability distribution with k parameters. This parametric uncertainty region U_{OL} defines a corresponding uncertainty region in the space of transfer functions which we denote $\mathcal{D}_{OL}$:

$$\mathcal{D}_{OL} = \{G(\theta) \mid G(\theta) \in \mathcal{M}_{OL} \text{ and } \theta \in U_{OL}\} \tag{4}$$

Properties of $\mathcal{D}_{OL}$.

$$G_0 \in \mathcal{D}_{OL} \text{ with probability } \alpha(k, \chi^2_{ol})$$

We have thus defined an uncertainty region $\mathcal{D}_{OL}$ which contains both the model G_{mod} and the true sytem G_0 with probability $\alpha(k, \chi^2_{ol})$ (e.g. $\alpha = 0.95$).

2.2 Indirect closed-loop identification

Let us now consider a controller K which stabilizes the true system G_0. In indirect closed-loop identification, we collect experimental data $[r_{id}\ y_{id}]$ on the closed loop composed of the true system G_0 and the stabilizing controller K in order to identify a model T_{mod} of the actual closed-loop transfer function $T_0 = (G_0K)/(1 + G_0K)$.

For the identification of the closed-loop transfer function, we consider a uniformly stable model set $\mathcal{M}_{CL}$

$$\mathcal{M}_{CL} = \{T(\xi) \mid T(\xi) = \frac{c_1 z^{-1} + ... + c_l z^{-l}}{1 + d_1 z^{-1} + ... + d_p z^{-p}} \text{ and } \xi^T = [d_1\ ...\ d_p\ c_1\ ...\ c_l] \in \mathbf{R}^f, f \triangleq l + p\} \tag{5}$$

We again make the important assumption that $T_0 \in \mathcal{M}_{CL}$. Therefore

$$T_0 = T(\xi_0) \in \mathcal{M}_{CL} \text{ for some } \xi_0 \in \mathbf{R}^f \tag{6}$$

A model $T_{mod} = T(\hat{\xi}) \in \mathcal{M}_{CL}$ of the closed-loop transfer function can now be identified using experimental data collected on the considered closed loop, together with an estimate of the covariance matrix P_ξ of $\hat{\xi}$. Just as in the open-loop case, we can define an ellipsoidal parametric uncertainty region U_{CL}:

$$U_{CL} = \{\xi \mid (\xi - \hat{\xi})^T P_\xi^{-1} (\xi - \hat{\xi}) < \chi^2_{cl}\} \tag{7}$$

From this set U_{CL}, we can deduce the set of corresponding open loop plants $G(\xi)$ defined as:

$$\mathcal{D}_{CL} = \{G(\xi) \mid G(\xi) = \frac{1}{K}\frac{T(\xi)}{1 - T(\xi)} \text{ and } \xi \in U_{CL}\} \tag{8}$$

The notation $G(\xi)$ used in (8) denotes the rational transfer function model whose coefficients are uniquely determined from ξ by the mapping

$$G(\xi) = \frac{1}{K}\frac{T(\xi)}{1 - T(\xi)}. \tag{9}$$

The nominal open-loop model derived from $T(\hat{\xi})$ is $G_{mod} = G(\hat{\xi})$.

Properties of U_{CL} and $\mathcal{D}_{CL}$. The probability level linked to the uncertainty regions U_{CL} and $\mathcal{D}_{CL}$ depends on the way the noise model of the closed-loop has been modelled [11, Chapter 9]. If the closed-loop noise model has been independently parametrized, then the following statements hold:

$$\xi_0 \in U_{CL} \textit{ with probability } \alpha(f, \chi^2_{cl})$$

$$G_0 = G(\xi_0) \in \mathcal{D}_{CL} \textit{ with probability } \alpha(f, \chi^2_{cl})$$

If the closed-loop noise model and $T(\xi)$ have common parameters and if the noise model set also contains the true noise model, then, denoting r $(r > f)$ the size of the total parameter vector ($T(\xi)$ + noise model), the uncertainty regions U_{CL} and $\mathcal{D}_{CL}$ have the following properties:

$$\xi_0 \in U_{CL} \textit{ with probability } \alpha(r, \chi^2_{cl})$$

$$G_0 = G(\xi_0) \in \mathcal{D}_{CL} \textit{ with probability } \alpha(r, \chi^2_{cl})$$

We have thus defined an uncertainty region $\mathcal{D}_{CL}$ which contains both the model G_{mod} and the true system G_0 with probability $\alpha(f, \chi^2_{cl})$ or $\alpha(r, \chi^2_{cl})$ (e.g. $\alpha = 0.95$).

3 The embedding coprime factor uncertainty set

In the previous section, it has been shown that an uncertainty region $\mathcal{D}$ can be constructed in the space of transfer function models both with open-loop ($\mathcal{D}_{OL}$) and with indirect closed-loop identification ($\mathcal{D}_{CL}$). This uncertainty region $\mathcal{D}$ contains both the true system G_0 and the model G_{mod}. We now say that a controller C, designed from G_{mod}, is *validated* if it stabilizes all models in this uncertainty region $\mathcal{D}$. Standard robust stability results infer the validation of a controller (i.e. the robust stabilization) from the stability of the nominal closed loop $[C\ G_{mod}]$, and some measure of the *size* of the uncertainty region $\mathcal{D}$, typically centered on G_{mod}. Unfortunately, no such results exist for the uncertainty regions $\mathcal{D}$ defined as above, and which result from prediction error identification. Indeed, robust control theory provides theorems about the stabilization of the true system G_0, but with the assumption that G_0 lies in uncertainty regions that are either an additive, or an (inverse) multiplicative, or a coprime factor perturbation of G_{mod}. In order to use standard robust control theory, we show how to embed the uncertainty region $\mathcal{D}$ into one of those particular uncertainty sets. In this paper, we have opted for the coprime uncertainty set described in [17], for the reasons presented in the introduction.

3.1 Coprime factor uncertainty and the Vinnicombe distance

Since G_{mod} will be used as our nominal model for control design, the embedding coprime factor uncertainty regions that we now construct will be centered at G_{mod}. We first consider a normalized coprime factor description of G_{mod}: $G_{mod} = ND^{-1}$, where N and D belong to the ring of rational proper stable transfer functions[1] and are such that $NN^* + DD^* = 1$ [13]. The coprime factor uncertainty set of size ϵ described in [17] is the set of all plants G_{in} which can be written as a perturbation of $[N\ D]$ with a perturbation $\Delta = \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \in \mathbf{L}_\infty$ such that $\| \Delta \|_\infty \leq \epsilon$:

$$\mathcal{G}(G_{mod}, \epsilon) = \left\{ G_{in} \mid G_{in} = \frac{N + \Delta_N}{D + \Delta_D}, \quad \| \Delta \|_\infty \leq \epsilon \quad \text{and} \quad \eta(G_{in}) = wno(D + \Delta_D) \right\}. \tag{10}$$

Here $\eta(G)$ denotes the number of open right half plane poles of G, and $wno(G)$ denotes the winding number about the origin of $G(s)$ as s follows the standard Nyquist D-contour indented into the right half plane around any imaginary axis poles and zeros of $G(s)$.

An alternative expression, easier to handle than definition (10), was proposed in [17]:

$$\mathcal{G}(G_{mod}, \epsilon) = \{ G_{in} \mid \delta_\nu(G_{mod}, G_{in}) \leq \epsilon \}. \tag{11}$$

where $\delta_\nu(G_{mod}, G_{in})$ is the Vinnicombe distance [16] between G_{mod} and G_{in} defined in (12) below. Expression (11) shows that the coprime factor uncertainty set is a ball of systems, centered at the model G_{mod} and whose radius is equal to ϵ.

$$\delta_\nu(G_{mod}, G_{in}) = \begin{cases} \max_\omega \kappa(G_{mod}(j\omega), G_{in}(j\omega)) = \max_\omega \frac{|G_{mod} - G_{in}|}{\sqrt{1+|G_{mod}|^2}\sqrt{1+|G_{in}|^2}} & if\ (13)\ is\ satisfied \\ 1 & otherwise \end{cases} \tag{12}$$

The condition to be fulfilled in order to have $\delta_\nu(G_{mod}, G_{in}) < 1$ is :

$$\begin{aligned} (1 + G^*_{mod} G_{in})(j\omega) \neq 0 \quad \text{for all } \omega \text{ and} \\ wno(1 + G^*_{mod} G_{in}) + \eta(G_{in}) - \tilde{\eta}(G_{mod}) = 0. \end{aligned} \tag{13}$$

where $G^*(s) = G(-s)$, $\tilde{\eta}(G)$ denotes the number of closed right half plane poles of G, while $\eta(G)$ and $wno(G)$ have been defined above.

If these last two conditions are satisfied, then the distance between two plants has a simple frequency domain interpretation (in the SISO case). Indeed, the quantity $\kappa(G_{mod}(j\omega), G_{in}(j\omega))$ is the chordal distance between the projections of $G_{mod}(j\omega)$ and $G_{in}(j\omega)$ onto the Riemann sphere of unit diameter [16]. The distance $\delta_\nu(G_{mod}, G_{in})$ between G_{mod} and G_{in} is therefore, according to (12), the supremum of these chordal distances over all frequencies.

These definitions also hold in discrete time via the use of the bilinear transform $s = (z-1)/(z+1)$ [17, page 259]. In the sequel, we will use a discrete time formalism since this formalism is used in the identification methods of Section 2.

[1] We consider here only scalar transfer functions.

3.2 The worst case Vinnicombe distance

Expression (11) shows that to embed an uncertainty region $\mathcal{D}$ into a coprime factor uncertainty set $\mathcal{G}(G_{mod}, \epsilon)$, one only has to find the smallest size ϵ such that $\mathcal{D} \subset \mathcal{G}(G_{mod}, \epsilon)$. In order to compute this smallest overbounding coprime factor uncertainty set, we introduce the notion of **worst case Vinnicombe distance** $\delta_{WC}(G_{mod}, \mathcal{D})$. This corresponds to the largest Vinnicombe distance between the model G_{mod} and any plant inside the set $\mathcal{D}$.

Definition of the worst case Vinnicombe distance.

$$\delta_{WC}(G_{mod}, \mathcal{D}) = \max_{G_{\mathcal{D}} \in \mathcal{D}} \delta_{\nu}(G_{mod}, G_{\mathcal{D}}) \tag{14}$$

Another important quantity is now defined: the **worst case chordal distance**. Its computation is the result of a convex optimization problem involving LMI constraints as will be shown in Section 3.3.

Definition of the worst case chordal distance at Ω. At a particular frequency Ω, we define $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D})$ as the maximum chordal distance between the projection on the Riemann sphere of $G_{mod}(e^{j\Omega})$ and the projections on the Riemann sphere of the frequency responses of all plants in $\mathcal{D}$ at the same frequency:

$$\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}) = \max_{G_{\mathcal{D}} \in \mathcal{D}} \kappa(G_{mod}(e^{j\Omega}), G_{\mathcal{D}}(e^{j\Omega})) \tag{15}$$

This last quantity can now be used to give an alternative expression of the worst case Vinnicombe distance, by making use of the following proposition.

Proposition 1 [17]. Given a metric space Λ, the set of all rational transfer functions $\mathcal{R}$, a mapping $\Lambda \to \mathcal{R} : \lambda \to G_{\lambda}$, continuous in the graph topology and a pathwise connected closed subset $\mathcal{U}$ of Λ with $\lambda_{mod} \in \mathcal{U} \to G_{mod}$, then

$$\max_{\lambda \in \mathcal{U}} \delta_{\nu}(G_{mod}, G_{\lambda}) = \max_{\Omega} \max_{\lambda \in \mathcal{U}} \kappa(G_{mod}(e^{j\Omega}), G_{\lambda}(e^{j\Omega})) \tag{16}$$

□

Lemma 1. The worst case Vinnicombe distance $\delta_{WC}(G_{mod}, \mathcal{D})$ defined in (14), can also be expressed in the following way using the worst case chordal distance:

$$\delta_{WC}(G_{mod}, \mathcal{D}) = \max_{\Omega} \kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}) \tag{17}$$

where $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D})$ is defined in (15).

Proof. This lemma is a direct consequence of Proposition 1 and of the definition of the worst case chordal distance. Indeed, the mappings between the parametric sets U (U_{OL} or U_{CL}) and the corresponding transfer function sets $\mathcal{D}$ ($\mathcal{D}_{OL}$ or $\mathcal{D}_{CL}$) are continuous in the graph topology since they can be expressed in the μ setup of Doyle [17, 3]. Therefore, the assumptions of Proposition 1 are fulfilled. We can thus write, for a set $\mathcal{D}$ of transfer functions such as $\mathcal{D}_{OL}$ or $\mathcal{D}_{CL}$, that

$$\delta_{WC}(G_{mod}, \mathcal{D}) = \max_{\Omega} \max_{G_{\mathcal{D}} \in \mathcal{D}} \kappa(G_{mod}(e^{j\Omega}), G_{\mathcal{D}}(e^{j\Omega}))$$

Using the definition (15) of the worst case chordal distance, this last expression is equivalent with (17). □

The notion of worst case Vinnicombe distance and worst case chordal distance having been introduced, the smallest embedding coprime factor uncertainty set that embeds the uncertainty region $\mathcal{D}$ is now defined.

Theorem 1 (smallest embedding coprime factor uncertainty set). The worst case Vinnicombe distance $\delta_{WC}(G_{mod}, \mathcal{D})$ defined in (14) is the smallest size ϵ of any coprime factor uncertainty set (11) such that $\mathcal{D} \subset \mathcal{G}(G_{mod}, \epsilon)$. Therefore, the smallest coprime factor uncertainty set which embeds $\mathcal{D}$ can be described as:

$$\mathcal{G}_{emb}(G_{mod}, \mathcal{D}) = \{G_{in} = \frac{N + \Delta_N}{D + \Delta_D} \ \mid \ \delta_{\nu}(G_{mod}, G_{in}) \leq \delta_{WC}(G_{mod}, \mathcal{D})\}. \tag{18}$$

where $[N \ D]$ is the normalized coprime factor description of the model G_{mod}.

Proof. All the plants in the uncertainty region $\mathcal{D}$ lie in the set (18) since the Vinnicombe distance between the model G_{mod} and any plant in $\mathcal{D}$ is smaller than (or equal to) $\delta_{WC}(G_{mod}, \mathcal{D})$ by the definition of the worst case Vinnicombe distance (14). Furtermore, a coprime factor uncertainty set of size $\epsilon < \delta_{WC}(G_{mod}, \mathcal{D})$ would not contain all the plants of $\mathcal{D}$ since there exists a plant in $\mathcal{D}$ whose Vinnicombe distance to the model is equal to $\delta_{WC}(G_{mod}, \mathcal{D})$. □

We now define a pointwise (i.e. frequency by frequency) version of the coprime factor uncertainty $\mathcal{G}(G_{mod}, \epsilon)$ defined in (11). This set $\mathcal{B}(G_{mod}, f)$ [17] will be called "pointwise coprime factor uncertainty set" in the sequel. It is centered at the model G_{mod} and its size is defined by the frequency function $f(\Omega)$.

$$\mathcal{B}(G_{mod}, f) = \{G_{in} \ \mid \ \kappa(G_{mod}(e^{j\Omega}), G_{in}(e^{j\Omega})) \leq f(\Omega) \ and \ \delta_{\nu}(G_{mod}, G_{in}) < 1\} \tag{19}$$

Just as was done for $\mathcal{G}(G_{mod}, \epsilon)$, we can define the smallest pointwise coprime factor uncertainty set such that $\mathcal{D} \subset \mathcal{B}(G_{mod}, f)$ using the notion of worst case chordal distance.

Corollary 1. If $\delta_{WC}(G_{mod}, \mathcal{D}) < 1$, then the worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D})$ defined in (15) is the smallest size $f(\Omega)$ of any pointwise coprime factor uncertainty set (19) such that $\mathcal{D} \subset \mathcal{B}(G_{mod}, f)$. Therefore, the smallest pointwise coprime factor uncertainty set which embeds $\mathcal{D}$ can be described as:

$$\begin{aligned} \mathcal{B}_{emb}(G_{mod}, \mathcal{D}) = \ \ & \{G_{in} \ \mid \ \kappa(G_{mod}(e^{j\Omega}), G_{in}(e^{j\Omega})) \leq \kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}) \ \forall \ \Omega \\ & \qquad and \ \delta_{\nu}(G_{mod}, G_{in}) < 1\} \end{aligned} \tag{20}$$

Proof. All the plants in the uncertainty region $\mathcal{D}$ lie in the set (20) since, for all Ω, the chordal distance between $G_{mod}(e^{j\Omega})$ and the frequency response at Ω of any plant in $\mathcal{D}$ is smaller than (or equal to) $\kappa_{WC}(G_{mod}(e^{j\Omega}),\mathcal{D})$ (see (15)) and since $\delta_{WC}(G_{mod},\mathcal{D}) < 1$.

Furthermore, a pointwise coprime factor uncertainty set such that $f(\Omega_0) < \kappa_{WC}(G_{mod}(e^{j\Omega_0}),\mathcal{D})$ for a given Ω_0 would not contain all the plants inside $\mathcal{D}$, since there exists a plant $G_{\mathcal{D}}$ in $\mathcal{D}$ such that

$$\kappa(G_{mod}(e^{j\Omega_0}), G_{\mathcal{D}}(e^{j\Omega_0})) = \kappa_{WC}(G_{mod}(e^{j\Omega_0}),\mathcal{D}) > f(\Omega_0).$$

$\square$

Since the true system G_0 lies in the uncertainty region $\mathcal{D}$ (with probability 0.95), it also lies in the coprime factor uncertainty set which embeds this uncertainty region $\mathcal{D}$ as well as in the pointwise embedding set (20). Therefore,

$$G_0 \in \mathcal{G}_{emb}(G_{mod},\mathcal{D}) \tag{21}$$

$$G_0 \in \mathcal{B}_{emb}(G_{mod},\mathcal{D}). \tag{22}$$

Since the true system is now included in the coprime factor uncertainty set $\mathcal{G}_{emb}(G_{mod},\mathcal{D})$, we can apply the different tools of mainstream robust control theory in order to design a controller that stabilizes all the plants in $\mathcal{G}_{emb}(G_{mod},\mathcal{D})$, and hence G_0, or to test the robust stabilization by a particular controller for all systems in this same set. This last point is developed in Section 4. In Section 4, it will also be shown that the pointwise set $\mathcal{B}_{emb}(G_{mod},\mathcal{D})$ can also be used to test the robust stabilization, leading to less conservative conditions. First we show how to compute the uncertainty sets defined in (18) and in (20).

3.3 Computation of the smallest embedding coprime factor uncertainty set

We have defined in (18) the smallest coprime factor uncertainty set $\mathcal{G}_{emb}(G_{mod},\mathcal{D})$ that embeds all models in an uncertainty region $\mathcal{D}$ that results from open or closed loop identification. This definition relies on the worst case Vinnicombe distance $\delta_{WC}(G_{mod},\mathcal{D})$ between the model and all members of such set $\mathcal{D}$. This worst case Vinnicombe distance is itself computed from the worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}),\mathcal{D})$ as shown in (17). In (20), we have also defined the smallest pointwise coprime factor uncertainty set $\mathcal{B}_{emb}(G_{mod},\mathcal{D})$ that embeds $\mathcal{D}$. Its definition also relies on the worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}),\mathcal{D})$. It remains to find a procedure for the computation of $\kappa_{WC}(G_{mod}(e^{j\Omega}),\mathcal{D})$ at each frequency. This is the object of the present section.

We show that the computation of the worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}),\mathcal{D})$ can be formulated as a convex optimization problem involving Linear Matrix Inequality (LMI) constraints [2]. An LMI is a matrix inequality of the form $F(\zeta) \triangleq F_0 + \sum_{i=1}^{q}\zeta_i F_i \leq 0$, where $\zeta \in \mathbf{R}^q$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{t\times t}$, $i = 0,\ldots,q$ are given. Several algorithms that have practical efficiency have been devised for solving these problems, see [15]. The

LMI problems can be solved using the free ware code SP [15] and its Matlab/Scilab interface `LMITOOL` [4] or the available commercial Matlab Toolbox, LMI Control Toolbox [5].

Given that the parametrizations of the transfer functions defining the (open loop) set $\mathcal{D}_{OL}$ and the (closed loop) set $\mathcal{D}_{CL}$ are different, the LMI problems will be specific to each case. Our computational procedure for $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D})$ is such that the ensuing uncertainty set $\mathcal{G}_{emb}(G_{mod}, \mathcal{D})$ (resp. $\mathcal{B}_{emb}(G_{mod}, \mathcal{D})$) will be the smallest possible uncertainty set (resp. pointwise coprime factor uncertainty set) that embeds all models in $\mathcal{D}$.

3.3.1 Worst case chordal distance for open-loop identification

With open-loop identification, a model $G_{mod} = G(\hat{\theta})$ is obtained together with a covariance matrix P_θ (see Section 2.1). The true system G_0 lies in the uncertainty region $\mathcal{D}_{OL}$ defined by (4) and (3) with a prescribed probability.

For ease of formulating the LMI problem, the model structure defined in (1) is rewritten at frequency Ω as follows:

$$G(\theta, \Omega) = \frac{b_1 z^{-1} + ... + b_m z^{-m}}{1 + a_1 z^{-1} + ... + a_n z^{-n}} = \frac{Z_2 \theta}{1 + Z_1 \theta} \quad with\ z = e^{j\Omega} \tag{23}$$

with

- $\theta^T = [a_1\ ...\ a_n\ b_1\ ...\ b_m] \in \mathbf{R}^k,\ \ k \triangleq n + m$
- $Z_1 = [e^{-j\Omega}\ e^{-2j\Omega}\ ...\ e^{-nj\Omega}\ 0\ ...\ 0] \in \mathbf{C}^k$
- $Z_2 = [0\ ...\ 0\ e^{-j\Omega}\ e^{-2j\Omega}\ ...\ e^{-mj\Omega}] \in \mathbf{C}^k$

Theorem 2 (open-loop). Consider the parameter estimate $\hat{\theta}$ and its corresponding covariance matrix P_θ, with $G(\hat{\theta}, \Omega)$ and $\mathcal{D}_{OL}$ defined in (23) and (4)-(3). The worst case chordal distance $\kappa_{WC}(G(\hat{\theta}, \Omega), \mathcal{D}_{OL})$ at frequency Ω is equal to $\sqrt{\gamma_{opt}}$ where γ_{opt} is the optimal value of γ in the following standard convex optimization problem involving LMI constraints:

$$\begin{array}{ll} minimize & \gamma \\ over & \gamma,\ \tau \\ subject\ to & \tau \geq 0\ and \end{array}$$

$$\begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} R & -R\hat{\theta} \\ (-R\hat{\theta})^T & \hat{\theta}^T R \hat{\theta} - \chi_{ol}^2 \end{pmatrix} \leq 0 \tag{24}$$

with

- $R = P_\theta^{-1}$
- $a_{11} = (Z_2^* Z_2 - Z_2^* x Z_1 - Z_1^* x^* Z_2 + Z_1^* x^* x Z_1) - \gamma(Z_2^* Q Z_2 + Z_1^* Q Z_1)$
- $a_{12} = -Z_2^* x + Z_1^* x^* x - \gamma(Z_1^* Q)$
- $a_{22} = x^* x - \gamma Q$
- $Q = 1 + x^* x$ and $x = G(\hat{\theta}, \Omega)$

Proof. First note that this optimization problem is convex since the LMI constraint is linear in its variables γ and τ. Therefore, this problem can be solved with the Matlab LMI Control Toolbox.

If we denote the frequency response of the identified model $G(\hat{\theta}, \Omega)$ by x, and that of any plant $G(\theta, \Omega)$ by $y(\theta)$, then a convenient way to state the problem of computing the worst case chordal distance at some frequency Ω is as follows (see (15)):

$$\textit{minimize } \gamma \textit{ such that } \kappa(x, y(\theta))^2 \leq \gamma \textit{ for all } y(\theta) \in \mathcal{D}_{OL}$$

The expression $\kappa(x, y(\theta))^2 \leq \gamma$ has to be transformed into an LMI. This can easily be done as proved in the following expressions.

$$\left(\frac{|x - y(\theta)|}{\sqrt{1+|x|^2}\sqrt{1+|y(\theta)|^2}}\right)^2 \leq \gamma \Longleftrightarrow$$

$$x^*x + y(\theta)^*y(\theta) - y(\theta)^*x - x^*y(\theta) - \gamma(1+x^*x)(1+y(\theta)^*y(\theta)) \leq 0 \Longleftrightarrow$$

$$\begin{pmatrix} y(\theta) \\ 1 \end{pmatrix}^* \begin{pmatrix} 1-\gamma(1+x^*x) & -x \\ -x^* & x^*x - \gamma(1+x^*x) \end{pmatrix} \begin{pmatrix} y(\theta) \\ 1 \end{pmatrix} \leq 0 \tag{25}$$

Now, $y(\theta)$ is replaced by its expression (23). By pre-multiplying (25) by $(1+Z_1\theta)^*$ and post-multiplying the same expression by $(1+Z_1\theta)$, we obtain:

$$\begin{pmatrix} Z_2\theta \\ 1+Z_1\theta \end{pmatrix}^* \begin{pmatrix} 1-\gamma(1+x^*x) & -x \\ -x^* & x^*x - \gamma(1+x^*x) \end{pmatrix} \begin{pmatrix} Z_2\theta \\ 1+Z_1\theta \end{pmatrix} \leq 0 \tag{26}$$

which is equivalent to the following constraint with variable γ and with $Q = 1 + x^*x$:

$$\begin{pmatrix} \theta \\ 1 \end{pmatrix}^* \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} \leq 0 \tag{27}$$

with

$$\begin{aligned} a_{11} &= (Z_2^*Z_2 - Z_2^*xZ_1 - Z_1^*x^*Z_2 + Z_1^*x^*xZ_1) - \gamma(Z_2^*QZ_2 + Z_1^*QZ_1) \\ a_{12} &= -Z_2^*x + Z_1^*x^*x - \gamma(Z_1^*Q) \\ a_{22} &= x^*x - \gamma Q \end{aligned}$$

Since θ is real, it can be shown that (27) is equivalent with

$$\overbrace{\begin{pmatrix} \theta \\ 1 \end{pmatrix}^T \begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix}}^{\sigma(\theta)} \leq 0 \tag{28}$$

This last expression is equivalent to stating that the square of the distance $\kappa(x, y(\theta))$ between $x = G(\hat{\theta}, \Omega)$ and $y(\theta) = G(\theta, \Omega)$ is smaller than γ. This must be true for all $\theta \in U_{OL}$. Denoting $R = P_\theta^{-1}$, this is equivalent with

$$(\theta - \hat{\theta})^T R(\theta - \hat{\theta}) < \chi^2_{ol} \tag{29}$$

which is equivalent to

$$\overbrace{\begin{pmatrix} \theta \\ 1 \end{pmatrix}^T \begin{pmatrix} R & -R\hat{\theta} \\ (-R\hat{\theta})^T & \hat{\theta}^T R\hat{\theta} - \chi^2_{ol} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix}}^{\rho(\theta)} \leq 0 \tag{30}$$

Let us now summarize. The problem of computing the worst case chordal distance $\kappa_{WC}(G(\hat{\theta}, \Omega), \mathcal{D}_{OL})$ is equivalent to finding the smallest γ such that $\sigma(\theta) \leq 0$ for all θ for which $\rho(\theta) \leq 0$. By the $\mathcal{S}$ procedure [10, 2] , this problem is equivalent to finding the smallest γ and a positive scalar τ such that $\sigma(\theta) - \tau\rho(\theta) \leq 0$, for all $\theta \in \mathbf{R}^k$ which is precisely (24).

To complete this proof, note that the worst case chordal distance at Ω is equal to $\sqrt{\gamma_{opt}}$ where γ_{opt} is the optimal value of γ. □

3.3.2 Worst case chordal distance for indirect closed-loop identification

With closed-loop identification, a model $T_{mod} = T(\hat{\xi})$ of the closed-loop transfer function is obtained together with a covariance matrix P_ξ (see section 2.2). An open-loop model $G(\hat{\xi})$ is derived by the mapping defined in (9). The true system $G_0 = G(\xi_0)$ lies in the uncertainty region $\mathcal{D}_{CL}$ defined by (8) and (7) with a prescribed probability. The model structure $T(\xi)$ at frequency Ω can therefore be rewritten as follows:

$$T(\xi, \Omega) = \frac{c_1 z^{-1} + ... + c_l z^{-l}}{1 + d_1 z^{-1} + ... + d_p z^{-p}} = \frac{Z_3 \xi}{1 + Z_4 \xi} \quad with\ z = e^{j\Omega} \tag{31}$$

with

- $\xi^T = [d_1\ ...\ d_p\ c_1\ ...\ c_l] \in \mathbf{R}^f,\ f \triangleq l + p$
- $Z_4 = [e^{-j\Omega}\ e^{-2j\Omega}\ ...\ e^{-pj\Omega}\ 0\ ...\ 0] \in \mathbf{C}^f$
- $Z_3 = [0\ ...\ 0\ e^{-j\Omega}\ e^{-2j\Omega}\ ...\ e^{-lj\Omega}] \in \mathbf{C}^f$

Theorem 3 (closed-loop). Consider the parameter estimate $\hat{\xi}$ and its corresponding covariance matrix P_ξ, with $T(\hat{\xi}, \Omega)$, $G(\hat{\xi}, \Omega)$ and $\mathcal{D}_{CL}$ defined in (31) and (8)-(7). The worst case chordal distance $\kappa_{WC}(G(\hat{\xi}, \Omega), \mathcal{D}_{CL})$ is equal to $\sqrt{\gamma_{opt}}$ where γ_{opt} is the optimal value of γ in the following standard convex optimization problem involving LMI constraints:

$$\begin{array}{ll} minimize & \gamma \\ over & \gamma,\ \tau \\ subject\ to & \tau \geq 0\ and \end{array}$$

$$\begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} W & -W\hat{\xi} \\ (-W\hat{\xi})^T & \hat{\xi}^T W\hat{\xi} - \chi^2_{cl} \end{pmatrix} \leq 0$$

with

- $W = P_\xi^{-1}$
- $a_{11} = (Z_5^* Z_5 - Z_5^* x Z_6 - Z_6^* x^* Z_5 + Z_6^* x^* x Z_6) - \gamma(Z_5^* Q Z_5 + Z_6^* Q Z_6)$
- $a_{12} = -Z_5^* x + Z_6^* x^* x - \gamma(Z_6^* Q)$
- $a_{22} = x^* x - \gamma Q$
- $Q = 1 + x^* x$, $x = \frac{1}{K} \frac{T(\hat{\xi},\Omega)}{1-T(\hat{\xi},\Omega)}$, $Z_5 = Z_3/K$ and $Z_6 = Z_4 - Z_3$

Proof. Using the expression (31) of $T(\xi)$ and the mapping (9) from $T(\xi)$ to $G(\xi)$, we get:

$$y = G(\xi, \Omega) = \frac{(Z_3/K)\ \xi}{1 + (Z_4 - Z_3)\xi} = \frac{Z_5\ \xi}{1 + Z_6 \xi} \ \textit{with } Z_5 = Z_3/K \textit{ and } Z_6 = Z_4 - Z_3 \tag{32}$$

The expression for y has the same structure as in Section 3.3.1. Denoting $x = G_{mod}(e^{j\Omega}) = G(\hat{\xi}, \Omega)$, the LMI problem can then be stated as:

$$\textit{minimize } \gamma \textit{ such that } \kappa(x, y)^2 \leq \gamma \textit{ for } y \in \mathcal{D}_{CL}$$

The proof now follows the same procedure as that of Theorem 2 by replacing θ by ξ, Z_1 by Z_6 and Z_2 by Z_5. □

3.3.3 Summarizing theorem

The methods for computing the worst case chordal distance having been presented for the two types of identification, we now summarize the results of this subsection in the following theorem and its corollary.

Theorem 4. The size of the smallest coprime factor uncertainty set embedding the uncertainty region $\mathcal{D}$ is given by the worst case Vinnicombe distance $\delta_{WC}(G_{mod}, \mathcal{D})$. This quantity can be computed as follows:

$$\delta_{WC}(G_{mod}, \mathcal{D}) = \max_{\Omega} \sqrt{\gamma_{opt}(\Omega)}$$

where $\gamma_{opt}(\Omega)$ is the solution at frequency Ω of the LMI based optimization problem of Theorem 2 if the model has been identified in open-loop, or of Theorem 3 if this model has been identified in closed-loop.

Proof. This theorem is a direct consequence of Lemma 1 and of Theorems 1, 2 and 3. □

Corollary 2. If $\delta_{WC}(G_{mod}, \mathcal{D}) < 1$, then the frequency function $f(\Omega)$ defining the size of the smallest pointwise coprime factor uncertainty set $\mathcal{B}(G_{mod}, f)$ that embeds the uncertainty region $\mathcal{D}$ is given by the worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D})$. This quantity can be computed at each frequency as follows:

$$\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}) = \sqrt{\gamma_{opt}(\Omega)}$$

where $\gamma_{opt}(\Omega)$ is the solution at frequency Ω of the LMI based optimization problem of Theorem 2 if the model has been identified in open-loop, or of Theorem 3 if this model has been identified in closed-loop.

Proof. This corollary is a direct consequence of Corollary 1 and of Theorems 2 and 3. □

4 Stability

In the previous section, we have shown how to embed an uncertainty region $\mathcal{D}$ arising from an ellipsoidal parameter confidence region into the smallest possible (pointwise) coprime factor uncertainty set. The Vinnicombe stability results [16] can now be applied to this (pointwise) coprime factor uncertainty set.

Before presenting the Vinnicombe stability results, let us first recall the considered problem. From the nominal model $G_{mod}(z)$, a controller $C(z)$ is designed. This controller stabilizes G_{mod} and achieves satisfactory performance with this model. However, this controller is not guaranteed to stabilize the true system G_0. We are therefore looking for robust stabilization conditions.

4.1 Min-Max type condition for robust stability

First we recall the definition of generalized stability margin for a closed-loop system made up of the negative feedback connection of a plant G and a controller C [17].

Definition of the stability margin.

$$b_{GC} = \begin{cases} \min_{\Omega} \kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) & if\ [C\ G]\ is\ stable \\ 0 & otherwise \end{cases} \tag{33}$$

where $\kappa(G_1, G_2)$ was defined in (12). Note that $0 \leq b_{GC} \leq 1$.

A main result of [16] is the following sufficient but not necessary condition to guarantee the stabilization of G_0 by C.

Proposition 2 [16]. Consider a model G_{mod} and a controller C that stabilizes G_{mod} with a stability margin $b_{G_{mod}C}$. Then C stabilizes G_0 if

$$G_0 \in \{G_{in} \ \mid \ \delta_\nu(G_{mod}, G_{in}) < b_{G_{mod}C}\}. \tag{34}$$

□

In an identification context, the true system G_0 is unknown, but we have shown that it lies, with probability 0.95, say, in the coprime factor uncertainty set $\mathcal{G}_{emb}(G_{mod}, \mathcal{D})$ defined by (18), where $\mathcal{D}$ is the uncertainty region defined by the parametric confidence region. Therefore, the following result can be stated.

Theorem 5. Let G_{mod} be an identified model and let $\mathcal{D}$ be a set of parametrized transfer functions containing G_{mod} and the true plant G_0. Then, a sufficient condition for the stabilization of the true system G_0 by a controller C, which stabilizes G_{mod} with a stability margin $b_{G_{mod}C}$, is to verify that:

$$\delta_{WC}(G_{mod}, \mathcal{D}) < b_{G_{mod}C}. \tag{35}$$

Proof. According to expression (21), G_0 lies in $\mathcal{G}_{emb}(G_{mod}, \mathcal{D})$ defined in (18). Therefore, by (35), $\delta_\nu(G_{mod}, G_0) \leq \delta_{WC}(G_{mod}, \mathcal{D}) < b_{G_{mod}C}$ and the stability then follows from (34). $\square$

The condition (34) of Proposition 2 is rather conservative, since $\delta_\nu(G_{mod}, G_{in}) = \max_\Omega \kappa(G_{mod}(e^{j\Omega}), G_{in}(e^{j\Omega}))$ while $b_{G_{mod}C} = \min_\Omega \kappa(G_{mod}(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})})$. Thus, it is a min-max type condition.

4.2 A less conservative condition for robust stability

In [16], a pointwise (i.e. frequency by frequency) and therefore less conservative version of the classical condition (34) is presented.

Proposition 3 [16]. Consider a model G_{mod} and a controller C that stabilizes G_{mod}. The stabilization of the true system G_0 by the controller C is guaranteed if

$$G_0 \in \{G_{in} \ \mid \ \kappa(G_{mod}(e^{j\Omega}), G_{in}(e^{j\Omega})) < \kappa(G_{mod}(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) \ \forall\ \Omega \ \mathit{and}\ \delta_\nu(G_{mod}, G_{in}) < 1\} \tag{36}$$

$\square$

Using this pointwise version of the robust stability result of Vinnicombe, and replacing the constraint (35) on the worst case Vinnicombe distance by a pointwise constraint on the worst case chordal distance, we can now state our main stability result for identified transfer functions.

Theorem 6 (main stability theorem). Let G_{mod} be an identified model and let $\mathcal{D}$ be a set of parametrized transfer functions containing G_{mod} and the true plant G_0. If $\delta_{WC}(G_{mod}, \mathcal{D}) < 1$, then the stabilization of the true system G_0 by a controller C (that stabilizes G_{mod}) is guaranteed if

$$\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}) < \kappa(G_{mod}(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) \quad \forall\ \Omega \in [0, \pi] \tag{37}$$

Proof. According to expression (22), G_0 lies in $\mathcal{B}_{emb}(G_{mod}, \mathcal{D})$ defined in (20). Therefore, by (37), $\kappa(G_{mod}(e^{j\Omega}), G_0(e^{j\Omega})) \leq \kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}) < \kappa(G_{mod}(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})})\ \forall \Omega$ and the stability then follows from (36). $\square$

Remark about the conservatism. The condition (37), even though much less conservative than (35), is still a sufficient but not necessary condition for the stabilization by C of all plants in $\mathcal{D}$. The remaining conservatism has two different reasons:

- the embedding of $\mathcal{D}$ into a larger set $\mathcal{B}_{emb}(G_{mod}, \mathcal{D})$.

- the fact that, outside the set defined in (36), there may still exist systems stabilized by C.

5 Example

Let us now illustrate the presented results. For this purpose, an example of controller validation for a model identified in closed-loop is considered. Following the procedure presented in the previous sections, an unbiased model $T(\hat{\xi})$ of the true closed-loop transfer function $T(\xi_0)$ is identified. Then, the worst case chordal distances at each frequency between the plant $G(\hat{\xi})$ corresponding to $T(\hat{\xi})$ and all the plants in the uncertainty region $\mathcal{D}_{CL}$ are computed by solving the convex optimization problem of Section 3.3.2. With these chordal distances, the main stability theorem of Section 4.2 is then used in order to validate a controller designed from $G(\hat{\xi})$.

Identification step. Let us consider the following true system G_0 with an Output Error structure:

$$y = \overbrace{\frac{z^{-1} + 0.5z^{-2}}{1 - 1.5z^{-1} + 0.7z^{-2}}}^{G_0} u + e$$

where e is a unit-variance white noise. The sampling time is 1 second.

We perform a closed-loop identification of an unbiased $T(\hat{\xi})$ with a controller $K = 0.5$ in the loop. This controller stabilizes G_0. We choose an ARMAX model structure since it is the structure of the actual closed loop $[K\ G_0]$. The number of collected data is equal to 1000. This identification yields:

$$y(\hat{\xi}) = \overbrace{\frac{0.5238z^{-1} + 0.2256z^{-2}}{1 - 1.0041z^{-1} + 0.951z^{-2}}}^{T(\hat{\xi})} r + \frac{1 - 1.4556z^{-1} + 0.6743z^{-2}}{1 - 1.0041z^{-1} + 0.951z^{-2}} e$$

As the noise model of an ARMAX model structure is not independently parametrized, the size χ^2_{cl} of the uncertainty region U_{CL} containing the parameters of the true closed-loop transfer function with probability 95 % is equal to 12.6. Indeed the size r of the total parameter vector of our ARMAX model structure is equal to 6 and $\alpha(6, 12.6) = Pr(\chi^2(6) \leq 12.6) = 0.95$.

The model $G(\hat{\xi})$ corresponding to $T(\hat{\xi})$ is equal to

$$G_{mod} = G(\hat{\xi}) = \frac{1}{K}\frac{T(\hat{\xi})}{1 - T(\hat{\xi})} = \frac{1.0476z^{-1} + 0.4511z^{-2}}{1 - 1.5279z^{-1} + 0.7254z^{-2}}$$

The set $\mathcal{D}_{CL}$ corresponding to the uncertainty region U_{CL} is now embedded into a (pointwise) coprime factor uncertainty set in order to use the related stability theorems.

Computation of the worst case chordal distances. Using P_ξ, the estimated covariance matrix of $\hat{\xi}$, the worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}_{CL})$ at each frequency can be computed with the optimization procedure of Section 3.3.2. It yields the solid line presented in Figure 1, where the worst case chordal distance is compared with the actual chordal distance $\kappa(G_{mod}(e^{j\Omega}), G_0(e^{j\Omega}))$ between G_{mod} and G_0.

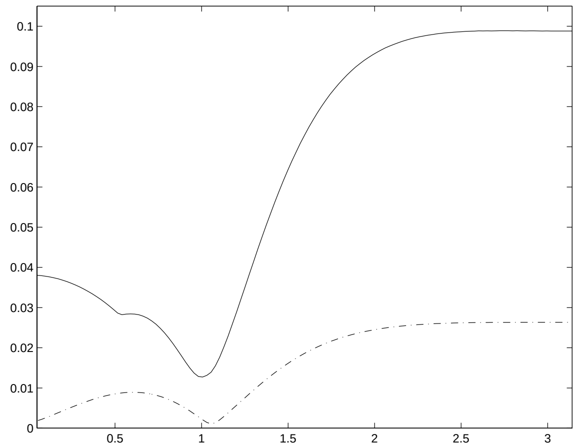

Figure 1: Worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}_{CL})$ (solid) and actual chordal distance $\kappa(G_{mod}(e^{j\Omega}), G_0(e^{j\Omega}))$ between G_{mod} and G_0 (dashdot) at each frequency

Stability test. Let us now consider two different controllers C_1 and C_2 which have been designed from the stable model G_{mod} and which stabilize it. The first controller C_1 results from a model reference control design with the specification that the static gain must be equal to 1. The desired closed-loop transfer function is chosen equal to $T_m(G_{mod}) = (1.1z^{-1})/(1 + 0.1z^{-1})$. The second controller C_2 is a simple proportional controller. These two controllers are:

$$C_1 = \frac{1.1z^{-1}(1 - 1.5279z^{-1} + 0.7254z^{-2})}{(1 - z^{-1})(1.0476z^{-1} + 0.4511z^{-2})} \; and \; C_2 = 0.605$$

We want to verify that these two controllers also stabilize the true system G_0. Therefore, we use Theorem 6 of Section 4.2 since it is the least conservative one. For these two controllers, we compute the pointwise stability margin $\rho(C_i, G_{mod}, \Omega)$ of the designed closed-loop composed of G_{mod} and C_i:

$$\rho(C_i, G_{mod}, \Omega) = \kappa(G_{mod}(e^{j\Omega}), -\frac{1}{C_i(e^{j\Omega})})$$

In order to validate these two controllers, these pointwise stability margins have to be compared at each frequency with the worst case chordal distance. As can be seen in Figure 2, the controller C_1 is guaranteed to stabilize the true system since we have:

$$\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}_{CL}) < \rho(C_1, G_{mod}, \Omega) \; \forall \Omega$$

The same can not be said for C_2 as can be seen in Figure 3. Therefore, this controller is not guaranteed to stabilize the true system. In fact, C_2 does indeed destabilize the true system as can be seen in the corresponding closed-loop transfer function which has two unstable poles of modulus equal to 1.0012.

$$\frac{G_0 C_2}{1 + G_0 C_2} = \frac{0.605(z^{-1} + 0.5z^{-2})}{1 - 0.895z^{-1} + 1.0025z{-2}}$$

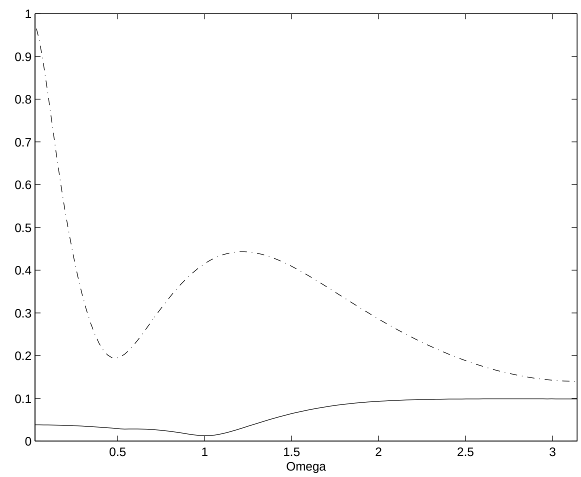

Figure 2: Comparison of worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}_{CL})$ (solid) and pointwise stability margin $\rho(C_1, G_{mod}, \Omega)$ (dashdot) at each frequency

6 Conclusions

In this paper, we have considered the robust stability problem of guaranteeing the stabilization of a true system by a controller designed from a model of this true system that has been identified either in open-loop or in closed-loop with prediction error methods. The proposed approach is to verify that the designed controller stabilizes all the plants in a set of parametrized transfer functions in which we can guarantee the presence of both the model and the true system with a certain probability level. This set of parametrized transfer functions is deduced either from the covariance matrix of the parameters of the plant model or from the covariance matrix of the parameters of the model of the closed-loop transfer function.

In order to use the Vinnicombe stability results, we have embedded this set of transfer functions into the smallest possible (pointwise) coprime factor uncertainty region. To perform this embedding, we propose attractive tools taking the form of convex optimization problems involving LMI constraints. The pointwise version of the Vinnicombe stability theorem adapted to our particular problem has been presented and illustrated by an example.

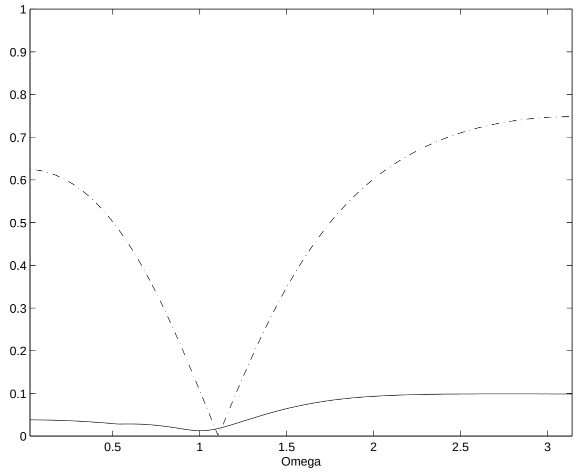

Figure 3: Comparison of worst case chordal distance $\kappa_{WC}(G_{mod}(e^{j\Omega}), \mathcal{D}_{CL})$ (solid) and pointwise stability margin $\rho(C_2, G_{mod}, \Omega)$ (dashdot) at each frequency

References

[1] X. Bombois, M. Gevers, and G. Scorletti. Controller validation for a validated model set. submitted to E.C.C. 1999.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*, volume 15 of *Studies in Appl. Math.* SIAM, Philadelphia, June 1994.

[3] J.C. Doyle. Analysis of feedback systems with structured uncertainties. *IEE Proc.*, 129-D(6):242–250, November 1982.

[4] L. El Ghaoui, R. Nikoukhah, and F. Delebecque. `LMITOOL`*: A front-end for LMI optimization, users's guide*, February 1995. Avalaible via anonymous ftp to `ftp.ensta.fr`, under `/pub/elghaoui/lmitool`.

[5] P. Gahinet, A. Nemirovsky, A. L. Laub, and M. Chilali. *LMI Control Toolbox.* The Mathworks Inc., 1995.

[6] M. Gevers. Towards a joint design of identification and control? *Essays on Control : Perspectives in the Theory and its Applications, Birkhauser, New York*, pages 111–151, 1993.

[7] M. Gevers, B. D.O. Anderson, and B. Codrons. Issues in modeling for control. In *Proc. American Control Conference*, pages 1615–1619, Philadelphia, USA, 1998.

[8] M. Gevers, B. Codrons, and F. De Bruyne. Model validation in closed-loop. Accepted for presentation at ACC 99.

[9] R.G. Hakvoort. *System Identification for Robust Process Control - PhD Thesis.* Delft University of Technology, Delft, The Netherlands, 1994.

[10] V.A. Jakubovič. The $\mathcal{S}$-procedure in nonlinear control theory. *Vestnik Leningrad Univ. (russian) Vestnik Leningrad Univ. Math. (amer.)*, 4 (amer.)(1 (russian)), 1971 (russian) 1977 (amer.).

[11] L. Ljung. *System Identification: Theory for the User.* Prentice-Hall, Englewood Cliffs, NJ, 1987.

[12] L. Ljung. Identification for control - what is there to learn. *Workshop on Learning, Control and Hybrid Systems, Bangalore*, 1998.

[13] D. M. McFarlane and K. Glover. A loop shaping design procedure using H_∞. *IEEE Trans. Aut. Control*, 37(6), June 1992.

[14] M. Milanese. Learning models from data: the set membership approach. In *Proc. American Control Conference*, pages 178–182, Philadelphia, USA, 1998.

[15] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.

[16] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *IEEE Trans Automatic Control*, AC-38:1371–1383, 1993.

[17] G. Vinnicombe. *Uncertainty and Feedback (H_∞ loop-shaping and the ν-gap metric).* book to be published, 1999.

[18] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Aut. Control*, AC-26(2):301–320, April 1981.

[19] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, New Jersey, 1995.