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Avoidance and intersection in the complex plane, a tool for simultaneous stabilization

V. Blondel, G. Campion and M. Gevers
Department of Automatic Control - University of Louvain
Place du Levant 3, B-1348 Louvain-La-Neuve, Belgium

Abstract

In this paper we study the following problem: "under what condition(s) is it possible to find a single controller which stabilizes k siso linear time invariant plants $p_i(s)$ $(i = 1, ..., k)$?". We show that the problem admits a solution if and only if an avoidance condition in the complex plane is satisfied and we use this last result to derive a sufficient condition for k plants to be simultaneously stabilizable.

1 Introduction

Simple questions can not always be simply answered. In this paper we give a very partial answer to a simple question in control theory which, for being open for ten years, does not seem to have a simple answer. The question is known under the name of *simultaneous stabilization problem* and is the following: "Under what condition(s) is it possible to find a single controller $c(s)$ which stabilizes k siso linear time invariant systems $p_i(s)$ $(i = 1, ..., k)$?". This question has already been solved for one or two plants: when $k = 1$, it is always possible to find a stabilizing controller, and when $k = 2$ a tractable necessary and sufficient condition, known as the *parity interlacing property*, exists (see [4]). The problem becomes harder when $k \geq 3$ (in fact no satisfactory answer exists even when $k = 3$) and most papers on the simultaneous stabilization problem deal either with necessary or with sufficient conditions ([1], [3], [7]).

In this paper we present the problem as an avoidance problem between complex valued functions. Roughly speaking, a set of k siso linear time invariant plants $\{p_1(s), ..., p_k(s)\}$ will be shown to be simultaneously stabilizable iff there exists a $k + 1^{th}$ plant $p_{k+1}(s)$ which avoids, in a sense that we will define, the plants $p_1(s), ..., p_k(s)$ for all s in the extended closed right half plane. With this view of the problem we will prove a new sufficient condition under which k plants are simultaneously stabilizable.

2 Notations-Definitions

$\mathbf{R}(s)$ is the set of real rational functions. $\mathbf{C}_\infty$ is the extended complex plane $\mathbf{C} \cup \{\infty\}$ adequately topologized. Ω is any subset of $\mathbf{C}_\infty$. We shall suppose troughout this note that Ω is symmetric with respect to the real axis (if $s \in \Omega$ then $\bar{s} \in \Omega$), that it is closed, simply connected and that its complement in $\mathbf{C}_\infty$ contains at least one value of $\mathbf{R} \cup \{\infty\}$. Ω is to be thought of as the complement in $\mathbf{C}_\infty$ of a region of stability. Classical examples of regions Ω are the closed unit disc and the extended closed right half plane which correspond respectively to the complement in $\mathbf{C}_\infty$ of the discrete and continuous time stability regions. A real rational function $f(s) \in \mathbf{R}(s)$ is *Ω-stable* if it has no poles in Ω. $S(\Omega)$ is the set of all Ω-stable functions. A real rational function $f(s) \in \mathbf{R}(s)$ belongs to $M(\Omega)$ if it has no zeros in Ω. Finally, we define $U(\Omega) := M(\Omega) \cap S(\Omega)$.

3 Avoidance and intersection

Immediate checking shows that, whatever Ω, $S(\Omega)$ is a commutative ring. It is also known that under our hypothesis on Ω, the field of fractions of $S(\Omega)$ is $\mathbf{R}(s)$ (see for example [5], p.50). This means that if $p(s) \in \mathbf{R}(s)$ then there exist $n(s), d(s) \in S(\Omega)$ such that $p(s) = \frac{n(s)}{d(s)}$ where $n(s)$ and $d(s)$ have no common zeros in Ω. Such a fractional decomposition of $p(s)$ is called an *Ω-coprime decomposition*. We may now define what we mean by the intersections of two functions $p_1(s), p_2(s) \in \mathbf{R}(s)$ in Ω.

Definition. *Let $p_1(s), p_2(s) \in \mathbf{R}(s)$ and let $n_i(s), d_i(s) \in S(\Omega)$ be fractional Ω-coprime decompositions of $p_i(s)$ $i = 1, 2$. The intersections of $p_1(s)$ and $p_2(s)$ in Ω are the zeros of $n_1(s)d_2(s) - d_1(s)n_2(s) \in S(\Omega)$ in Ω. If $n_1(s)d_2(s) - d_1(s)n_2(s) \in U(\Omega)$ then $p_1(s)$ and $p_2(s)$ have no intersections in Ω and we say that they avoid each other in Ω.*

This definition may look somewhat mysterious. In fact it is very natural and the procedure to compute the intersections between plants is very simple. From our assumptions on Ω it follows that the module of a function in $S(\Omega)$ has an upper bound on Ω and that of a function in $M(\Omega)$ has a lower bound on Ω. This allows us to prove the next lemma.

Lemma 1. *Let $p_1(s) \in S(\Omega)$ and $p_2(s) \in M(\Omega)$. Then there exists a real $L > 0$ such that $lp_2(s)$ avoids $p_1(s)$ in Ω $\forall l > L$.*

Proof. $p_1(s) \in S(\Omega)$ and $p_2(s) \in M(\Omega)$ and hence there exist trivial fractional Ω-coprime decompositions $p_1(s) = \frac{n_1(s)}{1}$ and $p_2(s) = \frac{1}{d_2(s)}$ where $n_1(s), d_2(s) \in S(\Omega)$. Define $L = \sup_{s \in \Omega} | n_1(s)d_2(s) | > 0$. L is finite because Ω is closed in the extended complex plane and $n_1(s)d_2(s)$ has no poles in Ω. It is clear that for every $l > L$, $n_1(s)d_2(s) - l$ is an element of $S(\Omega)$ which never takes the value zero when $s \in \Omega$. That is, $n_1(s)d_2(s) - l \in U(\Omega)$ $\forall l > L$. In other words, $lp_2(s)$ avoids $p_1(s)$ in Ω for every $l > L$. ■

4 Simultaneous stabilization

In the usual sense a controller $c(s) \in \mathbf{R}(s)$ is said to be a stabilizing controller for a plant $p(s)$ if $p(s)c(s)(1 + p(s)c(s))^{-1}$ is proper and has no poles with positive real part. In other words $c(s)$ stabilizes $p(s)$ if $p(s)c(s)(1 + p(s)c(s))^{-1} \in S(\mathbf{C}_{+\infty})$ where $\mathbf{C}_{+\infty}$ is the extended complex right half plane. It was shown in [5] that this is an ill-stated definition of stability and that it is necessary for practical purposes to ask for internal as well as external stability. A controller $c(s)$ is an *internal stabilizer* of a plant $p(s)$ if all the transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}, c(s)(1+p(s)c(s))^{-1}$ and $p(s)(1 + p(s)c(s))^{-1}$ are in $S(\mathbf{C}_{+\infty})$. Since we want to treat stabilization problems in a general framework, encompassing continuous as well as discrete time stability, we will say that a controller $c(s) \in \mathbf{R}(s)$ ***internally Ω-stabilizes*** (or is an *internal Ω-stabilizer* of) $p(s) \in \mathbf{R}(s)$ if all the transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}, c(s)(1+p(s)c(s))^{-1}$ and $p(s)(1+p(s)c(s))^{-1}$ are in $S(\Omega)$. This notion of internal Ω-stabilization is strongly connected to that of avoidance in Ω.

Lemma 2. *Let $p(s), c(s) \in \mathbf{R}(s)$. Then the controller $c(s)$ internally Ω-stabilizes $p(s)$ if and only if $-c^{-1}(s)$ avoids $p(s)$ in Ω.*

Proof. Let $p(s) = \frac{n_p(s)}{d_p(s)}$ and $c(s) = \frac{n_c(s)}{d_c(s)}$ be Ω-coprime decompositions of $p(s)$ and $c(s)$. It is well known that $c(s)$ internally Ω-stabilizes $p(s)$ iff $n_p(s)n_c(s) + d_p(s)d_c(s) \in U(\Omega)$ (see [5]). This last condition is satisfied if and only if $-c(s)^{-1}$ avoids $p(s)$ in Ω. ■

With this result we may reformulate the simultaneous stabilization problem under the form of an avoidance problem.

Corollary 1. *Let $p_i(s) \in \mathbf{R}(s)$ $(i = 1, ..., k)$. The plants $p_i(s)$ are simultaneously internally Ω-stabilizable if and only if there exists a $c(s) \in \mathbf{R}(s)$ such that $-c^{-1}(s)$ avoids $p_i(s)$ in Ω $(i = 1, ..., k)$.* ■

By using this last result and Lemma 1 it is straightforward to prove the next theorem.

Theorem 1. *Let $p_i(s) \in M(\Omega)$ $(i = 1, ..., k)$ and consider any $c(s) \in M(\Omega)$. Then there exists $\lambda \in \mathbf{R}$ such that $\lambda c(s)$ internally Ω-stabilizes $p_i(s)$ $(i = 1, ..., k)$.*

Proof. By using Lemma 1, for each $p_i(s)$ there exists $L_i > 0$ such that $-c^{-1}(s)$ avoids $lp_i(s)$ in Ω for every $l > L_i$. Define $L_{max} = max_{i=1,...,k} L_i$. Then clearly $-c^{-1}(s)$ avoids $lp_i(s)$ $\forall l > L_{max}$ $(i = 1, ..., k)$. Choose a $\lambda > L_{max}$ then $-c^{-1}(s)$ avoids $\lambda p_i(s)$ $(i = 1, ..., k)$ or, equivalentely, $-(\lambda c(s))^{-1}$ avoids $p_i(s)$ $(i = 1, ..., k)$ and by Corollary 1 the theorem is proved. ■

This last theorem is in fact a well known result in simultaneous stabilization when Ω is the extended right half plane (see for example [7]). If k plants are minimum phase and proper but not strictly proper, then there exists a controller, with arbitrarily specified poles and arbitrarily specified stable zeros, that internally stabilizes $p_i(s)$ $(i = 1, ..., k)$.

The condition of the theorem is a particular situation under which the simultaneous stabilization problem of k plants admits a solution. In the next theorem we prove a new result in the same vein; we give a sufficient condition under which k plants are simultaneously internally Ω-stabilizable. The underlying idea is the following: a finite set of plants is simultaneously stabilizable iff there exists an additional "plant" which avoids all of them (see Corollary 1). Suppose now that in a set of k plants $\{p_1, ..., p_k\}$ one of the plants (say p_1) avoids all the others. Then by Corollary 1 the plants $p_2, p_3, ..., p_k$ are simultaneously stabilized by $-p_1^{-1}$. In fact it is then possible to do more than that: it is then possible to find a stabilizing controller for the whole set $\{p_1, ..., p_k\}$. This is essentially what is contained in our next theorem which is the central result of this paper.

Theorem 2. *Let $p_i(s) \in \mathbf{R}(s)$ $(i = 1, ..., k)$ and suppose that there exist a j $(1 \leq j \leq k)$ such that $p_j(s)$ avoids $p_i(s)$ in Ω $(i = 1, ..., k$ and $i \neq j)$. Then the plants $p_i(s)$ $(i = 1, ..., k)$ are simultaneously internally Ω-stabilizable.*

Proof. Suppose without loss of generality that $j = 1$. Find an Ω-coprime fractional decomposition of $p_1(s)$, $p_1(s) = \frac{n_1(s)}{d_1(s)}$ with $n_1(s), d_1(s) \in S(\Omega)$. We know that under our assumptions on Ω (Ω is symmetric, simply connected and its complement contains at least one value in $\mathbf{R} \cup \{\infty\}$), $S(\Omega)$ is an Euclidean ring (see [5] for more details). Hence there exist $x(s), y(s) \in S(\Omega)$ such that $n_1(s)x(s) + d_1(s)y(s) = 1$. Since $p_1(s)$ avoids $p_i(s)$ in Ω $(i = 2, ..., k)$ we have that $n_i(s)d_1(s) - d_i(s)n_1(s) \in U(\Omega)$ $(i = 2, ..., k)$ and we define $u_i(s) = n_i(s)d_1(s) - d_i(s)n_1(s) \in U(\Omega)$ $(i = 2, ..., k)$. Finally we define $\delta = min_{i=2,...,k} \frac{inf_{s\in\Omega}|u_i(s)|}{sup_{s\in\Omega}|x(s)n_i(s)+y(s)d_i(s)|} > 0$ and choose ϵ with $0 < \epsilon < \delta$. We claim that $q(s) := \frac{n_1(s)-\epsilon y(s)}{d_1(s)+\epsilon x(s)} \in \mathbf{R}(s)$ avoids $p_i(s)$ in Ω $(i = 1, ..., k)$. Indeed, if $i = 1$ then $n_1(s)(d_1(s) + \epsilon x(s)) - d_1(s)(n_1(s) - \epsilon y(s)) = \epsilon(n_1(s)x(s) + d_1(s)y(s)) = \epsilon \in U(\Omega)$. Whereas for $i \geq 2$ we have $n_i(s)(d_1(s) + \epsilon x(s)) - d_i(s)(n_1(s) - \epsilon y(s)) = n_i(s)d_1(s) - d_i(s)n_1(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) = u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s))$. By construction of ϵ it is clear that $u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) \neq 0$ for every $s \in \Omega$ $(i = 2, ...k)$. This shows that $u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) \in U(\Omega)$ $(i = 2, ..., k)$ and thus $q(s) = \frac{n_1(s)-\epsilon y(s)}{d_1(s)+\epsilon x(s)}$ avoids $p_i(s)$ in Ω $(i = 2, ...k)$. Finally, $q(s)$ avoids $p_i(s)$ $(i = 1, ..., k)$ and, by applying Corollary 1, $-q^{-1}(s)$ is a simultaneous stabilizer for all $p_i(s)$ $(i = 1, ..., k)$. ■

5 Example

Let $p_1(s) = \frac{1}{s-1}, p_2(s) = \frac{-s}{3s+1}, p_3(s) = -\frac{s-2}{5s-1}$ and $p_4(s) = -\frac{s^2-3s+1}{7s^2-s+2}$. It is easy to see that $p_1(s)$ does not intersect any of the $p_i(s)$ in $\mathbf{C}_{+\infty}$ $(i = 2, 3, 4)$ and hence, by Theorem 2, the plants p_1, p_2, p_3 and p_4 are simultaneously internally $\mathbf{C}_{+\infty}$-stabilizable. It is even possible to say more. $p_1(s)$ intersects $p_i(s)$ $(i = 2, 3, 4)$ at the unique point $-1 \in \mathbf{C}$ and hence the plants p_1, p_2, p_3 and p_4 are simultaneously internally Ω-stabilizable for any region Ω that does not contains $\{-1\}$. For example $c(s) = \frac{3}{2}$ is a $\mathbf{C}_{+\infty}$-stabilizing controller for $p_i(s)$, $(i = 1, 2, 3, 4)$.

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