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An upper bound for the gain of stabilizing proportional controllers

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Abstract

A stable linear system controlled by a proportional controller is closed-loop stable provided the controller has sufficiently small gain. If the system has an unstable zero then any proportional controller with sufficiently large gain is destabilizing.

In this note we give an upper bound for the gain of stabilizing proportional controllers of stable systems that have one or more unstable zeros.

Keywords: Linear systems; Control; Stabilization; Proportional control; Static control; Stable systems; Non-minimum phase systems

1. Introduction

The following result, valid for stable systems, is an easy consequence of the small gain theorem (see, for example, [4]):

> A stable linear system controlled by a proportional controller is closed-loop stable provided the controller has sufficiently small gain (i.e., for some positive constant L, all controllers k satisfying $|k| < L$ are stabilizing).

Elementary manipulations show that for a system with rational transfer function $p(s)$ the largest possible bound L is equal to

$$L_{\text{opt}} = \frac{1}{\sup\{|p(s)|:\ \Re(s) \geqslant 0,\ \Im(p(s)) = 0\}},$$

where $\Re(\cdot)$ and $\Im(\cdot)$ are used to denote real and imaginary parts. This expression for L_{opt} is not convenient for computations because its evaluation is constrained by $\Im(p(s)) = 0$. If we define

$$\underline{L} = \frac{1}{\sup\{|p(s)|:\ \Re(s) \geqslant 0\}},$$

then $\underline{L} \leqslant L_{\text{opt}}$ and any proportional controller satisfying $|k| < \underline{L}$ is stabilizing. Moreover, the value $\underline{L}$ is considerably simpler to compute since it is the inverse of the H_∞ norm of $p(s)$.

A converse statement of the above property is

> A stable linear system that has one or more zeros in the open right half plane (ORHP), and that is controlled by a proportional controller is closed-loop unstable provided the controller has sufficiently large gain (i.e., for some finite constant U, all controllers k satisfying $U < |k|$ are destabilizing).

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As previously, the bound U depends on the system considered. In this case however the smallest possible bound U_{opt} is considerably harder to compute. In this note we give an upper bound for U_{opt} by using an argument from the geometric theory of analytic functions. More precisely, we show that if the system with transfer function $p(s)$ has a zero s_0 of multiplicity q in the ORHP and has a total number of zeros in the ORHP that does not exceed n, then

$$U_{\text{opt}} \leqslant \bar{U} := \frac{9(n+1)}{|2\Re(s_0)|^q |p^{(q)}(s_0)|},$$

where $p^{(q)}(s_0)$ is the qth derivative of $p(s)$ evaluated at s_0.

Any proportional controller k satisfying $\bar{U} < |k|$ is destabilizing.

A particular feature of the bound $\bar{U}$ is that it depends on the number of zeros of the system in the ORHP and on the qth derivative of the transfer function of the system evaluated at one particular zero, but it does not otherwise depend on the system. This renders the computation of $\bar{U}$ particularly easy to perform.

To summarise, we have the following ordering:

$$0 < \underline{L} \leqslant L_{\text{opt}} \leqslant U_{\text{opt}} \leqslant \bar{U} < \infty.$$

A proportional controller k satisfying $0 \leqslant |k| < L_{\text{opt}}$ is stabilizing, whereas it is destabilizing if $U_{\text{opt}} < |k| < \infty$. The bound $\bar{U}$ is described in this note.

The main result is proved and commented in Section 2. Examples are given in Section 3.

2. Result

We start with the needed property of analytic functions.

Theorem 2.1. *Suppose that the complex-valued function $f(z) = z + c_2 z^2 + \cdots$ is analytic in the open unit disc D and has n or less zeros there. Then the image of D under the mapping $\xi = f(z)$ completely covers the disc $|\xi| \leqslant 1/9(n+1)$.*

Proof. Let $f(z) = z + c_2 z^2 + \cdots$ be an analytic function in the open unit disc D with n or less zeros in D and let $\alpha \neq 0$ be a complex number such that $f(z) \neq \alpha$ for all z in D. We want to show that $|\alpha| > 1/9(n+1)$.

Consider the function defined by

$$g(z) := 1 - \frac{f(z)}{\alpha} = 1 - \frac{z}{\alpha} - \cdots.$$

Obviously, $g(z)$ is analytic, is never equal to zero and takes the value $\xi = 1$ less than n times in D. Since $g(z)$ is zero-free in D, we may define a function $h(z)$ by $h(z)^{n+1} = g(z)$ and $h(0) = 1$; then

$$h(z) := (g(z))^{1/(n+1)} = 1 - \frac{z}{\alpha(n+1)} - \cdots.$$

Since $g(z)$ assumes the value $\xi = 1$ less than n times in D, we can select a complex number ω such that $\omega^{n+1} = 1$ and $h(z) \neq \omega$ for all z in D. (Indeed, if such a value ω did not exist, $h(z)$ would assume the value $\xi = 1$ $n+1$ times, a contradiction.) As a final transformation consider the function defined by

$$l(z) := \frac{h(z)}{\omega} = \frac{1}{\omega} - \frac{z}{\alpha(n+1)\omega} + \cdots.$$

The function $l(z)$ is analytic in D and never takes the value $\xi = 0$ or the value $\xi = 1$ on D.

By Landau's theorem (see [2]), the first two coefficients of $l(z) = a_0 + a_1 z + \cdots$ are such that $|a_1| \leqslant 2|a_0|(|\log|a_0|| + A)$ where A can be taken to be equal to $\Gamma^4(0.25)/4\pi^2 = 4.377\ldots$ We thus obtain

$$\left|\frac{1}{\alpha(n+1)\omega}\right| \leqslant 2\left|\frac{1}{\omega}\right|\left(\left|\log\left|\frac{1}{\omega}\right|\right| + A\right).$$

Using $A < \frac{9}{2}$ and $|\omega| = 1$, we simplify the above inequality and get

$$|\alpha| > \frac{1}{9(n+1)}$$

as required. □

We now prove our theorem on proportional control.

Theorem 2.2. *Let $p(s)$ be the transfer function of a stable linear system that has at least one zero s_0 in the ORHP. Assume that k is a stabilizing proportional controller. Then*

$$|k| < \frac{9(n+1)}{|2\Re(s_0)|^q |p^{(q)}(s_0)|},$$

where n is equal to the number of zeros of $p(s)$ in the ORHP, q is the multiplicity of the zero s_0, $\Re(s_0)$ is *the real part of* s_0 *and* $p^{(q)}(s_0)$ *is the qth derivative of the transfer function* $p(s)$ *evaluated at* s_0.

Proof. Let n be the number of zeros of $p(s)$ in the ORHP and assume that s_0 is one such zero. For simplicity, assume first that s_0 has multiplicity one. Define $z_0 := (s_0 - 1)/(s_0 + 1) \in D$ and consider the following mappings of the complex plane:

$$\mu : \lambda \to \frac{\lambda - z_0}{\lambda \overline{z_0} - 1}$$

and

$$\sigma : \lambda \to \frac{1 + \lambda}{1 - \lambda}.$$

The linear transformation μ maps the unit disc D bijectively onto itself and is such that $\mu(0) = z_0$, whereas σ is the usual bilinear mapping between the open unit disc D and the ORHP. The value z_0 is chosen such that $\sigma(z_0) = s_0$. Due to these properties, the rational function $r(z)$ defined by

$$r(z) = p(\sigma(\mu(z)))$$

is such that

$$r(0) = p(\sigma(\mu(0))) = p(s_0) = 0.$$

By the chain rule we have

$$r'(z) = p'(\sigma(\mu(z)))\sigma'(\mu(z))\mu'(z)$$

and thus

$$r'(0) = p'(s_0)\sigma'(z_0)\mu'(0).$$

The quantities involved in this last expression can be computed as $\sigma'(z_0) = 2/(1 - z_0)^2$ and $\mu'(0) = (\overline{z_0} z_0 - 1)$. Using the definition of z_0 we obtain

$$r'(0) = -2p'(s_0)\Re(s_0)\frac{s_0 + 1}{\overline{s_0} + 1}.$$

The function $r(z)/r'(0) = z + c_2 z^2 + \cdots$ is analytic in D and has n zeros there. By the previous theorem, the image of D under the mapping $\xi = r(z)/r'(0)$ completely covers the disc $|\xi| < 1/9(n + 1)$ or, in other words, the image of D under the mapping $\xi = r(z)$ completely covers the disc

$$|\xi| \leqslant \frac{1}{9(n + 1)}\left|2p'(s_0)\Re(s_0)\frac{s_0 + 1}{\overline{s_0} + 1}\right|.$$

Due to the definition of $r(z)$ this means that, associated with any value ξ of modulus less than or equal to $(1/9(n + 1))|2p'(s_0)\Re(s_0)(s_0 + 1)/(\overline{s_0} + 1)|$, there is a point s^* of the ORHP, such that $p(s^*) = \xi$. It now remains to show the link between this property and proportional stabilization.

Let k be a proportional controller. If k stabilizes $p(s)$, then $kp(s)/(1 + kp(s))$ is stable and $p(s) \neq -1/k$ for all s in the ORHP. However, then, by the above condition, k must satisfy

$$\left|\frac{1}{k}\right| > \frac{1}{9(n + 1)}\left|2p'(s_0)\Re(s_0)\frac{s_0 + 1}{\overline{s_0} + 1}\right|.$$

Using $|(s_0 + 1)/(\overline{s_0} + 1)| = 1$ we obtain

$$|k| < \frac{9(n + 1)}{|2\Re(s_0)||p'(s_0)|}$$

and the theorem is proved for $q = 1$.

The proof is similar for $q \geqslant 2$. If s_0 is a zero of multiplicity q then the function $r(z)$ defined by $r(z) = p(\sigma(\mu(z)))$ is such that $r(0) = r'(0) = r^{(2)}(0) = \cdots = r^{(q-1)}(0) = 0$. The qth derivative of $r(z)$ evaluated at 0 is different from 0 and can be computed as $r^{(q)}(0) = p^{(q)}(s_0)(\sigma'(s_0)\mu'(0))^q$.

The function

$$\frac{r(z)}{r^{(q)}(0)} = z^q + c_{q+1} z^{q+1} + \cdots$$

is analytic in D and has n zeros in D.

By extending the result of Jenkins [2] it can be proved that, similarly as for the case $q = 1$, the image of D by a function $f(z) = z^q + c_{q+1} z^{q+1} + \cdots$ that has at most n zeros in D completely covers the disc $|\xi| \leqslant 1/9(n + 1)$. Applying this extended version of our previous theorem, we obtain that the range of $r(z)/r^{(q)}(0)$ on D covers the disc $|\xi| \leqslant +1/9(n + 1)$. The last steps are identical to those for the case $q = 1$; they lead to the following inequality for stabilizing proportional controllers:

$$|k| < \frac{9(n + 1)}{|2\Re(s_0)|^q |p^{(q)}(s_0)|}.$$

The theorem is proved. □

Remarks

(1) If $p(s)$ is factored as $n(s)/d(s)$ with $n(s)$ and $d(s)$ coprime polynomials, and if s_0 is a zero of multiplicity q of $p(s)$, then $p^{(q)}(s_0)$ is equal to $n^{(q)}(s_0)/d(s_0)$.

(2) A close look at the proof of the theorem reveals that the same bound is valid for systems that have poles on the imaginary axis. See Example 3.2 for an illustration of this.

(3) The expression $9(n+1)/|2\Re(s_0)|^q|p^{(q)}(s_0)|$ is increasing with n; it is therefore obvious that the statement of the theorem remains valid when n is substituted by a value that is larger than n. For example, n can be taken to be equal to the order of the system.

3. Examples

Example 3.1. Let

$$p(s) = \frac{(s-2)(s+1)}{2s^3+s^2+3s+1} = \frac{s^2-s-2}{2s^3+s^2+3s+1}.$$

The transfer function $p(s)$ has a zero of multiplicity one at $s_0 = 2$. The other two zeros are at -1 and at ∞; they are outside the ORHP. The first derivative of $p(s)$ evaluated at $s_0 = 2$ is equal to $p'(2) = \frac{3}{27} = \frac{1}{9}$ and thus any proportional stabilizing controller k must satisfy

$$|k| < \frac{9(n+1)}{|2\Re(s_0)|^q|p^{(q)}(s_0)|} = 40.5.$$

The controller $k = 41$ is destabilizing.

Example 3.2. Let

$$p(s) = \frac{6(s^2-5s+6)}{s^2+2}.$$

The transfer function $p(s)$ has all its poles on the imaginary axis. The zeros of $p(s)$ are at $s_0 = 2$ and at $s_0 = 3$, both are in the ORHP. We compute, for $s_0 = 2$, $|k| < 6.75$ and, for $s_0 = 3$, $|k| < 8.25$. Hence, any stabilizing proportional controller satisfies $|k| < 6.75$.

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