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A Rational Test for Strong Stabilization*

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Key Words—Stabilization; stability test; Routh's criterion.

Abstract—We propose a simple test for checking strong stabilizability. The test necessitates the construction of a single Routh table.

1. *Introduction*

Youla *et al.* (1974) showed that linear systems are strongly stabilizable (i.e. stabilizable by a stable controller) if and only if they have an even number of real poles between each pair of real zeros in the right half-plane. This condition is usually referred to as the parity interlacing condition.

Motivated by this result, Anderson and Jury (1976) exploited properties of the Cauchy index to show that it is possible to check the parity interlacing condition without explicitly computing the real poles and zeros of the system but by performing instead a finite number of rational operations on its coefficients. The test proposed by Anderson and Jury is given in terms of Cauchy indices.

In this note we express the parity interlacing condition as a condition on the difference between the number of open right and open left half-plane zeros of a polynomial constructed from the coefficients of the system. This formulation can be used to derive an explicit rational test for the parity interlacing condition. The test necessitates the construction of a single Routh table.

2. *Equivalent formulation of the parity interlacing condition*

Let p be as real polynomial. We use the following notation: $\deg(p)$ denotes the degree of p, $\rho_+(p)$ and $\rho_-(p)$ denote the number of open right and open left half-plane zeros of p, and $\rho(p) = \rho_+(p) - \rho_-(p)$ denotes the difference between these two quantities.

With this notation, we have the following theorem.

Theorem. Let

$$q(s) = \frac{n(s)}{d(s)} = \frac{a_k s^k + \ldots + a_1 s + a_0}{s^{k+1} + b_k s^k + \ldots + b_1 s + b_0}$$

be a linear system and assume that $\deg(n) \geq 1$ and $a_0 \neq 0$. Define the polynomial $p(s)$ by

$$p(s) := [n(-s^2) + sn'(-s^2)][n(-s^2) - sn'(-s^2)d(-s^2)].$$

Then the system $q(s)$ satisfies the parity interlacing condition if and only if $\rho(p) = 1$.

If $q(s)$ has no multiple zeros on the positive real axis, an equivalent condition is given by $\rho_+(p) = \frac{1}{2}[\deg(p) + 1]$.

Remark. Note that the last condition makes sense since p has always an odd degree.

Proof. It was shown in Anderson and Jury (1976, Main Result, p. 388) that the strictly proper system $q(s)$ satisfies the parity interlacing conditon iff

$$I_0^\infty \frac{n'(s)}{n(s)} = I_0^\infty \frac{n'(s)d(s)}{n(s)} \tag{1}$$

where I denotes the Cauchy index. In the following we use properties of the Cauchy index to reformulate the condition (1) as a condition on the zeros of p.

A rational function $f(s)$ satisfying $f(0) \neq \infty$ also satisfies $2I_0^\infty f(s) = I_{-\infty}^\infty sf(s^2)$. By assumption, $n(0) \neq 0$, and thus (1) is equivalent to

$$I_{-\infty}^\infty \frac{sn'(s^2)}{n(s^2)} = I_{-\infty}^\infty \frac{sn'(s^2)d(s^2)}{n(s^2)}. \tag{2}$$

By assumption, $d(s)$ is monic and $n(s)$ is of degree no less than one. Therefore the rational function on the right-hand side of (2) is positive when s approaches infinity from the positive axis and is negative when s approaches infinity from the negative axis. Using Property 5 from Gantmacher (1959, XV, § 12.1) (and correcting a misprint of the 1974 edition: on line 16 one should read $\frac{1}{2}(\epsilon_b - \epsilon_a)$), we rewrite the right-hand side of (2) and obtain

$$I_{-\infty}^\infty \frac{sn'(s^2)}{n(s^2)} = 1 - I_{-\infty}^\infty \frac{n(s^2)}{sn'(s^2)d(s^2)}. \tag{3}$$

The condition (3) can now be re-interpreted as a polynomial property. Define the polynomials p_1 and p_2 by

$$p_1(s) := n(-s^2) + sn'(-s^2),$$
$$p_2(s) := n(-s^2) - sn'(-s^2)d(-s^2).$$

By formula (20) from Gantmacher (1959, XV, § 3.4), we deduce

$$I_{-\infty}^\infty \frac{sn'(s^2)}{n(s^2)} = \rho(p_1) \tag{4}$$

and

$$I_{-\infty}^\infty \frac{n(s^2)}{sn'(s^2)d(s^2)} = \rho(p_2). \tag{5}$$

Hence, (3) can equivalently be expressed by

$$\rho(p_1) + \rho(p_2) = 1. \tag{6}$$

Since $p(s) = p_1(s)p_2(s)$, the condition (6) is also equivalent to

$$\rho(p) = 1, \tag{7}$$

and the first part of the theorem is proved.

For the second part, note that when $q(s)$ has no multiple zero on the positive real axis, p_1 and p_2 have no imaginary zeros. But then p has no imaginary zeros either, and hence $\rho(p) = 2\rho_+(p) - \deg(p)$. □

3. *A rational test*

The stabilizability of a system by a stable controller can be checked by computing the difference between the number of right and left open half-plane zeros of a polynomial constructed from the system. We now show how this quantity can be evaluated by observing sign changes in the first column of a Routh table.

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The number of open right half-plane zeros of a polynomial p is equal to the number of sign changes in the first column of the Routh table associated with p. If no sign change occurs, the polynomial is stable. According to the theorem in the previous section, this procedure can be used for checking strong stabilizability of systems with no multiple zeros on the positive real axis. In this case it suffices to construct the Routh table associated with

$$p(s) = [n(-s^2) + sn'(-s^2)][n(-s^2) - sn'(-s^2)d(-s^2)]$$

and to count the number of sign changes in the first column of the associated Routh table. There must be exactly $\frac{1}{2}[\deg(p) + 1]$ sign changes for the system $n(s)/d(s)$ to be strongly stabilizable.

In the more general situation where the system may have multiple zeros on the positive real axis we need to make a minor modification. The number of open left half-plane zeros of the polynomial $p(s)$ is equal to the number of open right half-plane zeros of $p(-s)$. This quantity is easily seen to be equal to the number of sign changes in the first column of a modified Routh table of $p(s)$ in which all the entries of the even- (or odd-) numbered lines have their sign reversed. Thus strong stabilizability can be checked by constructing the Routh table associated with

$$p(s) = [n(-s^2) + sn'(-s^2)][n(-s^2) - sn'(-s^2)d(-s^2)],$$

counting the number of sign changes in the first column of the associated Routh table and counting the number of sign changes in the first column of a modified table in which all the entries of the even- (or odd-) numbered lines have their sign reversed. The difference between these two quantities must be equal to one for the system $n(s)/d(s)$ to be strongly stabilizable.

4. *Examples*

Example 1. The system

$$q(s) = \frac{s-1}{s^2 - 5s + 6}$$

has its zeros at $s = 1$ and $s = \infty$ and its poles at $s = 2$ and $s = 3$. It therefore satisfies the parity interlacing condition. Indeed, following our procedure, we obtain a polynomial

$$p(s) = s^7 - s^6 + 6s^5 - 4s^4 + 10s^3 - 4s^2 + 5s + 1.$$

The Routh table associated with p is given by

1	6	10	5
−1	−4	−4	1
2	6	6	
−1	−1	1	
4	8		
1	1		
4			
1			

There are four sign changes in the first column $(1, -1, 2, -1, 4, 1, 4, 1)$. When entries on even-numbered lines have their signs reversed, the first column becomes $(1, 1, 2, 1, 4, -1, 4, -1)$, whose number of sign changes is three. The difference between these two quantities is one, and thus the system is strongly stabilizable.

The degree of p increases rapidly with those of n and d (we have $\deg(p) = 4\deg(n) + 2\deg(d) - 1$). The construction of the Routh table is therefore cumbersome even for systems of small order. In the next two examples we give only the final result, and skip the construction of the Routh table.

Example 2. From the system

$$q(s) = \frac{s-2}{s^3 - 5s^2 + 7s - 3},$$

we obtain the polynomial

$$p(s) = -s^9 + s^8 - 7s^7 + 5s^6 - 17s^5 + 8s^4 - 18s^3 + 7s^2 - 8s + 4$$

which has seven open right half-plane zeros and two open left half-plane zeros. It is therefore not strongly stabilizable, since $\rho(p) = 5$. Indeed, it can be checked that $q(s)$ has one zero at $s = 2$, a double zero at $s = \infty$, one pole at $s = 3$ and a double pole at $s = 1$, and it therefore does not satisfy the parity interlacing condition, since there is a unique zero (at $s = 3$) between the poles $s = 2$ and $s = \infty$.

Example 3. The system

$$q(s) = \frac{s^2 - 4s + 4}{s^3 - 7s^2 + 15s - 9}$$

gives rise to a 13th-order polynomial

$$\begin{aligned} p(s) = &-2s^{13} + 4s^{12} - 26s^{11} + 44s^{10} - 138s^9 + 189s^8 \\ &- 384s^7 + 396s^6 - 592s^5 + 408s^4 - 480s^3 \\ &+ 176s^2 - 160s + 16, \end{aligned}$$

which has four imaginary zeros, five open right half-plane zeros ands four open left half-plane zeros, and therefore satisfies $\rho(p) = 1$. The system $q(s)$ can thus be stabilized by a stable controller. Indeed, this can immediately be seen by checking that

$$q(s) = \frac{(s-2)^2}{(s-1)(s-3)^2}$$

satisfies the parity interlacing condition.

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