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# The birth of the joint spectral radius: An interview with Gilbert Strang

Vincent D. Blondel *

*Division of Applied Mathematics, Université Catholique de Louvain, 4 Avenue Georges Lemaitre, B-1348 Louvain-la-Neuve, Belgium*

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It may take many years for a notion to receive the attention it deserves. The definition of the joint spectral radius first appeared in 1960, in a three-pages long paper by Gian-Carlo Rota and Gilbert Strang.[1] When the paper was written, Rota was on the mathematics faculty at MIT where Strang had just arrived as a Moore Instructor. Their paper introduces the concept of the joint spectral radius and contains a proof that the joint spectral radius is equal to the infimum, over all matrix norms, of the largest norm of the matrices. The paper starts with the following sentence:

*The notion of joint spectral radius […], was obtained in the course of some work in matrix theory.*

There is no further mention in the paper of what work in matrix theory it relates to. Here is what Strang wrote about the paper in a recent volume of the collected works of Rota:

*Every few years, Gian-Carlo Rota would ask me whether anyone ever read our paper. After I had tenure, I could tell him the truth: 'not often'.*

And indeed, the paper went unnoticed for more than 30 years. In recent years, Gilbert Strang could change his answer. The notion of joint spectral radius has resurfaced in the last decade in a number of application areas, including wavelets, control theory, number theory, coding and sensor networks. It is now 45 years since the paper which introduced the subject of this special issue was published; so, let's try to find out how the joint spectral radius was actually born.[2]

* Tel.: +32 10 47 23 81; fax: +32 10 47 21 80.
*E-mail address:* vincent.blondel@uclouvain.be

[1] A commented list of the references cited in this paper appears at the end of the paper.

[2] This interview took place on April 24, 2006 at MIT. I express my sincere thanks to Prof. Strang for his collaboration. In addition to being one of the fathers of $\rho$, Gilbert Strang is the author of a number of highly influential scientific papers and textbooks in linear algebra and applied mathematics. Throughout his career he has also taken an active role for the promotion of applied mathematics. He was a President of SIAM (the Society for Industrial and Applied Mathematics) and is a member of the American Academy of Arts and Sciences.

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VB The 1960 paper introducing the joint spectral radius was one of your first papers, was it part of your PhD thesis?

GS No. At that point, I had arrived as a Moore Instructor at MIT and I got to know Rota. He was a new assistant professor. He was very friendly, a very nice person to know. He was a good friend over many years. I suppose I asked that question, that I brought the idea in a conversation and the question appealed to him. Of course, it wasn't major work. Quickly he and I had something to say and we decided to say it. He decided to write the paper. It was Gian-Carlo who formulated it as a problem in a normed linear space where I would have written it in matrix language. He also chose to publish it in the proceedings of a learned society with which he probably had contacts. The paper was published in a somewhat obscure journal and written in a slightly obscure way…

VB Did you have a particular application in mind for the concept?

GS Oh… That was almost 50 years ago! How did the idea come to mind? My memory doesn't produce the reason or the application. Of course for a single matrix, it is an important fact that the roots of the norm of the powers converge to the spectral radius. So probably I just somewhat naturally asked what happened if you had several matrices. My thesis was about finite difference methods; it had matrix analysis and so I was interested in several matrix problems. But serious applications, no. And as a result, we never thought of how to compute the joint spectral radius, and all these other questions that were deeper and much more interesting than what we did…

VB So it was a virgin result?

GS For us it was a virgin result, and a virgin birth. I'll perhaps just mention here a connection with a matrix theorem due to Kreiss that characterizes matrices whose powers are bounded. It is a major theorem, one of my favorites in matrix analysis.

VB What was it like to work with Rota?

GS With this small paper it was not really serious long work. In two or three conversations we could see the main ideas and then the paper was written. He was a great person. At that point he was still working in functional analysis. Later in his career he founded a big part of algebraic combinatorics; so he really changed subjects. He left functional analysis but not before this paper. Maybe that this is why he used this language.

VB The paper went unnoticed for about 30 years.

GS As far as I know, yes. And then I think it was the wavelet paper by Daubechies and Lagarias that used the concept. They had the application, they had particular matrices, and it happened to be a natural idea. Just by chance it was useful for wavelets. And then also algebraists became interested and there was this result by Berger and Wang that I really like, they showed that you could use the eigenvalues of matrix products in place of the norms. That was before the finiteness conjecture. It was a non-trivial result.

VB There have since then been a number of applications in other areas. Did you expect this?

GS Oh no! And that's terrific! Well, I guess that's the nice thing about mathematics; simple natural ideas appear in many places. And so I am really looking forward to this special issue of Linear Algebra and Its Applications!

Prof. G. Strang in his office at MIT.

## Commented list of references

The paper by Rota and Strang that introduces the joint spectral radius is

Gian-Carlo Rota and Gilbert Strang, A note on the joint spectral radius, Proc. Netherlands Academy, 22, pp. 379–381, 1960.

In the book of collected works of Rota, Strang has published a commentary on his joint paper with Rota. That paper also gives a list of about 125 references on the joint spectral radius; today that list would probably be twice as long. The commentary can be downloaded from `http://www-math.mit.edu/~gs/papers/papers.html`, the exact reference is

Gilbert Strang, The joint spectral radius, pp. 74–76 in “Gian-Carlo Rota on analysis and probability. Selected papers and commentaries”, Jean Dhombres, Joseph P.S. Kung and Norton Starr (Eds), Birkhäuser Boston, MA, 2003.

That book also contains the original article “Note on the joint spectral radius” (on pp. 74–76).

The paper of Daubechies and Lagarias to which Strang refers in the interview and that use the concept of joint spectral radius for analyzing the continuity of wavelets is

Ingrid Daubechies and Jeffrey C. Lagarias. Sets of matrices all infinite products of which converge. Linear Algebra Appl., 161, pp. 227–263, 1992.

That paper contains a number of errors and misprints that were later corrected in

Ingrid Daubechies and Jeffrey C. Lagarias. Corrigendum/addendum to: “Sets of matrices all infinite products of which converge”. Linear Algebra Appl., 327, pp. 69–83, 2001.

Lagarias and Wang have formulated in 1995 the finiteness conjecture that was later extensively studied, see

J.C. Lagarias and Yang Wang, The finiteness conjecture for the generalized spectral radius of a set of matrices, Linear Algebra Appl., 214, pp. 17–42, 1995.

As explained in the interview, Berger and Wang have proved that, for bounded sets of matrices, the norm appearing in the definition of the joint spectral radius can be replaced by a spectral radius. This is proved in

M.A. Berger and Yang Wang, Bounded semigroups of matrices, Linear Algebra Appl., 166, pp. 21–27, 1992.

One of Strang's favorite result in matrix analysis is Kreiss' Theorem. A good reference on the history of this result is

E. Wegert and L. N. Trefethen, From the Buffon needle problem to the Kreiss matrix theorem, Amer. Math. Monthly, 101, pp. 132–139, 1994.