III. Addendum II

The conditions that are necessary for the construction above are that the matrices

$$[A_{22},\, B_2]$$

and

$$\begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} \tag{10}$$

have full row and column rank, respectively (cf. p. 1508 of the paper[1]). If the system matrices satisfy a third condition, it will be possible to bypass the construction of a device to produce x_2 and still find a matrix K such that $A_{22} + B_2 K$ is nonsingular.

Let $K = R(I - N_1 N_1^+)$, where R is a matrix to be specified and N_1^+ is the Moore–Penrose generalized inverse of the matrix N_1 found in (8). The key property of $I - N_1 N_1^+ = \Phi$ is that $\Phi N_1 = 0$. Then it can be shown that the input

$$u = K(L_1 x_1 + M_1 y) + v \tag{11}$$

will yield a matrix $\Gamma_{22} = A_{22} + B_2 K = A_{22} + B_2 R \Phi$.

The matrix R can be chosen to make Γ_{22} nonsingular if the conditions in (10) are fulfilled and, in addition, the matrix

$$\begin{bmatrix} A_{22} \\ \Phi \end{bmatrix} \tag{12}$$

has full column rank. This condition arises from the dual factorizations

$$\begin{aligned} A_{22} + B_2 R \Phi &= [A_{22},\, B_2]\begin{bmatrix} I \\ R\Phi \end{bmatrix} \\ &= [I,\, B_2 R]\begin{bmatrix} A_{22} \\ \Phi \end{bmatrix}. \end{aligned}$$

If Γ_{22} is to be nonsingular, then all four factors must have full rank (cf., p. 1508 of the paper[1]). An appropriate matrix R may be found by using the algorithm described on page 1508 of the paper[1] by replacing K with R and C_2 by Φ. Condition (12) is essentially independent of those in (10).

IV. Addendum III

Now we consider a descriptor system containing the derivative of the control vector in the descriptor equation and/or the output equation

$$E\,\frac{dx}{dt} = Ax + B_0 u + B_1 \frac{du}{dt} \tag{13}$$

$$y = Cx + D_0 u + D_1 \frac{du}{dt}. \tag{14}$$

Here $E = \operatorname{diag}\{I_r,\, 0\}$, A, and C are partitioned as in (3)–(5), and

$$\begin{aligned} B_0 &= \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \\ B_1 &= \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \\ x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ W &= [D_0,\, D_1]. \end{aligned}$$

(For some aspects of controlling such systems, see [2] and [3].)

The problem is to find a matrix K for which the feedback law $u = Ky + v$ yields a nonsingular

$$\Gamma_{22} = A_{22} + J_2 K C_2. \tag{15}$$

We choose to require K to have the form $K = (I_p - B_1^+ B_1) R (I_m - WW^+)$, where R is yet to be specified. Note that $B_1 K = 0$ and also $KW = 0$ so that

$$\begin{aligned} K\left(D_0 u + D_1 \frac{du}{dt}\right) &= KW \begin{bmatrix} u \\ \dfrac{du}{dt} \end{bmatrix} \\ &= 0. \end{aligned}$$

Hence the second derivative of the input is eliminated from the equations resulting from feedback if either B_1 or D_1 is nonzero.

Now define $P = J_2(I_p - B_1^+ B_1)$ and $Q = (I_m - WW^+)C_2$ so that $J_2 K C_2 = PRQ$. From Section II of the paper[1] one concludes that $A_{22} + J_2 K C_2$ is nonsingular if and only if $[A_{22},\, P]$ and $[A_{22}^T,\, Q^T]^T$ have full row and column rank, respectively. Again, the algorithm on page 1508 of the paper[1] can be used to determine an appropriate R.

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A Note on Convex Combinations of Polynomials

Vincent Blondel

***Abstract*— We show an equivalence between conditions for linear systems to be stabilizable by stable controllers and conditions for convex combinations of polynomials to be stable.**

I. Introduction

In this paper, we point to an equivalence between conditions for linear systems to be stabilizable by stable controllers and conditions for convex combinations of polynomials to be stable.

For the proof of our result we shall need the following extension of the classical zero exclusion principle (see, e.g., [3, Th. 7.3.3]). Let Q be a pathwise connected subset of R^m, and suppose that the family of polynomials $\mathcal{P} := \{p(., q) : q \in Q\}$ has invariant degree and has continuous coefficient functions $a_i(q)$. Then the members of $\mathcal{P}$ all have the same number of zeros in the right-half plane iff $p(j\omega, q) \neq 0$ for all $q \in Q$ and $\omega \in R$.

Theorem 1: Suppose p_0, p_1 are two stable polynomials of identical sign (i.e., $p_0(0)p_1(0) > 0$). Let $p_0(s) = p_{00}(-s^2) + s p_{01}(-s^2)$, $p_1(s) = p_{10}(-s^2) + s p_{11}(-s^2)$ and define $d(s) = s[p_{01}(s)p_{10}(s) - p_{00}(s)p_{11}(s)]$, $n(s) = p_{00}(s)p_{10}(s) + s p_{01}(s)p_{11}(s)$. Then the following are equivalent.

1) $\lambda p_0(s) + (1 - \lambda)p_1(s)$ are stable for $0 \leq \lambda \leq 1$.

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2) $p_0(s)p_1(-s) + \mu$ have equally many zeros in the right-half plane when $\mu > 0$.
3) The system $n(s)/d(s)$ is stabilizable by a stable controller.

Proof ($1 \Leftrightarrow 2$): By the extended version of the zero exclusion principle given above, and by using the fact that p_0 and p_1 are stable, we deduce that the polynomials $\lambda p_0(s) + (1-\lambda)p_1(s)$ are stable for $0 \le \lambda \le 1$ iff

$$\lambda p_0(j\omega) + (1-\lambda)p_1(j\omega) \neq 0 \quad 0 < \lambda < 1,\ \omega \in R.$$

(The family $\lambda p_0(s) + (1-\lambda)p_1(s)$ has invariant degree when $0 < \lambda < 1$ since p_0, p_1 are stable and have same sign.) This last condition can be expressed equivalently by

$$p_0(j\omega) + \mu p_1(j\omega) \neq 0 \quad 0 < \mu,\ \omega \in R.$$

Multiplying both sides by $\overline{p_1(j\omega)} = p_1(-j\omega)$, we arrive at the equivalent condition

$$p_0(j\omega)p_1(-j\omega) + \mu \neq 0 \quad 0 < \mu,\ w \in R.$$

A second application of the extended zero exclusion principle leads us to the conclusion.

($2 \Leftrightarrow 3$): From the extended zero exclusion principle, the condition that $p_0(s)p_1(-s) + \mu$ have equally many zeros in the right-half plane when $\mu > 0$ is equivalent to

$$p_0(j\omega)p_1(-j\omega) + \mu \neq 0 \quad 0 < \mu,\ \omega \in R.$$

The decompositions $p_0(s) = p_{00}(-s^2) + sp_{01}(-s^2)$ and $p_1(s) = p_{10}(-s^2) + sp_{11}(-s^2)$ lead to $p_0(j\omega) = p_{00}(\omega^2) + j\omega p_{01}(\omega^2)$ and $p_1(-j\omega) = p_{10}(\omega^2) - j\omega p_{11}(\omega^2)$. Therefore, an equivalent condition is given by

$$\begin{aligned}&[p_{00}(\omega^2)p_{10}(\omega^2) + \omega^2 p_{01}(\omega^2)p_{11}(\omega^2) + \mu]\\ &\quad + j\omega[p_{01}(\omega^2)p_{10}(\omega^2) - p_{00}(\omega^2)p_{11}(\omega^2)] \neq 0,\\ &\quad 0 < \mu,\ \omega \in R.\end{aligned}$$

By looking at the real and imaginary parts of this expression, we deduce the equivalent condition that

$$p_{00}(\omega^2)p_{10}(\omega^2) + \omega^2 p_{01}(\omega^2)p_{11}(\omega^2) \ge 0$$

whenever

$$\omega[p_{01}(\omega^2)p_{10}(\omega^2) - p_{00}(\omega^2)p_{11}(\omega^2)] = 0, \quad w \in R.$$

In other words, the polynomial $n(s) = p_{00}(s)p_{10}(s) + sp_{01}(s)p_{11}(s)$ must take positive values whenever $d(s) = s[p_{01}(s)p_{10}(s) - p_{00}(s)p_{11}(s)]$ is equal to zero on the positive real axis. Since $p_{00}(0)p_{10}(0) = p_0(0)p_1(0) > 0$, this condition is satisfied if and only if $d(s)$ has an even number of zeros between each pair of positive real zeros of $n(s)$. By the parity interlacing condition (see [5]), this is equivalent to the requirement that $n(s)/d(s)$ is stabilizable by a stable controller. □

Example: Let $p_0(s) = 10s^3 + s^2 + 6s + 0.57$ and $p_1(s) = 10s^3 + 2s^2 + 8s + 1.57$. It is shown in [3] that although both polynomials are stable, not all convex combinations of p_0 and p_1 are stable. We verify this result by direct application of Theorem 1. From $p_0(s) = -(-s^2) + 0.57 + s[-10(-s^2) + 6]$ and $p_1(s) = -2(-s^2) + 1.57 + s[-10(-s^2) + 8]$ we construct $n(s) = 10s^3 - 14s^2 + 4.86s$ and $d(s) = 100s^3 - 138s^2 + 45.29s + 0.8949$. The polynomial $d(s)$ has a single zero (at 0.5991) between the pair (0, 0.6368) of zeros of $n(s)$, and the system n/d is therefore not stabilizable by a stable controller. The convex combinations of p_0 and p_1 are thus not all stable. This can be verified by checking that $2/3p_0 + 1/3p_1$ is unstable.

Remarks: The equivalence between statements 1) and 2) is also shown in [6]. Systems $n(s)/d(s)$ resulting from p_0 and p_1 may be nonproper. It may also be identically equal to infinity when $d(s) \equiv 0$. This special case happens only when $p_0(s) = kp_1(s)$ for some real k, which is a trivial case. (We thank one of the reviewers for this remark.)

The existence of a stable stabilizing controller for a system can be checked by the techniques described in [1] and [2]. These conditions can thus be used as alternatives to the conditions given in [4] for testing the stability of convex combinations of polynomials.

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On Variable Structure Output Feedback Controllers

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***Abstract*— In a recent work by Zak and Hui [1], a sliding-mode controller for multi-input/multi-output (MIMO) systems using static output feedback was proposed. Very nice geometric conditions for how to design sliding surfaces were given. However, there are two restrictive assumptions in it. One is that the uncertainties in the system must be bounded by a known function of outputs which excludes some possible uncertainties in the A matrix if the system is described by the triple (A, B, C). The other one requires a matrix equality [1, (4.3)] to be held which may also be very difficult to satisfy in many systems. In this paper, we propose a modification of the sliding mode controller for a class of single-input/single-output (SISO) systems which can eliminate the above-mentioned limitations and, under certain conditions, guarantee global closed-loop stability. Hence the range of applicability of the method in [1] can be greatly broadened.**

I. Introduction

Sliding mode control (or variable structure systems control) is a popular robust control method among control engineers. It is simple to design and completely robust to "matched" uncertainties. Its importance and applications can be seen from two recent special issues in the *International Journal of Control* [2] and the IEEE Transactions on Industrial Electronics [3]. One major drawback of sliding control is that states have to be available. Since in many

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