[8] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-35, pp. 131–140, 1990.

[9] P. V. Kokotovic and P. W. Sauer, "Integral manifold as a tool for reducing order modeling in nonlinear systems: A synchronous machine case study," in *Proc. 26th IEEE Conf. Decision Contr.*, Los Angeles, CA, 1987, pp. 873–878.

[10] P. Lucibello, "Nonlinear regulation, with internal stability, of a two-link flexible robot arm," in *Proc. 28th IEEE Conf. Decision Contr.*, Tampa, FL, 1989, pp. 1645–1650.

[11] P. Lucibello, "Inversion of linear square systems by learning," in *Proc. 30th IEEE Conf. Decision Contr.*, Brighton, UK, 1990, pp. 859–864.

[12] P. Lucibello and M. D. Di Benedetto, "Exact tracking with bounded elastic vibration for a flexible robot arm," *Diagnostics, Vehicle and Special Topics*, T. S. Stanker, S. Braun, S. Rakheja, Y. Stepanenko, M. Samaha, Eds. New York: ASME, pp. 245–249, 1989.

[13] B. Paden, D. Chen, R. Ledesma, and E. Bayo, "Exponentially stable tracking control for multijoint flexible-link manipulators," *ASME J. Dyn. Sys., Measurement Contr.*, to appear.

[14] M. K. Sain and J. L. Massey, "Invertibility of linear time-invariant dynamical systems," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 141–149, 1969.

[15] S. N. Silverman, "Inversion of multivariable linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 270–276, 1969.

[16] S. N. Singh, "A modified algorithm for invertibility in nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 595–598, 1981.

[17] ——, "Generalized decoupled control synthesis for invertible nonlinear systems," *IEE Proc.*, vol. 128, part D, pp. 157–161, 1981.

A Sufficient Condition for Simultaneous Stabilization

V. Blondel, G. Campion, and M. Gevers

Abstract—In this note, we study the following problem, "Under what condition(s) is it possible to find a single controller which stabilizes k single-input single-output linear time-invariant systems $p_i(s)$ ($i = 1, \cdots, k$)?". We introduce the concept of avoidance in the complex plane and use it to derive a sufficient condition for k systems to be simultaneously stabilizable. A method for constructing a simultaneous stabilizing controller is also provided and illustrated by an example.

I. INTRODUCTION

Simple questions cannot always be simply answered. In this note, we give a very partial answer to a simple question in control theory which, for being open for ten years, does not seem to have a simple answer. The question is known under the name of *simultaneous stabilization problem* and is the following, "Under what condition(s) is it possible to find a single controller $c(s)$ which stabilizes k SISO linear time-invariant systems $p_i(s)$ ($i = 1, \ldots, k$)?".

When $k = 2$ a tractable necessary and sufficient condition, known as the *parity interlacing property*, exists (see [12], [18], [15]). The problem becomes harder when $k \geq 3$ and most papers on the simultaneous stabilization problem deal either with necessary or with sufficient conditions ([2], [4], [9], [10], and [16]).

The connection between interpolation in the complex plane and the simultaneous stabilization problem was pointed out by various authors ([5]–[8]). In the same spirit, we present the problem in this note as an avoidance problem between complex valued functions. Roughly speaking, a set of k SISO linear time-invariant systems $\{p_1(s), \cdots, p_k(s)\}$ will be shown to be simultaneously stabilizable if and only if there exists a $k + 1$th system $p_{k+1}(s)$ which avoids, in a sense that we will define, the systems $p_1(s), \cdots, p_k(s)$ for all s in the extended closed right-half plane. With this view of the problem we will prove a new sufficient condition under which k systems are simultaneously stabilizable.

A method for constructing a simultaneous stabilizing controller is provided and illustrated with an example.

II. NOTATIONS–DEFINITIONS

$\mathbb{R}(s)$ is the set of real rational functions. $\mathbb{C}_\infty$ is the extended complex plane $\mathbb{C} \cup \{\infty\}$ adequately topologized. Ω is any subset of $\mathbb{C}_\infty$. We shall suppose throughout this note that Ω is symmetric with respect to the real axis (if $s \in \Omega$ then $\bar{s} \in \Omega$), that it is closed, simply connected, contains $\{\infty\}$ and that its complement in $\mathbb{C}_\infty$ contains at least one value of $\mathbb{R} \cup \{\infty\}$. Ω is to be thought of as the complement in $\mathbb{C}_\infty$ of a region of stability. A real rational function $f(s) \in \mathbb{R}(s)$ is Ω-*stable* if it has no poles in Ω (we draw the reader's attention to the fact that Ω-stability is defined by other authors in exactly the opposite way). $S(\Omega)$ is the set of all Ω-stable functions and $U(\Omega)$ is the set of functions that are in $S(\Omega)$ and that have their inverse in $S(\Omega)$: they are the units of the ring $S(\Omega)$.

III. AVOIDANCE AND INTERSECTION

Immediate checking shows that, whatever Ω, $S(\Omega)$ is a commutative ring. It is also known that under our hypothesis on Ω, the field of fractions of $S(\Omega)$ is $\mathbb{R}(s)$ (see e.g., [13, p. 50]). This means that if $p(s) \in \mathbb{R}(s)$ then there exist $n(s), d(s) \in S(\Omega)$ such that $p(s) = (n(s)/d(s))$ where $n(s)$ and $d(s)$ have no common zeros in Ω. Such a fractional decomposition of $p(s)$ is called an Ω-*coprime decomposition*. We may now define what we mean by the intersections of two functions $p_1(s), p_2(s) \in \mathbb{R}(s)$ in Ω.

Definition: Let $p_1(s), p_2(s) \in \mathbb{R}(s)$ and let $n_i(s), d_i(s) \in S(\Omega)$ be fractional Ω-coprime decompositions of $p_i(s)$ $i = 1, 2$. The intersections of $p_1(s)$ and $p_2(s)$ in Ω are the zeros of $n_1(s)d_2(s) - d_1(s)n_2(s) \in S(\Omega)$ in Ω. If $n_1(s)d_2(s) - d_1(s)n_2(s) \in U(\Omega)$, then $p_1(s)$ and $p_2(s)$ have no intersections in Ω and we say that they avoid each other in Ω.

This definition may look somewhat mysterious. In fact, it is very natural and the procedure to compute the intersections between systems is very simple. Consider $p_1(s), p_2(s) \in \mathbb{R}(s)$ and decompose $p_1(s) = (n_1(s)/d_1(s))$ and $p_2(s) = (n_2(s)/d_2(s))$, where $n_i(s), d_i(s)$ are polynomials with no common zeros ($i = 1, 2$). The finite intersections of $p_1(s)$ and $p_2(s)$ in Ω are simply the zeros in Ω of the polynomial $n_1(s)d_2(s) - d_1(s)n_2(s)$, whereas the possible additional intersections at infinity may be checked by inspection of the relative degree and gain of the functions. For example, the rational functions $p_1(s) = (2s/(s + 1)(s - 1))$ and $p_2(s) = (1/s - 3)$ have their intersections at the zeros of $2s(s - 3) - (s + 1)(s - 1) = s^2 - 6s + 1$ and at the point at infinity since $p_1(\infty) = p_2(\infty) = 0$.

IV. SIMULTANEOUS STABILIZATION

A controller $c(s) \in \mathbb{R}(s)$ is said to be an Ω-*stabilizing controller* of $p(s) \in \mathbb{R}(s)$ if and only if all the transfer functions $p(s)c(s)(1 + p(s)c(s))^{-1}$, $c(s)(1 + p(s)c(s))^{-1}$ and $p(s)(1 + p(s)c(s))^{-1}$ are in $S(\Omega)$. This notion of Ω-stabilization is strongly connected to that of avoidance in Ω.

Lemma: Let $p(s), c(s) \in \mathbb{R}(s)$. Then the controller $c(s)$ internally Ω-stabilizes $p(s)$ if and only if $-c^{-1}(s)$ avoids $p(s)$ in Ω.

Proof: Let $p(s) = (n_p(s)/d_p(s))$ and $c(s) = (n_c(s)/d_c(s))$ be Ω-coprime decompositions of $p(s)$ and $c(s)$. It is well known that $c(s)$ internally Ω-stabilizes $p(s)$ if and only if $n_p(s)n_c(s) + d_p(s)d_c(s) \in U(\Omega)$ (see [13]). This last condition is satisfied if and only if $-c(s)^{-1}$ avoids $p(s)$ in Ω. ∎

As a consequence of this lemma, the systems $p_i(s) \in \mathbb{R}(s)$ $(i = 1, \ldots, k)$ are simultaneously Ω-stabilizable if and only if there exists a $c(s) \in \mathbb{R}(s)$ such that $-c^{-1}(s)$ avoids $p_i(s)$ in Ω $(i = 1, \ldots, k)$. In the next theorem, we exploit this fact by providing a condition under which k systems are simultaneously Ω-stabilizable. The underlying idea is the following: a finite set of systems is simultaneously Ω-stabilizable if and only if there exists an additional "system" which avoids all of them in Ω. Suppose now that in a set of k systems $\{p_1, \cdots, p_k\}$ one of the systems (say p_1) avoids all the others in Ω. Then by the lemma the systems $p_2, p_3, \cdots, p_k$ are simultaneously Ω-stabilizable by $-p_1^{-1}$. In fact it is then possible to do more than that: it is then possible to find an Ω-stabilizing controller for the whole set $\{p_1, \cdots, p_k\}$. In addition to this, if one of the systems $p_1, \cdots, p_k$ is strictly proper then the resulting controller is proper. This is essentially what is contained in the next theorem.

Theorem: Let $p_i(s) \in \mathbb{R}(s)$ $(i = 1, \cdots, k)$ and suppose that there exists a j $(1 \leq j \leq k)$ such that $p_j(s)$ avoids $p_i(s)$ in Ω $(i = 1, \cdots, k$ and $i \neq j)$. Suppose also that one of the systems $p_i(s) \in \mathbb{R}(s)$ $(i = 1, \cdots, k)$ is strictly proper. Then the systems $p_i(s)$ $(i = 1, \cdots, k)$ are simultaneously Ω-stabilizable by a proper controller.

Proof: Suppose without loss of generality that $j = 1$. Find an Ω-coprime fractional decomposition of $p_1(s)$, $p_1(s) = (n_1(s)/d_1(s))$ with $n_1(s), d_1(s) \in S(\Omega)$. We know that under our assumptions on Ω, $S(\Omega)$ is an Euclidean ring (see [13] for more details). Hence, there exist $x(s), y(s) \in S(\Omega)$ such that $n_1(s)x(s) + d_1(s)y(s) = 1$. Since $p_1(s)$ avoids $p_i(s)$ in Ω $(i = 2, \cdots, k)$ we have $n_i(s)d_1(s) - d_i(s)n_1(s) \in U(\Omega)$ $(i = 2, \cdots, k)$ and we define $u_i(s) \triangleq n_i(s)d_1(s) - d_i(s)n_1(s) \in U(\Omega)$ $(i = 2, \cdots, k)$. The set Ω is closed in the extended complex plane $\mathbb{C}_\infty$ and therefore $\delta := \min_{i=2, \cdots, k}(\inf_{s \in \Omega}|u_i(s)|/\sup_{s \in \Omega}|x(s)n_i(s) + y(s)d_i(s)|)$ is well defined and strictly greater than zero. We choose ϵ with $0 < \epsilon < \delta$ and claim that $q(s) := (n_1(s) - \epsilon y(s))/(d_1(s) + \epsilon x(s)) \in \mathbb{R}(s)$ avoids $p_i(s)$ in Ω $(i = 1, \cdots, k)$. Indeed, if $i = 1$ then $n_1(s)(d_1(s) + \epsilon x(s)) - d_1(s)(n_1(s) - \epsilon y(s)) = \epsilon(n_1(s)x(s) + d_1(s)y(s)) = \epsilon \in U(\Omega)$. Whereas for $i \geq 2$ we have $n_i(s)(d_1(s) + \epsilon x(s)) - d_i(s)(n_1(s) - \epsilon y(s)) = n_i(s)d_1(s) - d_i(s)n_1(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) = u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s))$. By construction of ϵ it is clear that $u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) \neq 0$ for every $s \in \Omega$ $(i = 2, \cdots, k)$. This shows that $u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) \in U(\Omega)$ $(i = 2, \cdots, k)$ and thus $q(s) = (n_1(s) - \epsilon y(s))/(d_1(s) + \epsilon x(s))$ avoids $p_i(s)$ in Ω $(i = 2, \cdots, k)$. But $q(s)$ also avoids $p_1(s)$ in Ω and thus $-q^{-1}(s)$ is a simultaneous stabilizing controller for $p_i(s)$ $(i = 1, \cdots, k)$. It remains to show that $-q^{-1}(s)$ is proper, i.e., that $q(s)$ has no zeros at infinity. But this follows trivially from the fact that, by assumption, one of the $p_i(s)$ has a zero at infinity and that $q(s)$ avoids $p_i(s)$ at $\infty \in \Omega$. ∎

The assumption that one of the systems is strictly proper can

actually be removed without altering the final result (see [1] for this).

V. EXAMPLE

Let $p_1(s) = (1/s - 1)$, $p_2(s) = (-s/3s + 1)$, $p_3(s) = -(s - 2/5s - 1)$ and $p_4(s) = (s^2 - 3s + 1/7s^2 - s + 2)$. It is easy to see that $p_1(s)$ does not intersect any of the $p_i(s)$ in $\mathbb{C}_{+\infty}$ $(i = 2, 3, 4)$ and hence, by our theorem, the systems p_1, p_2, p_3, and p_4 are simultaneously $\mathbb{C}_{+\infty}$-stabilizable.

We construct a stabilizing controller for these systems by using the proof procedure of the theorem.

A coprime fractional decomposition of $p_1(s)$ is given by $p_1(s) = (n_1(s)/d_1(s)) = ((1/s + 1)/(s - 1/s + 1))$. A solution of the Bezout equation

$$n_1(s)x(s) + d_1(s)y(s) = 1$$

is given by $x(s) = 2$, $y(s) = 1$.

By the proof of the theorem, and for a small enough positive ϵ, we have that

$$q(s) := \frac{n_1(s) - \epsilon y(s)}{d_1(s) + \epsilon x(s)} = \frac{\dfrac{1}{s + 1} - \epsilon}{\dfrac{s - 1}{s + 1} + 2\epsilon} = \frac{1 - \epsilon(s + 1)}{(s - 1) + 2\epsilon(s + 1)}$$

avoids p_1, p_2, p_3, and p_4 in $\mathbb{C}_{+\infty}$. We take $\epsilon = 0.01$ and get $q(s) = (99 - s/101s - 99)$. Finally, using our lemma we have that

$$c(s) := -\frac{1}{q(s)} = \frac{101s - 99}{s - 99}$$

is a simultaneous stabilizing controller for p_1, p_2, p_3, and p_4.

It is even possible to say more. $p_1(s)$ intersects $p_i(s)$ $(i = 2, 3, 4)$ at the unique point $-1 \in \mathbb{C}$ and hence the systems p_1, p_2, p_3, and p_4 are simultaneously Ω-stabilizable for any region Ω that does not contain $\{-1\}$.

REFERENCES

[1] V. Blondel, "Simultaneous stabilization of linear systems: Mathematical solutions, related problems and equivalent formulations," Ph.D. dissertation, Catholic University of Louvain, Belgium, May 1992.

[2] E. Emre, "Simultaneous stabilization with fixed closed-loop characteristic polynomial," *IEEE Trans. Automat. Contr.*, vol. 28, pp. 103–104, 1983.

[3] B. Ghosh and C. Byrnes, "Simultaneous stabilization and pole-placement by nonswitching dynamic compensation," *IEEE Trans. Automat. Contr.*, vol. 28, pp. 735–741, 1983.

[4] B. Ghosh, "Some new results on the simultaneous stabilizability of a family of single–input single–output systems," *Syst. Contr. Lett.*, vol. 6, pp. 39–45, 1985.

[5] ——, "Transcendantal and interpolation methods in simultaneous stabilization and simultaneous partial pole placement problems," *SIAM J. Contr. Optimiz.*, vol. 24, pp. 1091–1109, 1986.

[6] ——, "An approach to simultaneous system design, part 1," *SIAM J. Contr. Optimiz.*, vol. 24, pp. 480–496, 1986.

[7] ——, "An approach to simultaneous system design, part 2," *SIAM J. Contr. Optimiz.*, vol. 26, pp. 919–963, 1988.

[8] J. Helton, "Worst-case analysis in the frequency domain," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 1154–1170, 1985.

[9] H. Kwakernaak, "A condition for robust stabilizability," *Syst. Contr. Lett.*, vol. 2, pp. 1005–1013, 1985.

[10] H. Maeda and M. Vidyasagar, "Some results on simultaneous stabilization," *Syst. Contr. Lett.*, vol. 5, pp. 205–208, 1984.

[11] G. Obinata and J. Moore, "Characterization of controllers in simultaneous stabilization," *Syst. Contr. Lett.*, vol. 10, pp. 333–340, 1988.
[12] R. Saeks and J. Murray, "Fractional representation, algebraic geometry and the simultaneous stabilization problem," *IEEE Trans. Automat. Contr.*, vol. 27, pp. 895–903, 1982.
[13] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: MIT Press, 1985.
[14] ——, Some results on simultaneous stabilization with multiple domains of stability, *Automatica*, vol. 23, no. 4, pp. 535–540, 1987.
[15] M. Vidyasagar and N. Viswanadham, "Algebraic design techniques for reliable stabilization," *IEEE Trans. Automat. Contr.*, vol. 27, pp. 1085–1095, 1982.
[16] K. Wei and B. Barmish, "An iterative design procedure for simultaneous stabilization of MIMO systems," *Automatica*, vol. 24, pp. 643–652, 1988.
[17] K. Wei, "Simultaneous pole assignment for a class of linear time-invariant siso systems," in *Proc. 28th Conf. Decision Contr.*, Tampa, FL, 1989, pp. 1247–1252.
[18] D. Youla, J. Bongiorno, and C. Lu, "Single-loop feedback stabilization of linear multivariable systems," *Automatica*, vol. 10, pp. 159–173, 1974.

The Pole Placement Map, its Properties, and Relationships to System Invariants

John Leventides and Nicos Karcanias

Abstract—A number of new properties of the complex and real pole placement map (PPM) are derived which relate to the dimension of their images and relate them to known system invariants. It is shown that those two dimensions are equal and that their computation is equivalent to a rank determination of the corresponding differential. A new expression of the differential of the PPM, allows the derivation of relationships between the Markov parameters and the Plucker matrix invariant of the system. Finally, conditions for pole assignability are derived, based on the relationships between the rank of the Plucker matrix and the rank of the differential of the PPM.

I. INTRODUCTION

The aim of this note is to establish a number of properties of the pole placement map under complex and real output feedback and especially properties of the image of this map. One of the important questions connected with the pole placement problem under constant (or dynamic) output feedback, is the derivation of a reasonable measure for the size of the set of polynomials, which for a system $S(A, B, C)$ of p-inputs, m-outputs, and n-states can be assigned. We choose as a measure of the size of this set, the dimension of the image of the real or the complex pole placement map (PPM). Although the structure of the image of the complex PPM is different than that of the real PPM (and in fact the complex case is nicer than the real), it is shown that both dimensions of the real and complex PPM (which are invariants of the system) are the same. The above dimensions are also shown to be equal to the rank of the differential of the corresponding PPM at a generic feedback K. The rank of this differential at $K = 0$ was shown [8] to be equal to the rank of $F_0 = [\text{col } CB, \text{col } CAB, \cdots, \text{col } CA^nB]$, (the 'col' operation on a matrix implies the formation of a composite

vector, obtained by superimposing the columns of the matrix) or at an arbitrary K [6] to be equal to the rank of $F_K = [\text{col } CB, \text{col } CHB, \cdots, \text{col } CH^nB]$, where $H = A + BKC$. The latter expression is not very convenient for the calculation of the rank at a generic K; instead, we propose an alternative expression of the form $(DT)_K \bar{P}_S$ where T is a function of K only and $\bar{P}_S$ is the reduced Plucker matrix of the system S and which is a complete invariant [4]. The relationship between the reduced Plucker matrix and the Markov parameters is established; in fact, it is shown that the Markov parameters may be computed by selecting certain rows of the Plucker matrix. It is shown that the rank of the Plucker matrix provides us with an upper bound for both the dimensions of the image of the complex and real PPM as well as an upper bound for the set {rank F_K}. As a result of the above properties, necessary tests for the pole assignability of a system $S(A, B, C)$ are derived.

II. STATEMENT OF THE PROBLEM

Let $S(A, B, C)$ be the state space description of a linear strictly proper system of p inputs, m outputs, and n states. Let also $G(s) = N(s)D(s)^{-1}$ be a coprime matrix fraction description of the transfer function of the system. The pole placement problem is to examine whether there is a solution to the equation

$$\det\left([I, K]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right) = \det([I, K]M(s))$$
$$= s^n + p_n s^{n-1} + \cdots + p_1 \quad (2.1)$$

where $M(s)$ is the column reduced and least degree composite matrix for S, or equivalently, to the equation

$$\det(Is - A - BKC) = s^n + p_n s^{n-1} + \cdots + p_1 \quad (2.2)$$

with respect to $K \in \mathbb{R}^{p \times m}$ and for a given $(p_n, \cdots, p_1) \in \mathbb{R}^n$. Of particular interest is to examine the size of this set of n-tuples. This is the same as in finding how large the image of the function χ is. The function, χ, from $\mathbb{R}^{pm}$ to $\mathbb{R}^n$, maps every K to $(p_n, \cdots, p_1)$ under the relation (2.1) or the equivalent relation (2.2) and is called the pole placement map (PPM) [2]. Its extension $\hat{\chi}$, from $\mathbb{C}^{pm}$ to $\mathbb{C}^n$, is called the complex pole placement map (CPPM). The image of CPPM can be examined more easily than that of PPM since there is sufficient algebraic geometry on the field of complex numbers.

Example 2.1: Consider the strictly proper system S whose transfer function $G(s)$ is expressed as a right coprime MFD as

$$G(s) = \begin{bmatrix} 0 & s^{-1} \\ -s^{-3} & s^{-2} \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ s & s^2 \end{bmatrix}^{-1}.$$

If we apply to $G(s)$ constant output feedback

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

then the closed-loop pole polynomial is given by

$$p(s) = s^4 + k_{21}s^3 + k_{22}s^2 - k_{12}s + k_{22}k_{11} - k_{12}k_{21}$$

and so, the pole placement map defined previously is given by

$$[k_{11}, k_{12}, k_{21}, k_{22}] \to [k_{21}, k_{22}, -k_{12}, k_{22}k_{11} - k_{12}k_{21}]. \quad \square$$

The size of this image is related to the rank of the differential of the CPPM (or PPM). This differential is strongly related to