[CL5]

10 Riccati Difference and Differential Equations: Convergence, Monotonicity and Stability

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10.1 Introduction

The main theme of this Chapter will be the connections between various Riccati equations and the closed loop stability of control schemes based on Linear Quadratic (LQ) optimal methods for control and estimation. Our presentation will encompass methods applicable both for discrete time and continuous time, and so we discuss concurrently the difference equations (discrete time) and the differential equations (continuous time) – the intellectual machinery necessary for the one suffices for the other and so it makes sense to dispense with both cases in one fell swoop.

Our strategy for the exploration of this subject is as follows:

- The connection between particular Riccati equations and their associated LQ optimal control and estimation problems is established. This is done for
  - Finite horizon LQ optimal control and least squares state estimation,
  - Infinite horizon LQ control and estimation,
  - Receding horizon LQ control and estimation.
- Asymptotic stability problems are posed for infinite and receding horizon variants of these closed loop systems.
- General Lyapunov stability methods for linear difference and differential equations are then presented in a form amenable to LQ applications using the Riccati equation.
- Convergence and monotonicity properties of Riccati equation solutions are then treated.
- Stability inherent to the infinite horizon problems is investigated using the Lyapunov methods.
- We then expand proceedings using monotonicity methods to establish stability for receding horizon LQ solutions.
- Two brief examples of the application of these methods are presented from adaptive control, where receding horizon LQ controllers have gained considerable currency because of their computational simplicity, and harmonic analysis, where Kalman filtering methods may be applied to reject noise in signals.

Bittanti, Laub, Willems (Eds.)
The Riccati Equation

The examples illustrate both the extent of applicability of these optimal methods and the facility of the tools connected with so-called Fake Algebraic Riccati Techniques, which are our central theme here.

10.2 Linear Quadratic Optimal Control Problems and Riccati Equations

10.2.1 Discrete Time

Our focus here will be on the feedback control of the linear system

$$x_{t+1} = Fx_t + Gu_t, \tag{10.1}$$

where x_t is the n-vector state process and u_t the m-vector control process. The matrices F and G are constant (and therefore the system (10.1) is time-invariant) and have appropriate dimensions $n \times n$ and $n \times m$.

We pose three LQ optimization problems distinguished by their criterion horizons: finite horizon LQ optimal control, infinite horizon LQ optimal control and receding horizon LQ optimal control. Each of these problems involves the specification of a quadratic performance objective for the linear system (10.1) – hence 'Linear Quadratic'. This is a mature subfield of Control Systems and the reader is referred to the excellent texts [1-7] for further material and derivation of the earlier results.

In many circumstances it is desirable to consider time-varying systems and time-varying LQ optimization criteria. This yields (expectedly) time-varying control laws. Our study here is, in general, restricted to the behaviour of LQ time-invariant control laws and so we consider initially an assortment of optimal control problems with constant, nonnegative definite weighting (penalty) matrices Q and R. This time-invariance of weighting matrices will then be relaxed at certain instances later connected with receding horizon problems. Throughout we will assume that the control weighting matrix, R or R_t, is positive definite and so is invertible.

Finite Horizon Linear Quadratic Optimal Control. Find the input sequence $\{u_t | t = 0, \ldots, N-1\}$ which minimizes the criterion

$$J(N, x_0, u) = x_N^T P_0 x_N + \sum_{j=0}^{N-1} \left\{ x_j^T Q_{N-j-1} x_j + u_j^T R_{N-j-1} u_j \right\}. \tag{10.2}$$

Here x_0 is the initial state, Q_j is a sequence of nonnegative definite matrices penalizing the excursions of the state from zero, R_j is a sequence of positive definite matrices penalizing the control energy, and P_0 is the nonnegative definite penalty matrix on the terminal state x_N.

The solution to this problem is given by a feedback control law as follows:

$$\begin{aligned} u_{N-j} &= -(G^T P_{j-1} G + R_{j-1})^{-1} G^T P_{j-1} F x_{N-j}, \\ &\triangleq K_{j-1} x_{N-j}, \quad j = 1, \ldots, N, \end{aligned} \tag{10.3}$$

where the control gain, K_j, is given by

$$K_j = -(G^T P_j G + R_j)^{-1} G^T P_j F \tag{10.4}$$

and P_j is the solution sequence of the following *Riccati Difference Equation* (RDE):

$$P_{j+1} = F^T P_j F - F^T P_j G (G^T P_j G + R_j)^{-1} G^T P_j F + Q_j, \tag{10.5}$$

solved forwards in j from the initial condition P_0.

We make several remarks concerning this problem and its solution.

- The control signal sequence, $\{u_j\}$, is a causal feedback of the system state sequence, $\{x_j\}$.
- The feedback gain sequence, $\{K_j\}$, is computed from the Riccati difference equation (10.5) solution sequence, $\{P_j\}$, which in turn is computed from P_0, $\{Q_j\}$, $\{R_j\}$, F and G via the iteration of the RDE.
- Given the above matrices, one iterates the RDE for the sequence $\{P_j\}$ from the initial condition P_0 (which is the final state penalty) effectively backwards in time relative to the evolution of the plant state. Hence the peculiar indexing in (10.3). These P_j and their corresponding gains K_j may be precomputed since they depend only on the model information and on the given penalty weightings. This contrasts to the state, x_t, which needs to be measured on-line. Alternatively, the state too can be precomputed but this then yields an open loop control law.
- The control sequence $\{u_t\}$ associated with the finite horizon LQ problem is defined only over a fixed time interval. It is easily demonstrated that the RDE has no finite escape properties and so the P_j are always bounded. Thus there is no sensible notion of stability which can be attributed to this LQ formulation.
- The minimal value of the criterion $J(N, x_0, u)$ is given by

$$J^*(N, x_0, u) = x_0^T P_N x_0.$$

Infinite Horizon Linear Quadratic Optimal Control. We next consider a variant of the finite horizon problem as we allow the horizon, N, to increase without bound. We now consider the case where the weighting matrices Q_j and R_j are constants, Q and R respectively, and examine the convergence of the finite horizon solution to a time-invariant control law.

Specifically, we consider the problem: Find the input sequence $\{u_t | t = 0, 1, \ldots\}$ which minimizes the criterion

$$\begin{aligned} J_\infty(x_0, u) &= \lim_{N \to \infty} J(N, x_0, u) \\ &= \lim_{N \to \infty} \sum_{j=0}^{N-1} \left\{ x_j^T Q x_j + u_j^T R u_j \right\}. \end{aligned} \tag{10.6}$$

Let us study this problem briefly.

- If $[F, G]$ is a stabilizable pair, i.e., all modes of F which are uncontrollable are associated with eigenvalues of magnitude strictly less than one, then a feedback control law, $u_t^s = K^s x_t$, exists which causes $x_t \to 0$ as $t \to \infty$.
- Evaluating the criterion $J(N, x_0, u_t^s)$ for $s = 0, \ldots, N$, we see from the optimality property that this value of $J(N, x_0, x_t^s)$ exceeds the optimal value, $J^*(N, x_0, u)$, whatever the particular choice of N. Further, since u_t^s causes the closed-loop plant to be exponentially stable, $J(N, x_0, u_t^s)$ converges to $J(\infty, x_0, u_t^s)$ as $N \to \infty$ and this latter quantity is finite. Thus the sequence of optimal costs, $\{J^*(N, x_0, u) | N = 1, 2, \ldots\}$, belongs to a compact set and so possesses a convergent subsequence with a finite limit point, $J_\infty^*(x_0)$. The problem stationarity plus optimality combine to ensure the uniqueness of this limit point.
- Since $J_\infty^*(x_0)$ exists and is finite and $R > 0$, the optimal control law tends to zero as $t \to \infty$. If $[F, Q^{1/2}]$ is detectable then this implies that $x_t \to 0$, i.e., asymptotic stability is achieved. This aspect of infinite horizon LQ control will be more formally treated shortly.

Just as this optimal control problem is posed as the limit of a finite horizon LQ optimal control problem, so too is the solution the limit of the finite horizon solution:

$$\begin{aligned} u_j &= -(G^T P_\infty G - R)^{-1} G^T P_\infty F x_j, \\ &\triangleq K_\infty x_j, \quad j = 1, 2, \ldots, \end{aligned} \tag{10.7}$$

where the constant control gain, K_∞, is given by

$$K_\infty = -(G^T P_\infty G + R)^{-1} G^T P_\infty F \tag{10.8}$$

and P_∞ is the maximal nonnegative definite solution of the following *Algebraic Riccati Equation* (ARE):

$$P_\infty = F^T P_\infty F - F^T P_\infty G (G^T P_\infty G + R)^{-1} G^T P_\infty F + Q. \tag{10.9}$$

Comparing this with the earlier remarks on the RDE,

- The control law (10.7) is still a causal linear state variable feedback.
- The optimal cost is $J^*(x_0, u) = x_0^T P_\infty x_0$.
- The control gain is now constant over all time. Therefore the feedback law plus the state process define a time-invariant closed loop for which asymptotic stability questions may readily be posed.
- The ARE (10.9) usually possesses many solutions, as may be seen by examining the scalar case. Our requirement here is that the maximal nonnegative definite solution be chosen.
- Under reasonable conditions on the LQ problem to be discussed later, one has that $P_N \to P_\infty$ as $N \to \infty$. That is, the maximal solution is also the limiting solution.

- The feedback gain, K_∞, is computed from the ARE solution, P_∞, but now this ARE solution is fixed in time. The ARE is an algebraic matrix equation, which is typically solved by symplectic eigenvector methods. Considerable effort was devoted to alternative procedures [8], [9] by which approximate ARE solutions or large N RDE solutions were computed. This computational difficulty of solving the ARE used to be rather burdensome in applications and so led to the proposal of a mixed-mode LQ optimal control problem between finite and infinite horizon. This we now discuss.

Receding Horizon Linear Quadratic Optimal Control. Historically speaking, the computational solution of the ARE posed a serious problem for the application of infinite horizon LQ control. Nowadays with better algorithms and hardware, this is less the case or, to draw on Yasser Arafat[1],

C'est caduque!

except in the important and recently resurgent field of LQ adaptive control where an instance of infinite horizon LQ control masquerading as finite horizon LQ control appears – *Receding Horizon LQ Control.* The numerical problem for infinite horizon LQ control hinged on the ability to solve the ARE (a symplectic eigenvector problem) reliably. In the adaptive control context this solution needs to be constructed at each adaptation step.

Receding horizon LQ control appears to be attributable to Thomas [10] and involves the following:

- At time t the plant is in state x_t and an N-step finite horizon LQ optimal control problem is posed. Find $\{u_{t+s} | s = 0, \ldots, N-1\}$ which minimizes

$$J(N, x_t, u) = x_{t+N}^T P_0 x_{t+N} + \sum_{j=0}^{N-1} \left\{ x_{t+j}^T Q_{N-j-1} x_{t+j} + u_{t+j}^T R_{N-j-1} u_{t+j} \right\}. \tag{10.10}$$

- The feedback control signal u_t only is applied.
- The N-step finite horizon problem is re-solved for time $t+1$ from state x_{t+1}.

The appellation 'Receding Horizon' refers to the fact that at each time a finite horizon problem is solved N steps into the future but that this horizon remains N steps distant. In the adaptive control context this translates into designing a controller with a fixed look-ahead or 'prediction horizon'.

Being but an N-step finite horizon with a sliding initial condition, the control solution here is given from (10.3) as

$$\begin{aligned} u_t &= -(G^T P_{N-1} G + R_{N-1})^{-1} G^T P_{N-1} F x_t \\ &\triangleq K_{N-1} x_t \end{aligned} \tag{10.11}$$

where P_{N-1} is the N'th element in the solution sequence of the RDE (10.5), commencing with initial condition P_0.

1 Paris, May 1989.

Predictably, some remarks are in order.

- The control law (10.11) is a causal linear state variable feedback. That is, the optimal control may be written as a causal feedback of the state process. It may also be written as an explicit (open-loop) function of time.
- The control gain, K_{N-1}, is constant. This is so in spite of the time variation of the gain in the finite horizon solution, and is due to the recession of the horizon.
- Computationally the solution of the receding horizon problem entails the iteration of the RDE for N steps.
- A major feature of this receding horizon solution is that, with relatively little increase in complexity, control constraints may be directly handled.
- Asymptotic stability questions arise here because, even though a finite horizon problem is involved, the control law (10.11) is applied on the infinite horizon.
- The LQ performance of the closed loop system is not easily related to the RDE solution applied, even with constant Q and R, except if N is taken very large.
- When the control horizon N is allowed to become large and Q and R are fixed, we know (and shall see) that $P_{N-1} \to P_\infty$ and the resolution of the closed loop stability and LQ performance is easily inferred from the infinite horizon LQ problem.

It is one of our aims here to focus on the stability and performance aspects of receding horizon LQ schemes.

10.2.2. Continuous Time

We consider the linear, time-invariant, continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{10.12}$$

where $x(t)$ is the n-vector state process and $u(t)$ an admissible m-vector control process. The matrices A and B have appropriate dimensions. Again we pose three LQ optimal control problems and provide solutions, although we do not dwell too long where matters correspond fully with discrete time.

Finite Horizon Linear Quadratic Optimal Control. Find the input function $\{u(t)|t \in (0,T)\}$ which minimizes the criterion

$$J(T, x(0), u) = x(T)^T P(0) x(T) \tag{10.13}$$

$$+ \int_0^T \{x(t)^T Q(T-t) x(t) + u(t)^T R(T-t) u(t)\}\, dt \tag{10.14}$$

This represents an integral version of the sum appearing in (10.2). The solution is given by

$$\begin{aligned} u(t) &= -R(T-t)^{-1} B^T P(T-t) x(t) \\ &\triangleq K(T-t) x(t), \end{aligned} \tag{10.15}$$

where the control gain, $K(\tau)$, is given by

$$K(\tau) = -R(\tau)^{-1} B^T P(\tau), \tag{10.16}$$

and $P(\tau)$ is the solution of the *Riccati Differential Equation* (also denoted RDE)

$$\dot{P}(t) = A^T P(t) + P(t) A - P(t) B R(t)^{-1} B^T P(t) + Q(t) \tag{10.17}$$

with initial condition $P(0) \geq 0$.

The similarity to the discrete-time case is mostly self-evident and so many of the earlier comments carry over *mutatis mutandis* to continuous time. Thus

- The optimal control may be written as the causal feedback of the state process via control gain $K(T-\tau)$, computed from $P(T-\tau)$, the solution of the RDE (10.17) with initial condition $P(0)$.
- An asymptotic stability question is not sensibly posed with respect to this finite horizon LQ problem.
- The optimal value of the performance criterion, $J(T, x(0), u)$, is given by

$$J^*(T, x(0), u) = x(0)^T P(T) x(0).$$

Infinite Horizon Linear Quadratic Optimal Control. Parallel to the discrete case, we introduce an infinite horizon, stationary LQ problem: Find the control function $\{u(t)|t \in (0,\infty)\}$ which minimizes the criterion

$$J_\infty(x(0), u) = \int_0^\infty \{x(t)^T Q x(t) + u(t)^T R u(t)\}\, dt. \tag{10.18}$$

(One may alternatively consider the case where $T \to \infty$ in (10.13) as was done in (10.6).) The solution is given by the feedback law

$$\begin{aligned} u(t) &= -R^{-1} B^T P x(t) \\ &\triangleq K x(t), \end{aligned} \tag{10.19}$$

where the constant control gain, K, is given by

$$K = -R^{-1} B^T P, \tag{10.20}$$

and P is the constant, maximally nonnegative definite solution of the (continuous) *Algebraic Riccati Equation* (also denoted ARE)

$$0 = A^T P + PA - PBR^{-1} B^T P + Q. \tag{10.21}$$

Again the earlier remarks still apply especially those directed towards the asymptotic stability problem. We shall address these shortly but firstly move on to consider the receding horizon problem.

Receding Horizon Linear Quadratic Optimal Control. Because of the increased reliance on digital computer solutions to control problems and the computational difficulties associated with the solution of the (continuous) ARE (a Hamiltonian eigenvector problem is involved), these continuous receding horizon problems are perhaps even more *caduque* than their discrete counterparts. Nevertheless and given some recent interest [11] in specifying a continuous version of the Adaptive Generalized Predictive Control we complete our catalogue by providing an appropriate specification.

- At time t the plant is in state $x(t)$ and a T-ahead finite horizon LQ optimal control problem is posed. Find $\{u(t+s)|s \in (0,T)\}$ which minimizes $J(T, x(t), u)$ defined by (10.13).
- Apply $u(t)$ and then re-solve for the next instant t^{+}.

Clearly, in parallel to earlier development, the solution to this problem is given equivalently by

$$u(t) = -R(T)^{-1}B^T P(T)x(t), \tag{10.22}$$

with $P(T)$ the solution of the continuous RDE (10.17) at time T with initial condition $P(0) \geq 0$.

The similarity with the discrete case is apparent, especially inasmuch as the asymptotic stability question is raised once again since (10.22) represents a time-invariant control law to be applied on the infinite timescale. We shall address means of resolving this very issue shortly but make the observation at this juncture that the discrete finite horizon problem, and hence the receding horizon problem, differ from the continuous versions in one key aspect. That is, the discrete finite horizon LQ problem is a finite dimensional optimization over the space of all possible N-step control m-vectors, i.e., $R^{N\times m}$, while the continuous version is an infinite dimensional problem. The conclusion that we draw from this is that, in continuous time, receding horizon LQ strategies are likely to be less appealing than in discrete time because of the inherent difficulty of incorporating constraints into the former the ability easily to impose conditions on, for example, control signal magnitudes or slewing rates is a major support for the use of receding horizon LQ controllers in discrete-time adaptive control.

10.3 Linear Optimal State Estimation and Riccati Equations

The material presented so far has concentrated on connections between LQ optimal control problems in either continuous time or discrete time and their respective Riccati equations (10.5), (10.9), (10.17), (10.21). As is widely known, these state control problems are dual to certain state estimation problems. Now we shall briefly treat these state estimation, or Kalman Filtering, ideas.

The correspondence between LQ control and least squares linear state estimation is strict duality but, for our needs here of addressing the asymptotic stability of LQ-based but suboptimal feedback control strategies (such as the receding horizon LQ control), we shall find that certain issues arise differently in each circumstance. These distinctions will be noted here.

10.3.1 Discrete Time

We consider the following system

$$x_{t+1} = Fx_t + Gw_t \tag{10.23}$$
$$y_t = Hx_t + v_t \tag{10.24}$$

where x_t is the state (no longer assumed measurable), y_t the measured output process, and w_t and v_t are zero mean, white, mutually independent gaussian noise processes with covariance

$$\mathrm{E}\begin{pmatrix} w_t \\ v_t \end{pmatrix}\begin{pmatrix} w_s^T & v_s^T \end{pmatrix} = \begin{pmatrix} Q_t & 0 \\ 0 & R_t \end{pmatrix}\delta_{ts}.$$

Our goal is to predict optimally the state, x_t, from output measurements $\{y_s|s = 1,\ldots,t-1\}$ from the initial data that our knowledge of x_0 is distributed as a gaussian random variable with mean x_0 and covariance Σ_0. The least squares solution, computed recursively, is the Kalman Predictor:

$$\hat{x}_{t+1|t} = F\hat{x}_{t|t-1} + K_t(y_t - H\hat{x}_{t|t-1}) \tag{10.25}$$

where the Kalman gain, K_t, is given by

$$K_t = F\Sigma_t H^T(H\Sigma_t H^T + R_t)^{-1}, \tag{10.26}$$

and Σ_t is the solution of the discrete filtering RDE

$$\Sigma_{t+1} = F\Sigma_t F^T - F\Sigma_t H^T(H\Sigma_t H^T + R_t)^{-1}H\Sigma_t F^T + Q_t, \tag{10.27}$$

solved forwards in time from initial conditions x_0, Σ_0. The state estimate is gaussian and has the following moments,

$$E(\hat{x}_{t|t-1}) = x_t,$$
$$E\left((x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})^T\right) = \Sigma_t.$$

The formulation of infinite horizon Kalman filtering problems follows directly from here by allowing $t \to \infty$ above. For stationary plants and noise covariance matrices Q and R, this infinite time or stationary solution is given by the following

$$\hat{x}_{t+1|t} = F\hat{x}_{t|t-1} + K(y_t - H\hat{x}_{t|t-1}) \tag{10.28}$$

where the fixed Kalman gain, K, is given by

$$K = F\Sigma_\infty H^T(H\Sigma_\infty H^T + R)^{-1}, \tag{10.29}$$

and Σ_∞ is the solution of the discrete filtering ARE

$$\Sigma_\infty = F\Sigma_\infty F^T - F\Sigma_\infty H^T(H\Sigma_\infty H^T + R)^{-1}H\Sigma_\infty F^T + Q. \tag{10.30}$$

Comparison of the control RDE (10.5) with the filtering RDE (10.27) establishes the duality. The major distinction between the control and filtering problems in practical terms is that the RDE of the latter is iterated forwards in time as opposed to the reverse iteration of (10.5). Issues of asymptotic stability of the closed loop control system translate into questions on the asymptotic stability of the undriven filters

$$\hat{x}_{t+1|t} = (F - K_t H)\hat{x}_{t|t-1}$$

or, in the stationary case,

$$\hat{x}_{t+1|t} = (F - KH)\hat{x}_{t|t-1}.$$

We shall center our attention on these stability questions shortly.

10.3.2 Continuous Time

Since the passage from LQ control to least squares state estimation involves the introduction of stochastic signals, in continuous time this is accompanied by the need also to convert to the Itô calculus. The state is presumed to evolve according to

$$dx(t) = Fx(t)dt + Gdw(t) \tag{10.31}$$
$$dy(t) = Hx(t)dt + dv(t), \tag{10.32}$$

where conditions above apply except that now $w(t)$ and $v(t)$ are mutually independent Brownian Motions with intensities $Q(t)$ and $R(t)$ respectively.

The least squares optimal state estimate now is given by the Kalman filter

$$d\hat{x}(t) = (F - K(t)H)\,\hat{x}(t)dt + K(t)dy(t) \tag{33}$$
$$K(t) = \Sigma(t)H^T R(t)^{-1} \tag{10.34}$$
$$\dot{\Sigma}(t) = F\Sigma(t) + \Sigma(t)F^T - \Sigma(t)H^T R(t)^{-1} H\Sigma(t) + Q(t), \tag{10.35}$$

with the obvious choice of initial conditions.

We shall not return explicitly to this Itô formulation of this problem since our task will be to analyse the stability properties of the unforced filter

$$d\hat{x}(t) = (F - K(t)H)\,\hat{x}(t)dt,$$

which is equally well treated by normal differential calculus.

10.4 Time Out

At this stage we have presented a collection of optimal control and optimal estimation problems in which Riccati difference equations, Riccati differential equations and Algebraic Riccati Equations arise. We shall be addressing the issues of asymptotic stability of the associated closed loop and unforced systems in various circumstances. These we summarize here:

1. Asymptotic stability of the stationary infinite horizon LQ closed loop,
$$x_{t+1} = \left(F - G(G^T P_\infty G + R)^{-1} G^T P_\infty F\right) x_t, \tag{10.36}$$
$$\dot{x}(t) = \left(F - GR^{-1}G^T P(\infty)\right) x(t) \tag{10.37}$$
2. Asymptotic stability of the stationary infinite horizon Kalman filter,
$$\hat{x}_{t+1} = \left(F - F\Sigma_\infty H^T (H\Sigma_\infty H^T + R)^{-1} H\right) \hat{x}_t, \tag{10.38}$$
$$\dot{\hat{x}}(t) = \left(F - H^T R^{-1} \Sigma(\infty) H\right) \hat{x}(t) \tag{10.39}$$
3. Asymptotic stability of the infinite horizon Kalman filter with arbitrary initial condition,
$$\hat{x}_{t+1} = \left(F - F\Sigma_t H^T (H\Sigma_t H^T + R)^{-1} H\right) \hat{x}_t, \tag{10.40}$$
$$\dot{\hat{x}}(t) = \left(F - H^T R^{-1} \Sigma(t) H\right) \hat{x}(t) \tag{10.41}$$
4. Asymptotic stability of the receding horizon LQ closed loop, with fixed N
$$x_{t+1} = \left(F - G(G^T P_N G + R)^{-1} G^T P_N F\right) x_t, \tag{10.42}$$
$$\dot{x}(t) = \left(F - GR^{-1}G^T P(N)\right) x(t) \tag{10.43}$$
5. Asymptotic stability of the Kalman filter frozen at a particular iteration,
$$\hat{x}_{t+1} = \left(F - F\Sigma_N H^T (H\Sigma_N H^T + R)^{-1} H\right) \hat{x}_t, \tag{10.44}$$
$$\dot{\hat{x}}(t) = \left(F - H^T R^{-1} \Sigma(N) H\right) \hat{x}(t) \tag{10.45}$$

We remark here that this set of stability questions displays some of the appealing variety of possible optimal estimation and control problems. The time-varying stability problem for Kalman filtering, problem 3, is noteworthy for the difficulty of posing a sensible dual in LQ control. This has been advanced in [12] (Section 5), however, by including a final state weighting in infinite horizon LQ. For the other pairs of problems, 1 & 2 and 4 & 5, duality means that the resolution of the one element deals immediately with the other. This holds even though the latter two problems are deliberately derived as suboptimal strategies. It is these last two problems which possess the more novelty because of their recent applications, upon which we shall comment later.

10.5 Asymptotic Stability Methods for Linear Equations

The range of asymptotic stability problems raised above refers to linear time invariant systems 1(2) and 4(5) and to the time varying system 3. We now present some tools suitable for the study of these issues. This presentation will follow the development in [13].

We begin by stating the discrete time Lyapunov stability theorem.

Theorem 10.1. *Consider the vector difference equation*

$$x_{t+1} = F_t x_t, \tag{10.46}$$

with transition function

$$\Phi(t+N,t) = F_{t+N-1}F_{t+N-2}\ldots F_{t+1}F_t.$$

Suppose there exists a positive definite matrix sequence, $\infty > \beta I \geq P_t \geq \alpha I > 0$ such that

$$F_{t+1}^T P_{t+1} F_{t+1} - P_t = -N_t N_t^T, \tag{10.47}$$

for some matrix sequence, N_t, and all t. Then (10.46) is stable in the sense of Lyapunov.

If further the pair $[F_t, N_t^T]$ is uniformly completely observable, i.e., there exists constants $T > 0$, $\gamma > 0$, $\delta > 0$ such that for all t,

$$\infty > \gamma I > \sum_{i=0}^{T-1} \Phi^T(t+i,t)N_{t+i}N_{t+i}^T\Phi(t+i,t) \geq \delta I > 0, \tag{10.48}$$

then (10.46) is exponentially asymptotically stable.

If F_t is a constant, F, then P_t may also be chosen to be constant, P, as may be N_t, N, and the uniform observability condition (10.48) may be replaced by detectability of the pair $[F, N^T]$.

Finally, the condition of uniform complete observability of $[F_t, N_t^T]$ may be replaced by the same condition on $[F_t - K_t N_t^T, N_t^T]$ for any bounded K_t and the same conclusion holds.

We do not provide a proof of this theorem here since it is available in [13], except for the final paragraph which is an easy extension. The key result to be drawn from this theorem is that a detectability condition arises in the assessment of stability and that this is needed to conclude the rate. The astute reader will have noticed the similarity to the earlier heuristic statements about infinite horizon LQ stability, where such a condition was foreshadowed.

In continuous time the result is the logical counterpart modulo the need to include regularization of the differential equation.

Theorem 10.2. *Consider the vector differential equation*

$$\dot{x}(t) = F(t)x(t), \tag{10.49}$$

with $F(\cdot)$ bounded and locally integrable and with transition function $\Phi(\tau,t)$. Suppose there exists a positive definite matrix function $\infty > \beta I \geq P(t) \geq \alpha I > 0$ such that

$$P(t)F(t) + F^T(t)P(t) = -N(t)N^T(t) \tag{10.50}$$

for some matrix function, $N(t)$, and all t. Then (10.49) is stable in the sense of Lyapunov.

If further the pair $[F(t), N^T(t)]$ is uniformly completely observable, i.e., there exists constants $T > 0$, $\gamma > 0$, $\delta > 0$ such that for all t,

$$\infty > \gamma I \geq \int_t^{t+T} \Phi^T(\tau,t)N(\tau)N^T(\tau)\Phi(\tau,t)\,d\tau \geq \delta I > 0, \tag{10.51}$$

then (10.49) is exponentially asymptotically stable.

If $F(t)$ is a constant, F, then $P(t)$ may also be chosen to be constant, P, as may be $N(t)$, N, and the uniform observability condition (10.51) may be replaced by detectability of the pair $[F, N^T]$.

Finally, the condition of uniform complete observability of $[F(t), N^T(t)]$ may be replaced by the same condition on $[F(t) - K(t)N^T(t), N^T(t)]$ for bounded, locally integrable $K(t)$ and the same conclusion holds.

Recall that detectability of $[F, N^T]$ corresponds to the condition that any unobservable modes of this pair be strictly stable. An algebraic test for detectability is that, for any eigenvector v of F with eigenvalue λ, i.e., $Fv = \lambda v$, if $N^T v = 0$ then $|\lambda| < 1$ (in the discrete case) or $\mathrm{Re}(\lambda) < 0$ (in the continuous case). Predictably, the development will next turn towards the application of these stability methods to the closed loop systems given by (10.36)–(10.45). This, in turn, will allow the connection to be made between problem specification and closed loop stability.

The next stage in our treatment of LQ stability problems will be to address the question of asymptotic stability for the stationary infinite horizon LQ optimal control problems (10.36), (10.37) or the dual stationary filtering problems (10.38), (10.39). As evidenced earlier, we need only consider explicitly either the control problems or the filtering problems to infer stability properties for the other. We begin with the discrete case and then present the continuous version. Before launching fully into the analysis of stability, we recognise the pivotal role of the ARE in these problems and consider firstly some properties of the Riccati equations and their solutions.

10.6 Riccati Equation Solution Properties: Convergence and Monotonicity

Now is the juncture in which those results concerned with the convergence and monotonicity properties of the Riccati equations will be presented. These results will, admittedly, tend to appear somewhat peripheral to our thrust toward stability but they provide the machinery underpinning the later work on stability. The issues in this section are to describe some pertinent properties of and connections between solutions of the Riccati equations and, further, to examine some dependencies of these solutions on parameters.

10.6.1 Discrete Time

Infinite horizon LQ and Kalman filtering problems are associated with algebraic Riccati equations (10.9), (10.30) and we have already commented briefly on the potential multiplicity of solutions. Indeed in specifying the desired solution we have referred to the maximal nonnegative definite solution. We now examine some properties of this solution.

Existence of Maximal Nonnegative ARE Solution.

Theorem 10.3. *[12] Consider the ARE associated with an infinite horizon LQ control problem,*

$$P = F^T PF - F^T PG(G^T PG + R)^{-1}G^T PF + Q \tag{10.52}$$

where

- *$[F, G]$ is stabilizable,*
- *$[F, Q^{1/2}]$ has no unobservable modes on the unit circle,*
- *$Q \geq 0$ and $R > 0$.*

Then there exists a unique, maximal, positive definite symmetric solution $\bar{P}$.

This theorem specifies conditions necessary for the existence of a positive definite maximal solution. If unobservable modes of $[F, Q^{1/2}]$ are permitted on the unit circle then the strict positivity of $\bar{P}$ gives way only to nonnegativity. The stabilizability condition is critical to the sense of an infinite horizon LQ problem. This result makes formal that which is heuristically reasonable. We next consider the extent to which the (infinite horizon) ARE solution might properly be regarded as the limiting value of the solution of the (finite horizon) RDE as the finite horizon grows without bound.

Convergence of RDE Solution to ARE Solution.

Theorem 10.4 *Consider the ARE (10.52) above and its maximal solution $\bar{P}$, and consider the RDE*

$$P_{t+1} = F^T P_t F - F^T P_t G(G^T P_t G + R)^{-1}G^T P_t F + Q. \tag{10.53}$$

Then, provided $[F, G]$ is stabilizable, $R > 0$, $[F, Q^{1/2}]$ is detectable and $P_0 \geq 0$, $P_t \to \bar{P}$ as $t \to \infty$.

Notice here, once again, that the theorem statement reinforces those earlier heuristics by which the validity of the infinite horizon solution was justified. The key condition in this theorem is that detectability on $[F, Q^{1/2}]$ is introduced. This is stronger than the 'no unobservable modes on the unit circle' condition of Theorem 10.3. Alternative complementary theorems may also be developed which trade such detectability strictions for more severe constraints on the initial condition matrix P_0, typically that $P_0 \geq \bar{P}$. For the moment, however, this form is best suited to our purposes.

It is worth remarking here that very revealing examples of the sufficiency of these theorem conditions may be simply developed by considering the scalar case with unstable $F = 2$, say. In this case, the ARE is a simple scalar quadratic equation for which existence of real solutions and their positivity are easily examined.

Comparison of RDE Solutions. deSouza [14] has recently provided a lovely extension of the results of Nishimura [15] and Poubelle [16], [17] on the comparative properties between solutions of like RDE's. An earlier version is attributed to Claude Samson. The proof is by substitution into the RDE.

Lemma 10.1. *[14] Consider two RDEs (10.53) with the same F, G and R matrices but possibly different Q's, Q^1 and Q^2 respectively. Denote their solution matrices P_t^1 and P_t^2 respectively. Then, the difference between the two solutions $\tilde{P}_t = P_t^2 - P_t^1$ satisfies the following equation*

$$\tilde{P}_{t+1} = \bar{F}_t^{1T}\tilde{P}_t\bar{F}_t^1 - \bar{F}_t^{1T}\tilde{P}_t G(G^T P_t^2 G + R)^{-1}G^T\tilde{P}_t\bar{F}_t^1 + \tilde{Q} \tag{10.54}$$

or,

$$\tilde{P}_{t+1} = \bar{F}_t^{1T}\tilde{P}_t\bar{F}_t^1 - \bar{F}_t^{1T}\tilde{P}_t G(G^T\tilde{P}_t G + \bar{R}_t)^{-1}G^T\tilde{P}_t\bar{F}_t^1 + \tilde{Q} \tag{10.55}$$

where

$$\bar{F}_t^1 = F - G(G^T P_t^1 G + R)^{-1}G^T P_t^1 F$$

$$\tilde{Q} = Q_2 - Q_1$$

$$\bar{R}_t = G^T P_t^1 G + R.$$

A wealth of useful results stems easily from this astute algebraic observation by deSouza. For example,

Theorem 10.5. *Under the conditions of Lemma 10.1, suppose that*

$$Q^1 \geq Q^2,$$

and, for some t we have

$$P_t^1 \geq P_t^2,$$

then for all $k > 0$

$$P_{t+k}^1 \geq P_{t+k}^2.$$

Proof. In (10.54) we have, under the theorem conditions, that $\tilde{P}_t \leq 0$, $\tilde{Q}_t \leq 0$ and $P_t^2 \geq 0$. Thus $\tilde{P}_{t+1} \leq 0$ and the result is established for $k = 1$. By induction the theorem follows for all positive k.

Note that the arbitrariness of the assignment of superscripts to solutions P_t^1 and P_t^2 means that complementary results are directly established with, say, $P_{t+k}^2 \geq P_{t+k}^1$. This feature will be seen again in the immediately following results.

Monotonicity Properties of RDE Solutions. We are now in a position to apply the deSouza Lemma 10.1 to derive far reaching monotonicity properties of the RDE solution which play a central part in stability analyses to follow. We have the following cascade of results flowing from clever application of the above lemma to a single RDE solution sequence but with differing time indices.

Theorem 10.6. *[16] If the non-negative definite solution P_t of the RDE (10.53) is nonincreasing at one time, i.e.,*

$$P_{t+1} \leq P_t \quad , \textit{for some } t,$$

then P_t is monotonically nonincreasing for all subsequent times,

$$P_{t+k+1} \leq P_{t+k} \quad , \textit{for all } k \geq 0.$$

Proof. Identify P_t^1 with P_t, P_t^2 with P_{t+1}, and take $Q^1 = Q^2$ in Lemma 10.1 or Theorem 10.5 where now $\tilde{Q}_t = 0$.

Theorem 10.7. *[16] If the non-negative definite solution P_t of the RDE (10.53) is nondecreasing at one time, i.e.,*

$$P_{t+1} \geq P_t \quad , \textit{for some } t,$$

then P_t is monotonically nondecreasing for all subsequent times,

$$P_{t+k+1} \geq P_{t+k} \quad , \textit{for all } k \geq 0.$$

Proof. Identify P_t^1 with P_{t+1}, P_t^2 with P_t, and take $Q^1 = Q^2$ in Lemma 10.1 or Theorem 10.5 where now $\tilde{Q}_t = 0$.

These two monotonicity theorems describe the effective sign definiteness of the change in successive solution values. The deSouza Lemma 10.1 may be equally well applied to derive a similar property of second differences of the RDE solutions.

Theorem 10.8. *[14] If the solution P_t of the RDE (10.53) has a nonpositive definite second difference at time t, i.e.,*

$$P_{t+2} - 2P_{t+1} + P_t \leq 0,$$

then for all $k \geq 0$,

$$P_{t+k+2} - 2P_{t+k+1} + P_{t+k} \leq 0.$$

Proof. Equation (10.55) states that $\Delta P_t \triangleq P_{t+1} - P_t$ satisfies an RDE with state weighting $\tilde{Q}_t = 0$ and control weighting $\tilde{R}_t$, which is greater than the original R.

Therefore ΔP_t itself obeys the monotonicity properties of any RDE solution. One then recognises that

$$\Delta P_{t+1} - \Delta P_t = P_{t+2} - 2P_{t+1} + P_t.$$

10.6.2 Continuous Time

The above results on discrete time Riccati equation properties carry over with but little alteration to the realm of continuous time. Indeed, the discrete case has historically been the more difficult from which to extract hard results because of issues such as transition function noninvertibility etc. We shall see, for example, that the continuous version of the deSouza Lemma is very much more easily established and that the history of monotonicity results is longer, going back at least to Kailath [18]. Because of this simplicity and similarity we shall attempt to be briefer.

Existence of Maximal Nonnegative ARE Solution.

Theorem 10.9. *Consider the ARE associated with an infinite horizon LQ control problem,*

$$0 = A^T P + PA - PBR^{-1}B^T P + Q. \tag{10.56}$$

where

- *$[A, B]$ is stabilizable,*
- *$[A, Q^{1/2}]$ has no unobservable modes on the imaginary axis,*
- *$Q \geq 0$ and $R > 0$.*

Then there exists a unique, maximal, positive definite symmetric solution $\bar{P}$.

Convergence of RDE Solution to ARE Solution.

Theorem 10.10. *Consider the ARE (10.56) above and its maximal solution $\bar{P}$, and consider the RDE*

$$\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q \tag{10.57}$$

Then, provided $[A, B]$ is stabilizable, $R > 0$, $[A, Q^{1/2}]$ is detectable and $P_0 \geq 0$, $P_t \rightarrow \bar{P}$ as $t \rightarrow \infty$.

Comparison of RDE Solutions, Monotonicity. Poubelle [17] now replaces deSouza as the fount of all wisdom as we move to continuous time. Here the central results pivot about the following lemma.

Lemma 10.2 *[17] Consider $P(t)$, the solution of the RDE*

$$\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q$$

with initial condition $P(0) \geq 0$ and denote the closed loop matrix

$$\bar{A}(t) = A - BR^{-1}B^TP(t).$$

Then $P(t)$ satisfies

$$\ddot{P}(t) = \dot{P}(t)[A - BR^{-1}B^TP(t)] + [A - BR^{-1}B^TP(t)]^T\dot{P}(t) \tag{10.58}$$
$$= \dot{P}(t)\bar{A}(t) + \bar{A}^T(t)\dot{P}(t). \tag{10.59}$$

Further, $\ddot{P}(t)$ satisfies

$$\dddot{P}(t) = \ddot{P}(t)[A - BR^{-1}B^TP(t)] + [A - BR^{-1}B^TP(t)]^T\ddot{P}(t) \tag{10.60}$$
$$- 2\dot{P}(t)B^TP(t)B\dot{P}(t)$$
$$= \ddot{P}(t)\bar{A}(t) + \bar{A}^T(t)\ddot{P}(t) - 2\dot{P}(t)B^TP(t)B\dot{P}(t) \tag{10.61}$$

Proof. By differentiating successively (10.17) and then differentiating (10.58).

The two higher order versions of the RDE (10.58) and (10.60) are themselves RDEs of sorts. In particular, their alternative descriptions (10.59) and (10.61) are deliberately displayed as Lyapunov equations. We have the following simple result concerning the solution of such equations.

Lemma 10.3. *Consider the time-varying Lyapunov equation*

$$\dot{S}(t) = S(t)M(t) + M^T(t)S(t) + W(t), \qquad S(0) = S_0. \tag{10.62}$$

Denote by $\Phi(t,\tau)$ the transition matrix associated with $M(t)$. Then the solution of (10.62) is given by

$$S(t) = \Phi^T(t,0)S_0\Phi(t,0) + \int_0^t \Phi^T(t,\tau)W(\tau)\Phi(t,\tau)\,d\tau. \tag{10.63}$$

One immediately derives the following monotonicity theorems.

Theorem 10.11. *If the non-negative definite solution $P(t)$ of the RDE (10.57) is nonincreasing at one time, i.e.,*

$$\dot{P}(t) \leq 0 \quad \text{, for some } t,$$

then $P(t)$ is monotonically nonincreasing for all subsequent times,

$$\dot{P}(t+s) \leq 0 \quad \text{, for all } s \geq 0.$$

Proof. Since $P(t)$ is the solution of the RDE (10.57), by Lemma 10.2, $P(t)$ satisfies (10.59), which is a Lyapunov equation with zero driving term. Appealing to Lemma 10.3, we have

$$\dot{P}(t+s) = \Phi^T(t+s,t)\dot{P}(t)\Phi(t+s,t).$$

Theorem 10.12. *If the non-negative definite solution $P(t)$ of the RDE (10.57) is nondecreasing at one time, i.e.,*

$$\dot{P}(t) \geq 0 \quad \text{, for some } t,$$

then $P(t)$ is monotonically nondecreasing for all subsequent times,

$$\dot{P}(t+s) \geq 0 \quad \text{, for all } s > 0.$$

Theorem 10.13. *If the solution $P(t)$ of the RDE (10.57) has a nonpositive definite second derivative at time t, i.e., $\ddot{P}(t) \leq 0$, then for all $s \geq 0$, $\ddot{P}(t+s) \leq 0$.*

Proof. Write $W(t) = -2\dot{P}(t)B^TP(t)B\dot{P}(t)$, which is clearly nonpositive definite, then (10.61) is a Lyapunov equation and has a solution $\ddot{P}(t)$ which is given by

$$\ddot{P}(t+s) = \Phi^T(t+s,t)\ddot{P}(t)\Phi(t+s,t) + \int_t^{t+s} \Phi^T(t,\tau)W(\tau)\Phi(t,\tau)\,d\tau.$$

The result follows.

10.6.3 Summary

With this section we have derived and presented a combination of convergence, comparison and monotonicity results for AREs and RDEs in discrete and continuous time, which will provide the technical machinery with which to assault the asymptotic stability problems when coupled to the Lyapunov stability techniques of the preceding section. These monotonicity results are delightfully general and, naturally enough, reflect the structural properties of optimal control and estimation problems. The truly remarkable feature of these results and their associates is that, despite the proclaimed similarity between continuous time and discrete time and the duality between filtering and control, distinct methodologies of proof are often required from each of these specific areas to establish readily many results.

10.7 Stability in Infinite Horizon LQ Problems

Our treatment here will concentrate on the infinite horizon stationary discrete LQ and Kalman filtering stability Problems 1 and 2, (10.36) and (10.38), and their continuous variants, (10.37) and (10.39). Additionally, we consider also the stability of the time-varying infinite horizon Kalman filter Problem 3 (10.40) and (10.41). Thereafter, in the next section, we shall move on to deal with the receding horizon Problems 4 and 5 (10.42)–(10.45). Some tools, such as the recasting of the RDE/ARE as a Lyapunov equation will be common and so we commence with this aspect.

We will appeal directly to our earlier Lyapunov analysis to derive the following results using a standard device of rewriting the RDE or ARE as a Lyapunov equation. Denote the LQ gain by

$$K_j = -(G^T P_j G + R)^{-1} G^T P_j F. \tag{10.64}$$

Then the RDE (10.5) may be written, following some simple arithmetic, as

$$\begin{aligned} P_{j+1} &= F^T P_j F - F^T P_j G(G^T P_j G + R_j)^{-1} G^T P_j F + Q_j \\ &= (F + GK_j)^T P_j (F + GK_j) + K_j^T R_j K_j + Q_j. \end{aligned} \tag{10.65}$$

The astute reader will have picked (10.65) as a Lyapunov equation, with P_j serving the role of the element with the same symbol in the stability theory for linear equations. This we shall exploit.

In continuous time we have the equivalent version of the RDE or ARE. Denote the LQ control gain

$$K(t) = -R(t)^{-1} B^T P(t). \tag{10.66}$$

Now rewrite the RDE as

$$\begin{aligned} \dot{P}(t) &= A^T P(t) + P(t)A - P(t)BR(t)^{-1} B^T P(t) - Q(t) \\ &= (A + BK(t))^T P(t) + P(t)(A + BK(t)) + K^T(t)RK(t). \end{aligned} \tag{10.67}$$

10.7.1 Discrete Problems

Our root problem here is to examine the asymptotic stability of

$$x_{t+1} = (F - G(G^T P_\infty G + R)^{-1} G^T P_\infty F)\, x_t, \tag{10.68}$$

where P_∞ is the maximal nonnegative definite solution of the ARE,

$$P_\infty = F^T P_\infty F - F^T P_\infty G(G^T P_\infty G + R)^{-1} G^T P_\infty F + Q. \tag{10.69}$$

Theorem 10.14. *Consider the time-invariant linear vector difference equation (10.68) representing the closed loop of an infinite horizon LQ controlled system, where P_∞ is the maximal nonnegative definite solution, $\bar{P}$, of the ARE (10.69). Subject to the conditions:*

- *$[F, G]$ is stabilizable,*
- *$[F, Q^{1/2}]$ is detectable,*
- *$Q \geq 0$ and $R > 0$,*

then (10.68) is exponentially asymptotically stable.

Proof. From the theorem conditions and Theorem 10.3, we see that the ARE possesses a positive definite maximal solution, $\bar{P}$. Recognising the correspondence between (10.68) and (10.46) in Theorem 10.1, comparing the writing (10.65) of the ARE as a Lyapunov equation and then invoking Theorem 10.1, we see that, provided $[F + GK_\infty, (K_\infty^T R K_\infty + Q)^{1/2}]$ is a detectable pair, (10.68) will be stable. Since

$$K_\infty^T R K_\infty + Q = \begin{pmatrix} K_\infty^T R^{1/2} & Q^{1/2} \end{pmatrix} \begin{pmatrix} R^{1/2} K_\infty \\ Q^{1/2} \end{pmatrix},$$

and

$$F + GK_\infty = F + \begin{pmatrix} GR^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} R^{1/2} K_\infty \\ Q^{1/2} \end{pmatrix};$$

detectability of $[F, Q^{1/2}]$ suffices to prove stability.

This is the fundamental discrete infinite horizon stability result which shall form the basis of our successive analysis. The key feature is that through the writing of the ARE as a Lyapunov equation (10.65) involving the closed loop matrix of (10.68), the positive definite ARE solution $\bar{P}$ now serves to define a quadratic Lyapunov function. The critical theorem condition is the detectability requirement – the other conditions are better associated with the well posedness of the LQ problem. It is this detectability which shall reappear as pivotal to the development of the receding horizon results.

10.7.2 Continuous Problems

We now consider the asymptotic stability of

$$\dot{x}(t) = (F - GR^{-1} G^T P(\infty))\, x(t) \tag{10.70}$$

where $P(\infty)$ is the maximal nonnegative definite solution of the ARE,

$$0 = A^T P(\infty) + P(\infty)A - P(\infty)BR^{-1} B^T P(\infty) + Q. \tag{10.71}$$

Theorem 10.15. *Consider the time-invariant linear vector differential equation (10.70) representing the closed loop of an infinite horizon LQ controlled system, where $P(\infty)$ is the maximal nonnegative definite solution, $\bar{P}$, of the ARE (10.71). Subject to the conditions:*

- *$[F, G]$ is stabilizable,*
- *$[F, Q^{1/2}]$ is detectable,*
- *$Q \geq 0$ and $R > 0$,*

then (10.70) is exponentially asymptotically stable.

Proof. Parallels the discrete case with the appeal to Theorem 10.2 using the (10.67) reformulation of the ARE.

10.7.3 Asymptotic Stability of the Time-varying Kalman Filter

We treat only the discrete case here since the continuous version is identical in form. Consider the time-varying Kalman filter

$$\hat{x}_{t+1} = \left(F - F\Sigma_t H^T (H\Sigma_t H^T + R)^{-1} H\right)\hat{x}_t, \tag{10.72}$$

with Σ_t being the solution of the filtering RDE (10.27)

$$\Sigma_{t+1} = F\Sigma_t F^T - F\Sigma_t H^T (H\Sigma_t H^T + R_t)^{-1} H\Sigma_t F^T + Q_t. \tag{10.73}$$

We have the following.

Theorem 10.16. *Consider the time-varying difference equation (10.72) representing the time-varying Kalman filter operating from initial condition, $\Sigma_0 \geq 0$. Subject to the conditions:*

- $[F, HR_t^{-1/2}]$ *is uniformly observable,*
- $[F, Q_t^{1/2}]$ *is uniformly controllable,*
- $Q_t \geq 0$ *and* $R_t > 0$,

then (10.72) is exponentially asymptotically stable.

Proof. Under the theorem conditions, the RDE solution Σ_t is a positive definite matrix sequence bounded above and below. The filtering version of (10.65)

$$\Sigma_{t+1} = (F - K_t H)\Sigma_t (F - K_t H)^T + K_t R_t K_t^T + Q_t \tag{10.74}$$

then admits direct appeal to Theorem 10.1 for asymptotic stability of (10.72) to follow from the uniform observability of the pair $[F^T, Q_t^{1/2}]$. This corresponds to the stated controllability condition.

The upshot of this section has been to enunciate the conditions for the asymptotic stability of infinite horizon closed loop optimal solutions. The key features, stabilizability and detectability essentially, concur with those presaged earlier in our heuristic development. We next turn to deal with sufficient conditions for receding horizon stability.

10.8 Stability in Receding Horizon LQ Problems

Recall from our discussion in earlier sections that the genealogy of receding horizon LQ problems is the application of finite horizon LQ methods in the infinite horizon context, with an aim to achieving computational savings. In stationary circumstances, i.e., Q and R constant, with sufficiently large value of horizon N, the RDE Convergence Theorem 10.4 and the ARE Stability Theorem 10.14 combine to produce an obvious asymptotic stability result.

Theorem 10.17. *Consider the receding horizon LQ closed loop system*

$$x_{t+1} = (F - G(G^T P_N G + R)^{-1} G^T P_N F)\, x_t, \tag{10.75}$$

where P_N is the N^{th} term in the solution sequence of the RDE (10.5) with constant weighting matrices $Q \geq 0$ and $R > 0$ with initial condition $P_0 \geq 0$. Then, provided $[F, G]$ is stabilizable and $[F, Q^{1/2}]$ is detectable, there exists an N_0 such that (10.75) is exponentially asymptotically stable for all $N \geq N_0$.

While this result gives hope for the eventual stability of receding horizon based control systems, it delivers no guide to selection of a suitable N. To study the stability of receding horizon closed loops with arbitrary N we introduce a new tool.

10.8.1 The Fake Algebraic Riccati Equation

Guided by the ease of establishing stability for infinite horizon LQ controllers directly from the ARE we harken back to the RDE and attempt to have it masquerade as a fictitious or frozen or fake ARE. The RDE (10.5)

$$P_{j+1} = F^T P_j F - F^T P_j G(G^T P_j G + R_j)^{-1} G^T P_j F + Q_j,$$

is a recursion for P_{j+1} given P_j. We rewrite this as

$$P_j = F^T P_j F - F^T P_j G(G^T P_j G + R_j)^{-1} G^T P_j F + \bar{Q}_j, \tag{10.76}$$

$$\bar{Q}_j = Q + P_j - P_{j+1}. \tag{10.77}$$

Here (10.76) appears no longer to be a recursion for P_{j+1} but rather to be an algebraic equation for P_j. Specifically, we have recast the RDE so that P_j satisifes an algebraic Riccati equation with the original value of R_j but with a different value of Q given by (10.77).

The continuous version of this construction yields

$$0 = A^T P(t) + P(t)A - P(t)BR(t)^{-1}B^T P(t) + \bar{Q}(t) \tag{10.78}$$

$$\bar{Q}(t) = Q - \dot{P}(t). \tag{10.79}$$

We shall not focus too greatly on continuous time where the results do not deviate significantly from their discrete time counterparts. These above elementary modifications to the RDEs will play a critical role in the stability arguments to follow.

10.8.2 Receding Horizon Stability via FARE and Monotonicity

The RDE reformed as an ARE (10.76) or (10.78) is known as the *Fake Algebraic Riccati Equation* (FARE). Reference to the infinite horizon stability Theorem 10.14, immediately yields the following receding horizon stability result.

Theorem 10.18. *Consider the receding horizon LQ closed loop system (10.75) with P_N being the solution of the FARE (10.76) (the RDE (10.5)). Provided:*

- *$[F, G]$ is stabilizable,*
- *$Q_N \geq 0$ and $R_N > 0$,*
- *$[F, Q_N^{1/2}]$ is detectable,*

then (10.75) is exponentially asymptotically stable.

Proof. The FARE with $\bar{Q}_N \geq 0$ and $R_N > 0$ is an ARE with these Q and R which is associated with an infinite horizon LQ problem with such weighting matrices. According to Theorem 10.14, the closed loop of this artificial infinite horizon problem is asymptotically stable. But this closed loop is, in fact, also that of the original receding horizon problem.

The means of achieving the stability of these receding horizon control loops is to recast them so that they appear as infinite horizon problem solutions. This, in itself, is not too surprising a feature and does not indicate too great a leap forward except that the construction required for ARE solution is replaced by a simpler test on the FARE. A closer analysis of $\bar{Q}_t$ and $\bar{Q}(t)$ indicates further beneficial properties. Recall,

$$\begin{aligned}\bar{Q}_t &= Q + P_t - P_{t+1}\\ \bar{Q}(t) &= Q - \dot{P}(t).\end{aligned}$$

We may now pose and answer some questions concerning the potential detectability properties of these $\bar{Q}$.

Lemma 10.4. *Suppose that $[F, Q^{1/2}]$ is detectable. Then*

$$P_{N+1} \leq P_N \quad \Rightarrow \quad Q_N \geq Q \quad \Rightarrow \quad [F, Q_N^{1/2}] \text{ is detectable},$$

or, in continuous time,

$$\dot{P}(t) \leq 0 \quad \rightarrow \quad Q(t) \geq Q \quad \rightarrow \quad [F, Q^{1/2}(t)] \text{ is detectable}.$$

This result establishes the connection between nonincreasing solutions of the RDE and $[F, Q^{1/2}]$ detectability. All that remains is to invoke the monotonicity properties of the RDE already derived in Theorems 10.6 and 10.1 pertaining to the constant weighting RDEs (10.53) and (10.57). This we now do.

Theorem 10.19. *Consider the discrete time closed loop (10.75) derived from a receding horizon LQ problem with constant weighting matrices, $Q \geq 0$ and $R > 0$, and horizon N. Suppose that $[F, G]$ is stabilizable and $[F, Q^{1/2}]$ is detectable. Then if, for some N_0,*

$$P_{N_0+1} \leq P_{N_0},$$

(10.75) is exponentially asymptotically stable for any $N \geq N_0$.

Theorem 10.20. *Consider the continuous time closed loop derived from a receding horizon LQ problem with constant weighting matrices, $Q \geq 0$ and $R > 0$, and horizon T,*

$$\dot{x}(t) = \left(F - GR^{-1}G^T P(T)\right) x(t). \tag{10.80}$$

Suppose that $[F, G]$ is stabilizable and $[F, Q^{1/2}]$ is detectable. Then if, for some T_0,

$$\dot{P}(T_0) < 0,$$

(10.80) is exponentially asymptotically stable for any $T \geq T_0$.

These two theorems rely upon the montonicity properties of the RDE to establish that $\bar{Q}$ of the FARE always produces $[F, \bar{Q}^{1/2}]$ detectable and, thereby, a stable closed loop. The theorem statements rely upon the monotonic nonincreasing aspects of the RDE solutions. As a consequence of this, if P_j is always nonincreasing and converges to a constant nonnegative definite value, P_∞, then clearly one must satisfy $P_j \geq P_\infty$ for all j and Q_j exceeds Q for all j.

To admit the application of FARE results in a similar fashion in the circumstance of $P_j \leq P_\infty$, one may appeal to the second difference results on the RDE solution, Theorems 10.8 and 10.13, to ensure that Q never is permitted to become nonpositive definite even though $\bar{Q}_j \leq Q$. Specifically,

Theorem 10.21. *[14] Consider the discrete time closed loop (10.75) derived from a receding horizon LQ problem with constant weighting matrices, $Q \geq 0$ and $R > 0$, and horizon N. Suppose that, for some N_0:*

- *$[F, G]$ is stabilizable,*
- *$Q_{N_0} \geq 0$,*
- *$[F, Q_{N_0}^{1/2}]$ is detectable,*
- *$P_{N_0+2} - 2P_{N_0+1} + P_{N_0} \leq 0$,*

then (10.75) is asymptotically stable for any $N \geq N_0$.

Proof. We appeal to Theorem 10.8 to show that the final theorem condition above implies that the second difference of P_N is nonpositive for $N \geq N_0$. This, in turn, implies that the first difference, $P_{N+1} - P_N$, is nonincreasing for such N. Thus Q_N is a nondecreasing function of N.

The continuous version is older but similar.

Theorem 10.22. *[17] Consider the continuous time closed loop (10.80) derived from a receding horizon LQ problem with constant weighting matrices, $Q \geq 0$ and $R > 0$, and horizon T. Suppose that, for some T_0:*

- *$[F, G]$ is stabilizable,*
- *$\bar{Q}(T_0) \geq 0$,*
- *$[F, \bar{Q}(T_0)^{1/2}]$ is detectable,*
- *$\ddot{P}(T_0) \leq 0$,*

then (10.75) is asymptotically stable for any $T \geq T_0$.

We have now wended our way through an expanse of results on the stability of LQ optimal control and filtering systems combining: Lyapunov stability, monotonicity properties of RDEs, the FARE etc. The key observation has been to use the FARE coupled with monotonicity arguments to establish sensible stability requirements for the receding horizon LQ strategy. Further, more technically demanding variations on this theme are contained in the recent book [20]. We shall now present two examples of the application of these methods.

10.9 Examples

10.9.1 Stability of Generalized Predictive Control

Generalized Predictive Control (GPC) is a very popular process control design procedure due (*inter alia*) to Clarke, Mohtadi and Tuffs [19]. This control design especially has been successful in the field of Adaptive Control where many practical applications have been reported. At the heart of GPC is a receding horizon LQ problem. At time t the LQ criterion

$$J = \sum_{0}^{N} y_{t+j}^T y_{t+j} + \lambda \sum_{0}^{N_u} u_{t+j}^T u_{t+j}, \tag{10.81}$$

subject to $\{u_{t+j} = 0 | j = N_u + 1, \ldots, N\}$, is minimized and u_t is applied. Here N is the prediction horizon, N_u is the control horizon and $\lambda > 0$ is a control weighting.

For the case $N_u = N$, the GPC formulation may be viewed as receding horizon LQ with

$$Q = H^T H, \quad P_0 = H^T H, \quad R = \lambda I.$$

After identifying the GPC specification as receding horizon LQ, one appeals to the foregoing theory to determine stability of the controlled system. It is here that one strikes a snag – The above choiced for Q, R and P_0 do not admit simple affirmation of stability. Indeed, by inspection, one may replace the initial condition $P_0 = H^T H$ by the equivalent initial condition $P_{-1} = 0$. Thus $P_0 > P_{-1}$ and, by Theorem 10.7, P_j is then destined always to increase. This makes it more problematic to ensure that $Q_j \geq 0$. It is then not surprising that GPC exhibits difficulties in assuring designed closed loop stability. Further modifications to GPC are evaluated in [20] using some more technical monotonicity devices.

Given that the GPC receding horizon LQ control meets stability difficulties because the P_j sequence is always monotonic nondecreasing, one might ask whether other general strategies have more success in forcing P_j to decrease. In [20], techniques which effectively select P_0 infinite are examined using methods of this chapter.

10.9.2 Harmonic Analysis in Noise

It is frequently the case in many signal processing problems that one wishes to evaluate or estimate the harmonic components of a (slowly varying) periodic signal in noise. Here we shall consider the case where the period of the signal is known. In such circumstances, one may write a signal model for the measured signal, y_t, as follows.

$$x_{t+1} = F x_t + G w_t \tag{10.82}$$

$$y_t = H x_t + v_t, \tag{10.83}$$

where x_t is a vector of harmonic cosine and sine terms,

$$F = 1 \oplus \left\{ \begin{matrix} \text{block diag} \\ k = 1, 2, \ldots, n/2 - 1 \end{matrix} \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \right\} \oplus (-1),$$

$$H = (1/\sqrt{2} \quad 1 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0 \quad -1/\sqrt{2})^T,$$

w_t and v_t are independent noise processes with covariances Q and R. Note that F here has all of its eigenvalues on the unit cirlce. This formulation is examined in [21]. We consider two cases; the matrix Q is zero, and the matrix Q is nonzero. Harmonic analysis is identically equivalent to state estimation for this model.

When there is no state noise in the model (10.82) ($Q = 0$), the above set of equations describes a truly periodic signal corrupted by noise. If one attempts to set up the Kalman filter to estimate the state x_t optimally, then the limiting covariance Σ_∞ for the problem is zero, and thus the limiting Kalman gain is zero. This concurs with the intuitive solution to optimal estimation of a purely periodic signal in noise, i.e., average the answers over many periods. When F has all its eigenvalues on the unit circle and $Q = 0$, the stability results of earlier fail because $[F, Q^{1/2}]$ possesses many uncontrollable modes on the unit circle.

One technique to force stability into the solution of such problems is to set the input noise variance Q to a nonzero value arbitrarily. There are many other methods as well, see [2]. The summary choice of nonzero Q still obliges the designer to solve an ARE for a positive definite Σ_∞ to design the Kalman filter. An alternative approach explored in [21] is to select a positive value ϵ and determine a Kalman-like gain

$$K = F\epsilon I H^T (H\epsilon I H^T + R)^{-1},$$

associated with a $\Sigma_\infty = \epsilon I$. The question then arises: Does such a choice yield filter stability?

The answer is affirmative because, by construction of the FARE

$$\epsilon I - F\epsilon I F^T + F\epsilon I H^T (H\epsilon I H^T + R_t)^{-1} H\epsilon I F^T = \bar{Q},$$

one easily establishes that $\bar{Q} > 0$ for any $\epsilon > 0$. As a side remark, we mention that these filters degenerate to the Discrete Fourier Transform when $\epsilon \to \infty$.

10.10 Conclusion

We have led the reader through a tutorial development of the stability properties of Linear Quadratic feedback control systems and least squares state estimation in several guises: continuous and discrete time; finite, infinite and receding horizon. Our aim has been to reveal the connections between Lyapunov stability theory and the Riccati equations. The additional disclosures have then stemmed from these connections when coupled to properties purely of the Riccati equations themselves, namely convergence and monotonicity. These new tools provide simple stability tests for large classes of receding horizon LQ problems which are finding increasing practical application. The novelty here is the construction and application of Fake Algebraic Riccati Techniques.

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