Complexity of Control on Finite Automata

Jean-Charles Delvenne
Division of Applied Mathematics, Université catholique de Louvain,
4, avenue Georges Lemaître, B-1348 Louvain-la-Neuve, Belgium
delvenne@inma.ucl.ac.be

Vincent D. Blondel
Division of Applied Mathematics, Université catholique de Louvain,
4, avenue Georges Lemaître, B-1348 Louvain-la-Neuve, Belgium
blondel@inma.ucl.ac.be

Abstract

We consider control questions for finite automata viewed as input/output systems. In particular, we find estimations of the minimal number of states of an automaton able to control a given automaton. We show that on average, feedback control automata are not smaller than open-loop automata.

1 Introduction

In control theory feedback is used to obtain a desired behavior from a system.

A wide variety of arguments have been developed to explain why feedback is so useful. Most of them rely on the notions of uncertainty and lack of information: the system we want to control is not perfectly known, or the signals between the system and its environment suffer from noise, for example. In this chapter, following Egerstedt and Brockett in [3], we study the problem under another light: the complexity argument.

We assume that the system we wish to control is perfectly known and we would like to steer it from a given state to another state in the simplest possible way. Can availability of feedback lead to smaller, less complex controlling devices, as we could expect? And how can we measure complexity?

To formalize these concepts we use automata, which, as finite objects, are easy to understand. Here we consider automata not as devices conceived to simply accept words, but rather as transducers transforming a word to another word of same length. An input letter is read and an output letter is produced at every step. An automaton is thus an input-output discrete-time system with a finite state space and finite input and output alphabets. In the context of this paper, the complexity of an automaton is simply identified with its number of states. Automata may be viewed as an approximations of continuous state space systems, where space, inputs and outputs are quantized by finitely many values.

An automaton can be controlled by another automaton, either in open-loop or in feedback. We find that feedback control automata can be much less complex than open-loop control automata to steer an automaton in particular cases, but not on average. More precisely, we prove the following result. The average number of states needed to drive an n-state automaton from one state to another is

- in the order of $\ln n$ for an open-loop control automaton;
- between the order of $\ln^{1-\epsilon} n$ and the order of $\ln n$ for a feedback control automaton, for any $\epsilon > 0$.

We see that while it is not proved that open-loop and feedback have the same order of complexity, little room is left for a spectacular improvement.

In other words, measuring the output is of little use on a system with a random structure. However it is easy to find examples with particular structures on which feedback leads to controllers of very low complexity. We suggest that it could also be the case for classes of automata, such as quantized linear systems.

Our results are in contrast with those presented in [3], where it is also asked whether feedback is less complex than open loop when controlling automata. However, the results presented in [3] apply to a quite sophisticated variant of automata called free-running feedback automata. Essentially, these are automata that do not read an input letter at every time step, and where an input letter contains a description of the feedback law applied to the automaton. Thus the input alphabet may be quite large. The complexity of control is the length of the input word realizing the objective, multiplied by the logarithm of the input alphabet cardinality. Then for such devices, with hypotheses on the structure of the automaton, a strategy based on mix of open-loop and feedback is proved to be less complex than pure open-loop.

This work is also in the spirit of [1], [2], [4] (among others) where systems have a continuous state space, but discrete time, discrete inputs and discrete outputs. Those articles study the amount of information needed to control the system. We pursue similar goals with quite different methods.

Before we can control a system, we must sometimes identify it. In 1956, Moore [5] proposed algorithms to identify a black-box automaton, on which we can test inputs and observe the corresponding outputs. Then Trakhtenbrot, Barzdin and Korshunov developed a probabilistic point of view and showed that automata are much easier to identify on average than in the worst case. They also designed efficient algorithms able to identify "most" automata; see [6]. We stick to a similar framework, but explore one step beyond: supposing that the automaton is correctly identified, how can we control it in the least complex way?

The main theorem is formulated in Section 2, along with preliminary definitions. It is proved in Sections 3, 4 and 5. Conclusions and ideas for future work are presented in Section 6.

2 Formulation and results

In this section we give precise definitions of automaton, control, complexity of control. Then we describe our main result.

Definition 1 *An* automaton *is a sextuple* $(Q, q, X, Y, \delta, \gamma)$*, where*

- Q *is a finite set, the elements of which are called the* states*;*
- X *is a finite set, called the* input alphabet*;*
- $q \in Q$ *is the* initial state*;*
- Y *is a finite set, called the* output alphabet*;*
- $\delta \in Q^{Q\times X}$ *is the* transition function*;*
- $\gamma \in Y^{Q\times X}$ *is the* output function.

An automaton can be seen as an input-output dynamical system. Given an input word $x_0x_1\dots x_n$, the automaton starts from the initial state q, moves to state $q' = \delta(q, x_0)$ and emits the output letter $y_0 = \gamma(q, x_0)$. Then it reads the input letter x_1, moves to state $q'' = \delta(q', x_1)$ and emits $y_1 = \delta(q', x_1)$. Finally, the output word corresponding to $x_0x_1\dots x_n$ is $y_0y_1\dots y_n$ (see Figure 1 for an example).

We set $\delta(q, x_0x_1\dots x_k) = \delta\Big(\delta\big(\dots\delta(\delta(q, x_0), x_1),\dots\big), x_k\Big)$, for any $x_0x_1\dots x_k$ in X^*. Recall that X^* is the set of finite words on alphabet X.

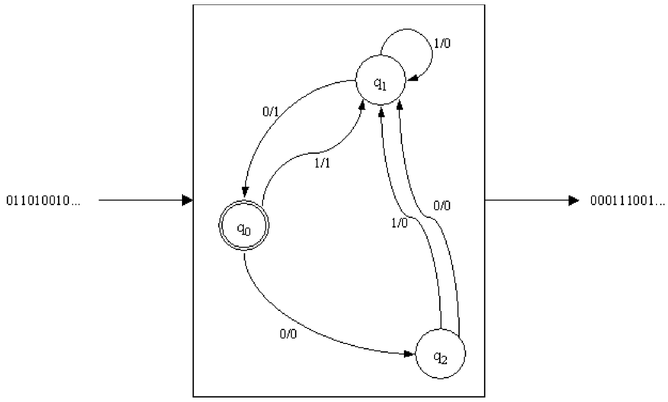


Figure 1: An example of automaton. The vertices are the states, and the edges have labels of the form x/y, where x is an input symbol and y an output symbol. The initial state is q_0. Upon the input of the word 011010010, the automaton is successively in the states q_0, q_2, q_1, q_1, q_0, q_1, q_0, q_2, q_1, q_1, q_0 and outputs the word 000111001.

If X has only one symbol then we say that the automaton is *inputless*, because the input contains no information. The length of the sequence is simply the time variable.

Suppose we have an automaton and we would like to drive it from one state to another state, by feeding the automaton with an appropriate word. This is what we call the reachability problem.

Definition 2 *The reachability problem:*

- *Instance: An automaton A (called the* system automaton*) and a state of A (the* target state*).*
- *Problem: Find a word (the* control word*) of X^* that makes A pass —at least once— through the target state.*

The control word can be chosen to be the output of another automaton. We compare two kinds of strategies in order to solve a reachability problem: open-loop and feedback.

- *The open-loop strategy:* An inputless automaton O (the *control* automaton) is connected to the input of the target automaton, as indicated on the top of Figure 2.

- *The feedback strategy:* An automaton F (the *control* automaton) is connected in feedback (bottom of Figure 2). We can see it as a game played between the control automaton and the system automaton; the goal of the former is to force the latter to enter the target state. Let us mention a detail: for the process to be completely specified, the first output of F must be specified, too.

Thus an *open-loop solution* for an instance of the reachability problem is an inputless automaton that makes the system automaton evolve and reach the target state, when connected in open-loop as described above. A *feedback solution* is composed of a control automaton and a symbol of its output alphabet, that makes the system automaton reach the target state when the control automaton is connected in feedback with the system automaton and gives the specified symbol as first output.

Our goal is to find for both strategies an estimate of the number of states of the control automaton needed to steer a given system automaton A. We would like also to know whether open-loop control is more complex than feedback control. In other words, we ask the following question: does the possibility to make a measurement on the system lead to less complex controllers?

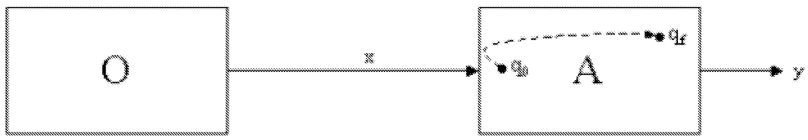
Given a solvable instance of the reachability problem, we naturally define the *open-loop complexity* of the instance as the number of states of the smallest open-loop solution. The *feedback complexity* of the instance is the number of states of the smallest feedback solution.

For the sake of simplicity, we will focus on automata with binary input and binary output alphabets. However our results have immediate extension to alphabets of arbitrary fixed cardinality.

The following example shows that feedback can be much less complex than open-loop.

Example 1 *We want to solve the reachability problem for the n-state automaton represented on Figure 3 and the target state q_{n-1}. We assume that the word $x_0x_1 \dots x_{n-2}$ is not eventually periodic, except trivially. An inputless automaton producing this word as output must therefore have at least $n-1$ states. And the complexity of the open-loop strategy for this instance is $n-1$.*

But the particular form of the output function allows a one-state automaton put in feedback to solve the problem. Actually, even a zero-state automaton suffices! It is enough to connect the output of the automaton directly to the input.



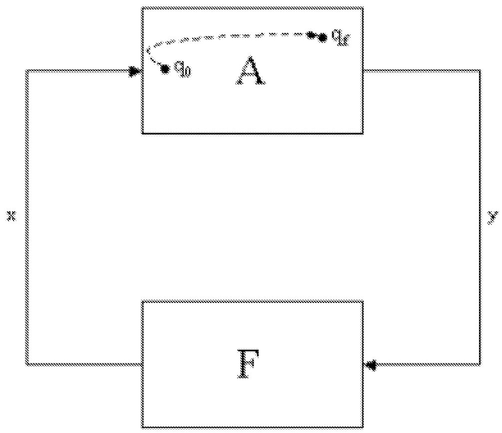


Figure 2: We want to drive A from q_0 to q_f. Above: The open-loop strategy. Note that no input has been drawn towards O, since only one input word is possible, up to the length: it may be seen as the time variable. Below: The feedback strategy. The control automaton F is supposed "to play first": the first value of $y \in Y$ must be given.

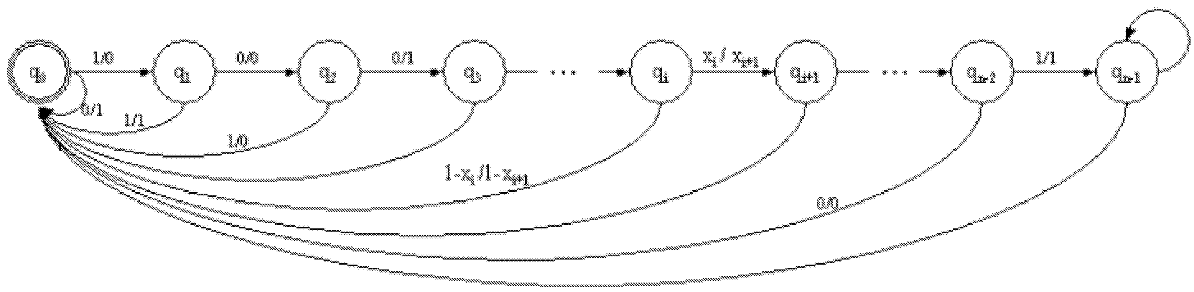


Figure 3: An example of automaton for which the feedback strategy is much less complex then open-loop. The initial state is q_0, and we want to reach state q_{n-1}. The word $x_0 x_1 x_2 \dots x_{n-2}$ is not eventually periodic. In the open-loop strategy, an n_1-state control automaton is needed. In the feedback strategy, a zero-state automaton is enough.

As announced in the introduction, the gap of complexity between open-loop and feedback is quite small on average.

To make this statement precise, we must endow the set of n-state automata with a probability measure.

Let us compute the number of different n-state automata with input and output alphabets of cardinalities x and y respectively. From every state x edges must be drawn and for each of these there are n possible end states and y possible outputs. This leads to $(yn)^{xn}$ possible graphs. For any of them there is a choice of n possible initial states, and so there are finally $n(yn)^{xn}$ different automata. For simplicity, we suppose that the set of states is always $\{0, 1, \dots, n-1\}$.

Of course, many different automata are *isomorphic*, i.e., they are identical up to a permutation of the states. And even more automata define the same input/output function on words.

However, sticking to the formalism of [6], we shall do averages on the class of $n(yn)^{xn}$ different automata. For that purpose we endow this set with uniform probability measure. A random n-state instance of the reachability problem is composed of a random n-state automaton and a random state chosen uniformly among $\{0, 1, \dots, n-1\}$. From now we will also restrict ourselves to the case $x = y = 2$, as already mentioned.

When we say that a statement is true for "almost all n-state automata", we mean that it is true for "a proportion of n-state automata that tends towards 1 as n increases".

We are now able to state our main theorem. A *solvable* n-state instance of the reachability problem is made of an n-state system automaton and a target state that is reachable from the initial state.

Theorem 1 *Consider the uniform probability measure on n-state automata.*

There are constants C_1, C_2 such that for any large enough n, the expected number of states of the smallest open-loop control automaton that solves a solvable n-state instance reachability problem is between $C_1 \ln n$ and $C_2 \ln n$.

For every $\epsilon > 0$, there is a constant C_ϵ such that the expected number of states of the smallest feedback control automaton that solves a solvable n-state instance reachability problem is between $C_\epsilon \ln^{1-\epsilon} n$ and $C_2 \ln n$.

In other words, the expected complexity of open-loop is in $\Theta(\ln n)$, while the expected complexity of feedback is in $\Omega(ln^{1-\epsilon})$ (for all $\epsilon > 0$) and $\mathcal{O}(\ln n)$.

Proof. Corollary 1 of Section 4 proves that the complexity for both strategies is in $\mathcal{O}(\ln n)$. Proposition 3 of Section 5 proves that the open-loop complexity is in $\Omega(\ln n)$. Proposition 5 of Section 5 proves that the complexity of feedback is in $\Omega(\ln^{1-\epsilon} n)$. $\square$

Sections 3, 4 and 5 are devoted to the proof of Theorem 1, and may be skipped in a first reading.

3 Solvable instances of the reachability problem

It is clear that the expected complexity can be computed only over solvable instances. The following proposition essentially says that more than one third of the instances of the reachability problem are solvable. Recall that we consider only automata with binary input and output alphabets.

Proposition 1 *For any $\alpha < \frac{1}{e}$ and any large enough n, a random instance of the reachability problem is solvable with probability at least α. More precisely, for almost all n-state automata at least $\lfloor \alpha n \rfloor$ states are reachable from the initial state.*

Proof. Let $Q = \{0, 1, \ldots, n-1\}$ the set of states of a random automaton.

A subset $R \subseteq Q$ is *invariant* if, starting from a state of R, it is impossible to leave R. In other words, no edge leaves R.

We are going to prove that in almost all n-state automata, there is no invariant subset of states of size at most $\lfloor \alpha n \rfloor$, for $0 < \alpha < 1/e$. As the set of states that are reachable from the initial state is an invariant set, this is enough to prove the proposition.

The probability for any set R of size r to be invariant is $(\frac{r}{n})^{2r}$, since $2r$ edges start from R, and each of them arrive in R with probability r/n.

For any $R \subseteq Q$, consider the random variable I_R that is equal to 1 if the set R is invariant and 0 otherwise. The expectation of I_R is the probability of the event $I_R = 1$, which is $(r/n)^{2r}$. The probability that there is a invariant set of size at most αn is equal to

$$\mathbb{P}\left(\sum_{\substack{R \subseteq Q \\ 0 < \mathrm{card}(R) \leq \alpha n}} I_R \geq 1 \right),$$

where $\mathbb{P}$ denotes the probability of an event.

For any non-negative integer-valued random variable X, the well-known Markov's inequality

$$\mathbb{P}(X > 0) \leq \mathbb{E}X$$

holds, where $\mathbb{E}$ is the expectation. Indeed, $\mathbb{E}X = \sum_{n>0} n\mathbb{P}(X = n) \geq \sum_{n>0} \mathbb{P}(X = n) = \mathbb{P}(X > 0)$.

Here we take $X = \sum_{0<\mathrm{card}(R)\leq \alpha n} I_R$. The probability that there is an invariant set of size at most αn is

$$\begin{aligned}
\mathbb{P}(X > 0) &\leq \mathbb{E}X \\
&= \sum_{0<\mathrm{card}(R)\leq \alpha n} \mathbb{E}I_R \\
&= \sum_{r=0}^{\lfloor \alpha n \rfloor} \binom{n}{r} \left(\frac{r}{n}\right)^{2r}.
\end{aligned}$$

If we prove that this last quantity converges to 0 as $n \to \infty$, then we conclude from Markov's inequality that almost all automata of size n have have no invariant set of states of size at most αn. This argument is an example of the so-called *first moment method.*

It remains to prove that $\lim_{n\to\infty} \sum_{r=1}^{\lfloor \alpha n \rfloor} \binom{n}{r} (\frac{r}{n})^{2r} = 0$. For every n and for every term of the series, the following relations hold:

$$\begin{aligned}
\binom{n}{r} \left(\frac{r}{n}\right)^{2r} &\leq \frac{r^{2r}}{r! n^r} \\
&\leq \frac{e^r r^r}{n^r} \leq (e\alpha)^r.
\end{aligned}$$

We used Stirling's approximation $r! \geq (r/e)^r$, and the fact that $r \leq \alpha n$. Hence

$$\sum_{r=1}^{\lfloor \alpha n \rfloor} \binom{n}{r} (\frac{r}{n})^{2r} \leq \sum_{r \geq 1} (e\alpha)^r = \frac{1}{1 - e\alpha} - 1.$$

Moreover, for any fixed r, the sequence $\binom{n}{r}(\frac{r}{n})^{2r} \leq \frac{e^r r^r}{n^r}$ converges to 0 as $n \to \infty$. Thus we may apply Lebesgue's dominated convergence theorem to swap limit and summation as follows:

$$\lim_{n\to\infty} \sum_{r=1}^{\lfloor \alpha n \rfloor} \binom{n}{r} (\frac{r}{n})^{2r} = \sum_{r=1}^{\lfloor \alpha n \rfloor} \lim_{n\to\infty} \binom{n}{r} (\frac{r}{n})^{2r} = 0.$$

□

It is easy to improve the ratio $1/e$ to $1/2$, using finer approximations of binomial coefficients. But we don't know how far this ratio can be improved. It might be the case that all states are reachable for almost all automata. As far as we know this is an open problem.

4 An upper bound on complexity

If a state of an n-state automaton is reachable from the initial state, then it is so with an input word of length at most n, and so the complexity of feedback and open loop are both bounded by n. On average much more can be said.

Theorem 2 *There is a constant C such that for almost all n-state automata, if a state is reachable from another state, then it is reachable with an input word of length at most $C \ln n$.*

We do not prove this result here; the interested reader is referred to [6].

Corollary 1 *There is a constant C such that for almost all n-state instances of the reachability problem, feedback and open loop control are both bounded by $C \ln n$. In particular, the expected complexity of a solvable instance is in $\mathcal{O}(\ln n)$ for both strategies.*

5 Lower bounds on complexity

Let us a take a random instance of the reachability problem of size n, i.e., a random n-state automaton (with binary input and output alphabets) with a random target state. Take a random p-state inputless automaton (with binary output alphabet). We will denote $\mathcal{P}^{ol}(n,p)$ the probability that the p-state control automaton solves the n-state instance of the reachability problem in open-loop.

Now take a random p-state automaton (with binary input and output alphabets). We will denote $\mathcal{P}^{f}(n,p)$ the probability the the p-state control automaton solves the n-state instance in feedback.

In this section, we compute estimates of these quantities that allows us to give lower bounds on complexity.

First of all, a little technical lemma:

Lemma 1 *There is a constant $c > 0$ such that for any $p \in \mathbb{N}$, any $n \geq p^2$ and $k = \sqrt{2pn \ln n}$ we have*

$$\frac{e^{-k}}{(1-\frac{k}{pn})^{pn-k+\frac{p}{2}}} \leq c\frac{1}{n}.$$

Proof. Taking the logarithm:

$$\begin{aligned}
\ln \frac{e^{-k}}{(1-\frac{k}{pn})^{pn-k+\frac{p}{2}}} &= -\sqrt{2pn\ln n} - \big(pn - \sqrt{2pn\ln n} + \frac{p}{2}\big)\ln\big(1 - \sqrt{2\frac{\ln n}{pn}}\big) \\
&= -\sqrt{2pn\ln n} - \big(pn - \sqrt{2pn\ln n} + \frac{p}{2}\big)\big(-\sqrt{2\frac{\ln n}{pn}} - \frac{\ln n}{pn} + \mathcal{O}((2\frac{\ln n}{pn})^{3/2})\big) \\
&= -\ln n - \sqrt{\frac{2\ln n}{pn}}\ln n + \sqrt{\frac{p\ln n}{2n}} + \frac{\ln n}{2n} + \mathcal{O}((2\frac{\ln n}{pn})^{3/2}pn).
\end{aligned}$$

The second equality used the Taylor expansion $\ln(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^3)$. All terms of the last member except the first are bounded by a constant, provided that $n \geq p^2$. Thus

$$\frac{e^{-k}}{(1-\frac{k}{pn})^{pn-k+\frac{p}{2}}} \leq c\frac{1}{n},$$

for some $c > 0$.□

5.1 A lower bound for open-loop complexity

We may obtain the following bound on $\mathcal{P}^{ol}(n,p)$:

Proposition 2 *For any large enough p and any $n \geq p^2$, the probability $\mathcal{P}^{ol}(n,p)$ for a random inputless p-automaton to solve in open-loop a random n-state instance of the reachability problem is bounded by*

$$\mathcal{P}^{ol}(n,p) \leq \sqrt{\frac{p}{n}\ln n}.$$

Proof. We fix an arbitrary inputless control automaton of p states.

In a first step we give an upper bound on the probability for the path described in the random target automaton to pass through at least k of the n states.

In a second step we use Lemma 1 to compute a bound on $\mathcal{P}^{ol}(n,p)$.

Step 1 To explore at least k different states, we must explore at least k different edges. Each time a new edge is being explored, the end state of the edge is randomly chosen among the n states. If this new edge occurs after l steps (i.e., when the path already drawn has length l), and if less than k states are explored at that moment, then at most $n - \lfloor l/p \rfloor$ are allowed for the end state of the edge (in order not to loop before having seen k states). This is because if the couple (state of the control automaton, state of the system automaton) is the same at two steps, then the system enters a loop; thus $\lfloor l/p \rfloor$ states are "forbidden". Moreover, the ith new edge is discovered after a length $l \geq i$. Thus the probability to explore at least k edges is less than the following product of k factors:

$$\overbrace{\frac{n}{n}\cdots\frac{n}{n}}^{p}\overbrace{\frac{n-1}{n}\cdots\frac{n-1}{n}}^{p}\cdots\cdots\overbrace{\frac{n-\lfloor k/p\rfloor}{n}\cdots\frac{n-\lfloor k/p\rfloor}{n}}^{\leq p}, \tag{1}$$

where each factor —except maybe the last— is repeated p times. This is smaller than:

$$\begin{aligned}\left(\frac{n!}{(n-\lfloor k/p\rfloor)!n^{\lfloor k/p\rfloor}}\right)^p &\leq \left(\big(1+\mathcal{O}(1/n)\big)\frac{e^{-\lfloor k/p\rfloor}}{(1-\frac{1}{n}\lfloor\frac{k}{p}\rfloor)^{n-\lfloor k/p\rfloor+\frac{1}{2}}}\right)^p \\ &\leq \mathcal{O}(1)\frac{e^{-k'}}{(1-\frac{k'}{pn})^{pn-k'+\frac{p}{2}}},\end{aligned}$$

where $k' = k - p$. Stirling's formula has been used to derive the first inequality. The error factor is in $\mathcal{O}(1)$ thanks to the assumption $n \geq p^2$.

Step 2 Let k be equal to $\sqrt{2pn\ln n}+p$. Then Lemma 1 ensures that

$$\frac{e^{-k'}}{(1-\frac{k'}{pn})^{pn-k'+\frac{p}{2}}} \le \mathcal{O}(1)\frac{1}{n}.$$

Then the expected number of states visited by the control automaton in the system automaton is at most

$$\big(1-\mathcal{O}(1)\frac{1}{n}\big)\big(\sqrt{2pn\ln n}+p\big)+\mathcal{O}(1)\frac{1}{n}n \le 2\sqrt{pn\ln n}$$

for p large enough. Thus, $\mathcal{P}^{ol}(n,p) \le 2\sqrt{\frac{p}{n}\ln n}$ for p large enough. □

In fact, the proof shows that the result holds not only for a random control automaton but for any control automaton.

We now derive a lower bound on open-loop complexity.

Proposition 3 *The expected complexity of the open-loop strategy for an n-state solvable instance of the reachability problem is in $\mathcal{O}(\ln n)$.*

Proof. The behavior of an inputless automaton is completely characterized by the infinite word it produces as output. This output word is eventually periodic.

The number of inputless p-state automata generating different output sequences is $p2^p$. Indeed, p is the sum of lengths of the non-periodic part and the period, and there are p possibilities for the length of the period.

So, using Proposition 2, the probability that an n-state instance of the reachability problem is solved by at least one p-state open-loop automaton is (provided that $n \ge p^2$ and p is large enough) bounded above by

$$4p2^p\sqrt{\frac{p}{n}\ln n}.$$

Assume this quantity is lower than 1/6:

$$4p\,2^p\sqrt{\frac{p}{n}\ln n} \le \frac{1}{6},$$

which may be written

$$4p^{3/2}2^p \le \frac{1}{6}\sqrt{\frac{n}{\ln n}}.$$

This is satisfied if

$$p \le c_0 \ln n,$$

for some $c_0 > 0$.

Thanks to Proposition 1, we know that for n large enough, at least one third of the n-state instances of the reachability problem are solvable. If $p \leq c_0 \ln n$, it means that at least half of the n-state solvable instances are not solved by any open-loop control automaton of at most p states. Hence the average complexity of an n-state solvable instance is at least $\frac{c_0}{2} \ln n$. So the result follows with $c = c_0/2$. $\Box$

5.2 A lower bound for feedback complexity

We now adapt the arguments of the preceding subsection to the case of feedback.

Proposition 4 *For any large enough p and for any $n \geq 4p^2$, the probability $\mathcal{P}^f(n,p)$ for a random p-state automaton to solve in feedback a random n-state instance of the reachability problem is bounded by*

$$\mathcal{P}^f(n,p) \leq 2\sqrt{2pn \ln n}.$$

Proof. First we fix an arbitrary feedback control automaton.

The main idea of the proof is the same as in proof of Proposition 2. There we used the fact that if the system automaton is twice in the same state while the open-loop control automaton is also in the same state, then the whole system falls into a loop and will not discover new states. The case of the feedback strategy is slightly more complicated. Indeed, knowing the states of the control and system automata is not sufficient to determine the evolution of the system. Suppose that the system automaton A "has just played" (remember the game-theoretic terminology): it went from q to q' according to the transition $\delta_A(q,x) = q'$. Let $y = \gamma_A(q,x)$ the output produced during this transition. During this time the control automaton F is in state, say, s. But now F must play! It is going to enter the state $\delta_F(s,y)$. So to compute the next states, s, q' and y are needed. Since y can take two values, A (having played) and F (going to play) can only be twice in state q' and s respectively.

Heuristically, we thus expect the same result as in Proposition 2, except that that p becomes $2p$. The rest of the proof rigorously establishes this fact.

Let the states of F be denoted by $s_1, \ldots, s_p$.

We want to estimate the probability that at least k states of the system automaton are visited when F is connected in feedback to it. To explore at least k different states of the system automaton, we must explore at least

k different edges. Every time a new edge is being explored, the end state of the edge is chosen randomly among the n states. Suppose that after l steps (i.e., when the path already drawn has length l), less than k states have been reached, the system automaton is in state q_{i_l}, the output of F is x_{m_l} and the edge starting from q_{i_l} labelled by x_{m_l} has never been used: we must choose an end state for this new edge.

Suppose in addition that the current state of F is s_{j_l}. The end state of the new edge (i.e., $\delta_A(q_{i_l}, x_{m_l})$) is chosen among three kinds of states:

- Those that have been reached once while F was in state s_{j_l}. Let a_{j_l} be the number of such states. If one of these states is chosen, then the probability for the trajectory to fall into a loop is at least $1/2$, because one of the two possible values for $\gamma_A(q_{i_l}, x_{m_l})$ is "forbidden".

- Those that have been reached (exactly) twice while F was in state s_{j_l}. Let b_{j_l} the number of such states. They may not be chosen as end state of the edge, since the whole system would loop (remember we do not want to loop before having explored k states).

- Those that have not been explored yet when F was in state s_{j_l} (they might have been explored when F was in another state, of course). Their number is $r_{j_l} = n - a_{j_l} - b_{j_l}$. They may be chosen without restriction.

In the first case, a_{j_l} is decreased by one ("$a_{j_l} := a_{j_l} - 1$") and b_{j_l} is increased by one. In the third case, r_{j_l} is decreased by one and a_{j_l} is increased by one.

So the probability to choose a "good" end state for a new edge (i.e., not to fall into a loop) is less than $\frac{1}{2}\frac{a_{j_l}}{n} + \frac{r_{j_l}}{n}$, and the quantity $\frac{1}{2}a_{j_l} + r_{j_l}$ decreases by $1/2$, in every case (starting from n, for $l = 0$).

So the probability of discovering at least k edges is bounded by a weighted average of terms of the form

$$\prod_{l \geq 0 |\ \text{new edge drawn at step } l \text{ and } < k \text{ states reached}} \frac{1}{2}\frac{a_{j_l}}{n} + \frac{r_{j_l}}{n},$$

taken over all the paths visiting k states. For each of these terms, we denote t_i the number of occurrences of state s_i in the corresponding path. Then the term may be expressed (by a permutation of factors) :

$$\prod_{1 \leq i \leq p} \frac{n}{n}\frac{n - \frac{1}{2}}{n}\frac{n - \frac{2}{2}}{n} \cdots \frac{n - \frac{t_i}{2}}{n}.$$

Moreover, $\sum_{1\le i\le p} t_i \ge k$. So it is routine to see that each term is bounded by

$$\Big(\frac{n}{n}\frac{n-\frac{1}{2}}{n}\frac{n-\frac{2}{2}}{n}\cdots\frac{n-\lfloor\frac{k}{p}\rfloor\frac{1}{2}}{n}\Big)^p,$$

itself bounded by

$$\Big(\frac{n}{n}\frac{n}{n}\frac{n-1}{n}\frac{n-1}{n}\cdots\frac{n-\lfloor\frac{k}{2p}\rfloor}{n}\frac{n-\lfloor\frac{k}{2p}\rfloor}{n}\Big)^p,$$

where each factor is repeated twice.

Comparing with Equation 1, we see that nothing has changed, except that p has become $2p$. So the end of the proof remains unchanged in its principle: the conclusion is that the probability that $\mathcal{P}^f(n,p)$ is less than $2\sqrt{\frac{2p}{n}\ln n}$. $\square$

Again, the proof holds not only for a random feedback control automaton but for any feedback control automaton.

Thus:

Proposition 5 *For every $\epsilon > 0$, the expected number of states of the smallest feedback control automaton that solves an n-state solvable instance of the reachability problem is, for some constant C_ϵ and all large enough n, greater than $C_\epsilon \ln^{1-\epsilon} n$.*

Proof. The number of different p-state automata is $p(2p)^{2p}$. So, using the preceding proposition, the probability that a random n-state instance of the reachability problem is solved by at least one p-state feedback automaton is bounded above by:

$$2p(2p)^{2p}\sqrt{\frac{2p}{n}\ln n}.$$

We want this quantity to be lower than 1/6:

$$2p(2p)^{2p}\sqrt{\frac{2p}{n}\ln n} \le \frac{1}{6}.$$

After some manipulations it becomes

$$(2p)^{2p+\frac{3}{2}} \le \frac{1}{6}\sqrt{\frac{n}{\ln n}},$$

or

$$(2p+3/2)\ln(2p) \le \frac{1}{2}(\ln n - \ln\ln n) - \ln 6.$$

This is satisfied if

$$p \leq c_0 (\ln n)^{1-\epsilon},$$

for some $c_0 > 0$, any $\epsilon > 0$ and n large enough.

Thanks to Proposition 1, we know that for n large enough, at least one third of the n-state instances of the reachability problem are solvable. If $p \leq c_0 \ln^{1-\epsilon} n$, it means that at least half of the solvable n-state instances are not solved by any p-state open-loop control automaton. Hence the expected complexity of an n-state instance is at least $\frac{c_0}{2} \ln n$. So the result follows with $c = c_0/2$. □

6 Conclusions

We studied finite automata as input/output systems, and the complexity needed to control them. It happens from our results that making measurements on the output of a system is not very useful for automata with a completely random structure.

Of course, feedback cannot be more complex than open-loop, and we gave an example where it is actually much less complex.

We are lead to the conclusion that feedback is useful to gain complexity only in connection with a particular structure of the automaton.

Future work could be devoted to finding classes of particular systems for which feedback is particularly interesting. For instance, it would be natural to see what happens if the automaton results from the quantization of a continuous linear system.

7 Acknowledgements

J.-C. D. is a Research Fellow with the Belgian National Research Fund (FNRS). This paper presents research results of the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.

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