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On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval[☆]

Georges Bastin[a,*], Jean-Michel Coron[b]

[a] Department of Mathematical Engineering, Université catholique de Louvain, 4, Avenue G. Lemaitre, 1348 Louvain-La-Neuve, Belgium

[b] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Institut Universitaire de France, 4, Place Jussieu, 75252 Paris Cedex, France

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ABSTRACT

Conditions for boundary feedback stabilizability of non-uniform linear 2×2 hyperbolic systems over a bounded interval are investigated. The main result is to show that the existence of a basic quadratic control Lyapunov function requires that the solution of an associated ODE is defined on the considered interval. This result is used to give explicit conditions for the existence of stabilizing linear boundary feedback control laws. The analysis is illustrated with an application to the boundary feedback stabilization of open channels represented by linearized Saint–Venant equations with non-uniform steady-states.

1. Introduction

In this paper we discuss the boundary feedback stabilization of linear 2×2 hyperbolic systems over a bounded interval and its application to nonlinear systems with non-uniform steady-states.

We consider the general class of linear 2×2 hyperbolic systems in the characteristic form

$$\begin{aligned} &\partial_t z_1 + \lambda_1(x)\partial_x z_1 + \gamma_1(x) z_1 + \delta_1(x) z_2 = 0, \\ &\partial_t z_2 - \lambda_2(x)\partial_x z_2 + \gamma_2(x) z_1 + \delta_2(x) z_2 = 0, \end{aligned} \tag{1}$$

under the boundary conditions

$$\begin{aligned} &z_1(t, 0) = \tilde{u}_1(t), \\ &z_2(t, L) = \tilde{u}_2(t), \end{aligned} \tag{2}$$

where $t \in [0, +\infty)$ is the time variable, $x \in [0, L]$ is the space variable, the functions λ_1, λ_2 are in $C^1([0, L]; (0, +\infty))$ and the functions γ_i, δ_i are in $C^1([0, L]; \mathbb{R})$.

This is a control system where, at time t, the state is $(z_1(t, \cdot), z_2(t, \cdot))^T \in L^2(0, L)^2$ and the control is $(\tilde{u}_1(t), \tilde{u}_2(t))^T \in \mathbb{R}^2$. Our concern is to analyze, by using a control Lyapunov function, the stabilizability of this system with linear decentralized boundary feedback control laws. The so-called *uniform* case (i.e. when the coefficients λ_i, γ_i and δ_i are constants), has been previously studied in [1–4] by using quadratic Lyapunov functions of the form

$$V(z) := \int_0^L \left(A z_1^2(t, x) \mathrm{e}^{-\mu x} + B z_2^2(t, x) \mathrm{e}^{\mu x}\right) \mathrm{d}x,$$

where A, B, μ are positive constants. The same kind of Lyapunov function has also been used in [5] for symmetric 2×2 models of heat exchangers and in [6] for gas pipelines represented by isentropic Euler equations.

In the present paper, our purpose is to complement these previous results by considering the *non-uniform* case where the coefficients are functions of x. More precisely we address the issue of the existence of basic quadratic control Lyapunov functions in this context (see e.g. [7, Section 12.1] for the classical concept of control Lyapunov function and Section 3 for a definition of basic quadratic control Lyapunov functions).

Conditions for boundary feedback stabilizability are established in Section 3. Our main result is to show that the existence of a basic quadratic control Lyapunov function requires that the solution of an associated ODE is defined on the considered interval. This result is then used to give explicit conditions for the existence of linear boundary feedback control laws in two cases: (i) when the control is available on both sides of the system; (ii) when the control is available only on one side of the system.

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* Corresponding author.
E-mail address: Georges.Bastin@uclouvain.be (G. Bastin).

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Behind this analysis, our motivation is in fact to investigate the stabilization of physical hyperbolic systems with non-uniform steady-states. In Section 4, as a matter of illustration, we show how our analysis can be applied to the design of stabilizing control laws for open-channels represented by linearized Saint–Venant equations with a non-uniform steady-state.

Finally, in Section 5, we analyze the case of a system that corresponds to the linearization of Saint–Venant equations for a channel with non-zero slope and uniform steady-state. This special case is interesting because it allows to explicitly apprehend the exact limitation of the use of basic quadratic control Lyapunov functions for linear hyperbolic systems of balance laws.

A preliminary proposition regarding the existence of functions satisfying certain differential inequalities is a key result for our analysis and is first given in Section 2.

2. A preliminary proposition

Let $L > 0$, let $a \in C^0([0, L])$ and $b \in C^0([0, L])$. We are interested in the existence of $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that

$$f > 0 \quad \text{in } [0, L], \tag{3}$$

$$g > 0 \quad \text{in } [0, L], \tag{4}$$

$$f' < 0 \quad \text{a.e. in } [0, L], \tag{5}$$

$$g' > 0 \quad \text{a.e. in } [0, L], \tag{6}$$

$$-f'g' > (af + bg)^2 \quad \text{a.e. in } [0, L]. \tag{7}$$

A necessary and sufficient condition for the existence of (f, g) is given in the following proposition.

Proposition 1. *There exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(7) hold if and only if the maximal solution η of the Cauchy problem*

$$\eta' = \left|a + b\eta^2\right|, \qquad \eta(0) = 0, \tag{8}$$

is defined on $[0, L]$.

Remark 1. The function

$$(x, s) \in [0, L] \times \mathbb{R} \mapsto \left|a(x) + b(x)s^2\right| \in \mathbb{R}$$

is continuous in $[0, L] \times \mathbb{R}$ and locally Lipschitz with respect to s. Hence the Cauchy problem (8) has a unique maximal solution.

Proof of Proposition 1. We start with the "only if" part. Let $f \in C^1([0, L])$ and $g \in C^1([0, L])$ be such that (3)–(7) hold. Let us define $h \in C^1([0, L])$ by

$$h(x) := \frac{1}{f(x)}, \quad \forall x \in [0, L]. \tag{9}$$

(Note that, by (3), $f(x) \neq 0$ for every $x \in [0, L]$.) Then (3), (5) and (7) become respectively

$$h > 0 \quad \text{in } [0, L], \tag{10}$$

$$h' > 0 \quad \text{a.e. in } [0, L], \tag{11}$$

$$g'h' > (a + bgh)^2 \quad \text{a.e. in } [0, L]. \tag{12}$$

Note that, by (4) and (10), $g(x)h(x) > 0$ for every $x \in [0, L]$. This allows to define $w \in C^1([0, L])$ by

$$w(x) := \sqrt{g(x)h(x)}, \quad \forall x \in [0, L]. \tag{13}$$

We have

$$w(0) > 0. \tag{14}$$

Note that

$$w' = \frac{1}{2\sqrt{gh}}\left(g'h + gh'\right). \tag{15}$$

From (4), (6), (10), (11) and (15), we have

$$w' > 0 \quad \text{a.e. in } [0, L]. \tag{16}$$

From (15) we have

$$w'^2 = \frac{1}{4gh}\left(g'h + gh'\right)^2 = g'h' + \frac{1}{4gh}\left(g'h - gh'\right)^2. \tag{17}$$

From (12), (13), (16) and (17), we have

$$w' > \left|a + bw^2\right| \quad \text{a.e. in } [0, L]. \tag{18}$$

From (8), (14), (18) and a classical theorem on ordinary differential equations, we have, on the interval of definition $I \subset [0, L]$ of η,

$$\eta < w. \tag{19}$$

This shows that $I = [0, L]$ and concludes the proof of the "only if" part of Proposition 1.

Let us now turn to the "if" part of Proposition 1. We assume that the maximal solution of the Cauchy problem (8) is defined on $[0, L]$. Then, if $\varepsilon > 0$ is small enough, the solution η_ε of the Cauchy problem

$$\eta_\varepsilon' = \left|a + b\eta_\varepsilon^2\right| + \varepsilon, \qquad \eta_\varepsilon(0) = \varepsilon, \tag{20}$$

is defined on $[0, L]$. We choose such a $\varepsilon > 0$. Note that

$$\eta_\varepsilon > 0 \quad \text{in } [0, L]. \tag{21}$$

Let us define $f \in C^1([0, L])$ and $g \in C^1([0, L])$ by

$$f(x) := \frac{1}{\eta_\varepsilon(x)}, \quad \forall x \in [0, L], \tag{22}$$

$$g(x) := \eta_\varepsilon(x), \quad \forall x \in [0, L]. \tag{23}$$

(Note that, by (21), f is well defined.) Clearly (3) and (4) hold. From (22) and (23), we have

$$f' = -\frac{\eta_\varepsilon'}{\eta_\varepsilon^2}, \tag{24}$$

$$g' = \eta_\varepsilon'. \tag{25}$$

From (20), (24) and (25), we have (5), (6) and

$$-f'g' = \frac{\eta_\varepsilon'^2}{\eta_\varepsilon^2}. \tag{26}$$

From (22) and (23), we have

$$(af + bg)^2 = \frac{1}{\eta_\varepsilon^2}\left(a + b\eta_\varepsilon^2\right)^2. \tag{27}$$

From (20), (26) and (27), we get (7).

This concludes the proof of the "if" part of Proposition 1. □

Remark 2. With the proof of the "if" part of Proposition 1, we have in fact proved that if the maximal solution η of the Cauchy problem $\eta' = \left|a + b\eta^2\right|$, $\eta(0) = 0$ is defined on $[0, L]$, then there exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that

$$f > 0 \quad \text{in } [0, L],$$

$$g > 0 \quad \text{in } [0, L],$$

$$f' < 0 \quad \text{in } [0, L], \tag{28}$$

$$g' > 0 \quad \text{in } [0, L], \tag{29}$$

$$-f'g' > (af + bg)^2 \quad \text{in } [0, L]. \tag{30}$$

The point is that inequalities (28)–(29)–(30) hold in $[0, L]$ instead of a.e. in $[0, L]$ for inequalities (5)–(6)–(7). Now it is obvious that the existence of $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(4)–(28)–(29)–(30) hold implies the existence of $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(4)–(5)–(6)–(7) hold. Hence we have in fact established the following more general result.

Proposition 2. *The three following statements are equivalent:*

- *There exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(4)–(5)–(6)–(7) hold.*
- *There exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(4)–(28)–(29)–(30) hold.*
- *The maximal solution η of the Cauchy problem $\eta' = \left|a + b\eta^2\right|$, $\eta(0) = 0$ is defined on $[0, L]$.*

3. Existence condition for a basic quadratic control Lyapunov function

In order to carry on the analysis, we first make a coordinate transformation inspired by [8, Chapter 9]. We introduce the notations

$$\varphi_1(x) = \exp\left(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}\mathrm{d}s\right),$$

$$\varphi_2(x) = \exp\left(-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}\mathrm{d}s\right),$$

$$\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)},$$

and the new coordinates

$$y_1(t, x) = \varphi_1(x) z_1(t, x), \qquad y_2(t, x) = \varphi_2(x) z_2(t, x). \tag{31}$$

Then the system (1) is transformed into the following system expressed in these new coordinates:

$$\begin{aligned} &\partial_t y_1 + \lambda_1(x)\partial_x y_1 + a(x) y_2 = 0,\\ &\partial_t y_2 - \lambda_2(x)\partial_x y_2 + b(x) y_1 = 0 \end{aligned} \tag{32}$$

with

$$a(x) = \varphi(x)\delta_1(x), \qquad b(x) = \varphi^{-1}(x)\gamma_2(x).$$

We consider this system under the boundary conditions

$$\begin{aligned} &y_1(t, 0) = u_1(t) := \varphi_1(0)\tilde{u}_1(t),\\ &y_2(t, L) = u_2(t) := \varphi_2(L)\tilde{u}_2(t). \end{aligned} \tag{33}$$

Eqs. (32) and (33) form a control system where, at time t, the state is $y(t, \cdot) = (y_1(t, \cdot), y_2(t, \cdot))^T \in L^2(0, L)^2$ and the control is $u(t) = (u_1(t), u_2(t))^T \in \mathbb{R}^2$.

We introduce the following control Lyapunov function candidate

$$V(y) := \int_0^L \left(q_1(x) y_1^2(t, x) + q_2(x) y_2^2(t, x)\right)\mathrm{d}x, \tag{34}$$

where $q_1 \in C^1([0, L]; (0, +\infty))$ and $q_2 \in C^1([0, L]; (0, +\infty))$ have to be determined. The time derivative $\dot{V}$ of V along the trajectories of (32)–(33) is

$$\begin{aligned} \dot{V}(y, u) &= \int_0^L \left(2q_1 y_1 \partial_t y_1 + 2q_2 y_2 \partial_t y_2\right)\mathrm{d}x\\ &= -\int_0^L \left(2q_1 y_1(\lambda_1 \partial_x y_1 + a y_2)\right.\\ &\quad \left. + 2q_2 y_2(-\lambda_2 \partial_x y_2 + b y_1)\right)\mathrm{d}x\\ &= -B - \int_0^L I \mathrm{d}x, \end{aligned} \tag{35}$$

with

$$\begin{aligned} B := {} & \lambda_1(L) q_1(L) y_1^2(t, L) - \lambda_2(L) q_2(L) u_2^2 - \lambda_1(0) q_1(0) u_1^2\\ & + \lambda_2(0) q_2(0) y_2^2(t, 0), \end{aligned} \tag{36}$$

$$I := (-(\lambda_1 q_1)_x) y_1^2 + 2(q_2 b + q_1 a) y_1 y_2 + ((\lambda_2 q_2)_x) y_2^2. \tag{37}$$

We have the following definition.

Definition 1. A function $V(y)$ with given q_1 and q_2 is a control Lyapunov function for the control system (32)–(33) if and only if

$$\forall y \in H^1(0, L)^2, \quad \exists u \in \mathbb{R}^2 \text{ such that } \dot{V}(y, u) \leqslant 0. \tag{38}$$

It is a *strict* control Lyapunov function if and only if

$$\forall y \in H^1(0, L)^2 \setminus \{(0, 0)^T\}, \quad \exists u \in \mathbb{R}^2 \text{ such that } \dot{V}(y, u) < 0. \tag{39}$$

A function $V(y)$ satisfying this definition will be called the "*basic quadratic (strict) control Lyapunov function*".

Our main result is then given in the following theorem.

Theorem 1. *There exists a basic quadratic strict control Lyapunov function for the control system (32)–(33) if and only if the maximal solution η of the Cauchy problem*

$$\eta' = \left|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2\right|, \qquad \eta(0) = 0 \tag{40}$$

is defined on $[0, L]$.

Proof.

(a) "*Only if*" *condition*. A necessary condition for $V(y)$ to be a (strict) control Lyapunov function is that I is a strictly positive quadratic form with respect to (y_1, y_2) for almost every x in $[0, L]$, i.e.

$$-(\lambda_1 q_1)_x > 0 \quad \text{a.e. in } [0, L], \tag{41}$$

$$(\lambda_2 q_2)_x > 0 \quad \text{a.e. in } [0, L], \tag{42}$$

$$-(\lambda_1 q_1)_x (\lambda_2 q_2)_x > (q_2 b + q_1 a)^2 \quad \text{a.e. in } [0, L]. \tag{43}$$

We define the functions $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that

$$f(x) := \lambda_1(x) q_1(x), \quad \forall x \in [0, L], \tag{44}$$

$$g(x) := \lambda_2(x) q_2(x), \quad \forall x \in [0, L]. \tag{45}$$

The quadratic form $V(y)$ is coercive with respect to $(y_1, y_2)^T \in L^2(0, L)^2$ (i.e. $\exists \sigma > 0$ such that $V(y) \geqslant \sigma \int_0^L (y_1^2 + y_2^2)\mathrm{d}x$) if and only if (3) and (4) hold. Note that (41) is equivalent to (5) and that (42) is equivalent to (6). Property (43) is equivalent to (7) with a and b defined by

$$a(x) := \frac{a(x)}{\lambda_1(x)}, \qquad b(x) := \frac{b(x)}{\lambda_2(x)}, \quad \forall x \in [0, L]. \tag{46}$$

Following Proposition 1, we consider the maximal solution η of the Cauchy problem

$$\eta' = \left|\frac{a}{\lambda_1} + \frac{b}{\lambda_2}\eta^2\right|, \qquad \eta(0) = 0.$$

It follows from Proposition 1 that a necessary condition for the existence of a control Lyapunov function $V(y)$ of the form (34) is that η is defined on $[0, L]$.

(b) "*If*" *condition*. Let us assume that η is defined on $[0, L]$. Then there is a strict control Lyapunov function $V(y)$ of the form (34). Indeed, by Proposition 2, there exist $q_1 \in C^1([0, L]; (0, +\infty))$ and $q_2 \in C^1([0, L]; (0, +\infty))$ such that (41)–(43) hold everywhere in $[0, L]$ (instead a.e. in $[0, L]$). Let us define the following decentralized feedback control laws

$$u_1(t) := k_1 y_2(t, 0), \qquad u_2(t) := k_2 y_1(t, L). \tag{47}$$

Then for any constant k_1 and k_2 selected such that

$$k_2^2 \leqslant \frac{\lambda_1(L) q_1(L)}{\lambda_2(L) q_2(L)}, \qquad k_1^2 \leqslant \frac{\lambda_2(0) q_2(0)}{\lambda_1(0) q_1(0)}, \tag{48}$$

we have

$$\dot{V} \leqslant -\rho(|y_1|^2_{L^2(0,L)} + |y_1|^2_{L^2(0,L)}), \tag{49}$$

for some $\rho > 0$ independent of (y_1, y_2). This leads to exponential stability with a rate depending on ρ and σ, themselves depending on q_1 and q_2. □

Remark 3. The proof of Proposition 1 provides a way to construct "good" coefficients q_1 and q_2 for the Lyapunov function: take $\varepsilon > 0$ small enough and consider the solution of the Cauchy problem

$$\eta_\varepsilon' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta_\varepsilon^2 \right| + \varepsilon, \qquad \eta_\varepsilon(0) = \varepsilon, \tag{50}$$

and then define q_1 and q_2 by

$$q_1(x) := \frac{1}{\lambda_1(x) \eta_\varepsilon(x)}, \quad \forall x \in [0, L], \tag{51}$$

$$q_2(x) := \frac{\eta_\varepsilon(x)}{\lambda_2(x)}, \quad \forall x \in [0, L]. \tag{52}$$

Of course (50) can be replaced by some similar Cauchy problem whose solution could be simpler to compute. For example, if $a \geqslant 0$ and $b \geqslant 0$, one can replace (50) by

$$\eta_\varepsilon' = (1 + \varepsilon) \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta_\varepsilon^2 \right|, \quad \eta_\varepsilon(0) = \varepsilon. \tag{53}$$

Remark 4. With Theorem 1, if the two controls u_1 and u_2 are available, we see that the stabilization of the system may be achieved with the control laws (47) provided that the tuning parameters k_1 and k_2 satisfy inequalities (48). This is true for any L such that η is defined on $[0, L]$.

In contrast, if only the control u_1 is available and if the boundary conditions are of the form

$$y_1(t, 0) = u_1(t) \quad \text{and} \quad y_2(t, L) = M y_1(t, L) \tag{54}$$

with M an a-priori given constant, there is a limitation to the Lyapunov approach even if η is defined on $[0, L]$ as stated in the following theorem (with an evident appropriate modification of Definition 1).

Theorem 2. *There exists a basic quadratic strict control Lyapunov function for the control system (32)- (54) if and only if the maximal solution η of the Cauchy problem (40) is defined on $[0, L]$ and $|M| < [\eta(L)]^{-1}$.*

Proof. We first observe that in order to have $\dot{V} \leqslant 0$ (with $u_2 := M y_1(t, L)$ in B) we must have

$$M^2 \leqslant \frac{f(L)}{g(L)}. \tag{55}$$

It follows from our proof of Proposition 1 (and with the notations therein) that

$$\frac{g(L)}{f(L)} = g(L) h(L) = w^2(L) > \eta^2(L), \tag{56}$$

(see in particular (19)). Let us first treat the case where $a \neq 0$. Then $\eta(L) > 0$ and it follows from (55) and (56), that a necessary condition for the existence of a control Lyapunov of the form (34) is that

$$|M| < \frac{1}{\eta(L)}. \tag{57}$$

Conversely, let us assume that (57) holds. Then, it follows from Proposition 1 that there exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(7) and (55) hold. Then it suffices to take the feedback law $u_1(t) := k_1 y_2(t, 0)$ with

$$k_1^2 \leqslant \frac{\lambda_2(0) q_2(0)}{\lambda_1(0) q_1(0)} = \frac{g(0)}{f(0)}. \tag{58}$$

For the case $a = 0$, it follows from Proposition 2 that there exist $f \in C^1([0, L])$ and $g \in C^1([0, L])$ such that (3)–(4)–(28)–(29)–(30) and (55) hold (in fact in this case the existence of such f and g is straightforward). We then take the feedback law $u_1(t) := k_1 y_2(t, 0)$ with k_1 satisfying (58) as above. □

4. Application to Saint–Venant equations

We consider a pool of a prismatic horizontal open channel with a rectangular cross section and a unit width. The dynamics of the system are described by the Saint–Venant equations

$$\begin{aligned} &\partial_t H + \partial_x (HV) = 0, \\ &\partial_t V + \partial_x \left(\frac{V^2}{2} + gH \right) + gC \frac{V^2}{H} = 0, \end{aligned} \tag{59}$$

with the state variables $H(t, x)$ = water depth and $V(t, x)$ = water velocity. C is a friction coefficient and g the gravity acceleration.

The channel is provided with hydraulic control devices (pumps, valves, mobile spillways, sluice gates, etc.) which are located at the two extremities and allow to assign the values of the flow-rate on both sides:

$$\begin{aligned} &Q_1(t) = H(t, 0) V(t, 0), \\ &Q_2(t) = H(t, L) V(t, L). \end{aligned} \tag{60}$$

The system (59)–(60) is a control system with state $H(t, x)$, $V(t, x)$ and controls $Q_1(t), Q_2(t)$. This system is clearly open-loop unstable. The objective is to design decentralized control laws, with $Q_1(t)$ function of $H(t, 0)$ and $Q_2(t)$ function of $H(t, L)$, in order to stabilize the system about a constant flow-rate set point Q^*.

A steady-state (or equilibrium profile), corresponding to the set-point Q^*, is a couple of time-invariant *non-uniform* (i.e. space-varying) state functions $H^*(x)$, $V^*(x)$ such that $H^*(x) V^*(x) = Q^*$ which satisfy the differential equations

$$\begin{aligned} &\partial_x (H^* V^*) = 0, \\ &\partial_x \left(\frac{V^{*2}}{2} + gH^* \right) + gC \frac{V^{*2}}{H^*} = 0. \end{aligned}$$

These equations may also be written as

$$V^* \partial_x H^* = -H^* \partial_x V^* = -\frac{gCV^{*3}}{gH^* - V^{*2}}. \tag{61}$$

In this section, as a first stage toward a more comprehensive study of the problem, we shall focus on the stabilizability of the *linearized* system by using the analysis of the previous section.

In order to linearize the model, we define the deviations of the states $H(t, x)$ and $V(t, x)$ with respect to the steady-states $H^*(x)$ and $V^*(x)$:

$$h(t, x) \triangleq H(t, x) - H^*(x), \qquad v(t, x) \triangleq V(t, x) - V^*(x).$$

Then the linearized Saint–Venant equations around the steady-state are

$$\partial_t h + V^*\partial_x h + H^*\partial_x v + (\partial_x V^*)h + (\partial_x H^*)v = 0,$$
$$\partial_t v + g\partial_x h + V^*\partial_x v - gC\frac{V^{*2}}{H^{*2}}h + \left[\partial_x V^* + 2gC\frac{V^*}{H^*}\right]v = 0.$$

The characteristic (Riemann) coordinates are defined as follows:

$$\begin{aligned} z_1(t,x) &= v(t,x) + h(t,x)\sqrt{\frac{g}{H^*(x)}},\\ z_2(t,x) &= v(t,x) - h(t,x)\sqrt{\frac{g}{H^*(x)}}, \end{aligned} \tag{62}$$

and the inverse transformation is as follows:

$$h(t,x) = \frac{z_1(t,x) - z_2(t,x)}{2}\sqrt{\frac{H^*(x)}{g}},$$
$$v(t,x) = \frac{z_1(t,x) + z_2(t,x)}{2}.$$

With these definitions and notations, the linearized Saint–Venant equations are written in the characteristic form

$$\partial_t z_1 + \lambda_1(x)\partial_x z_1 + \gamma_1(x)z_1 + \delta_1(x)z_2 = 0,$$
$$\partial_t z_2 - \lambda_2(x)\partial_x z_2 + \gamma_2(x)z_1 + \delta_2(x)z_2 = 0$$

with the characteristic velocities

$$\lambda_1(x) = V^*(x) + \sqrt{gH^*(x)}, \qquad -\lambda_2(x) = V^*(x) - \sqrt{gH^*(x)},$$

and the coefficients

$$\gamma_1(x) = g\frac{CV^{*2}}{H^*}\left[-\frac{3}{4(\sqrt{gH^*}+V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}}\right],$$
$$\delta_1(x) = g\frac{CV^{*2}}{H^*}\left[-\frac{1}{4(\sqrt{gH^*}+V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right],$$
$$\gamma_2(x) = g\frac{CV^{*2}}{H^*}\left[\frac{1}{4(\sqrt{gH^*}-V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}}\right],$$
$$\delta_2(x) = g\frac{CV^{*2}}{H^*(x)}\left[\frac{3}{4(\sqrt{gH^*}-V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right].$$

The steady-state flow is subcritical (or fluvial) if the following condition holds

$$gH^*(x) - V^{*2}(x) > 0 \quad \forall x. \tag{63}$$

Under this condition, the system is strictly hyperbolic with

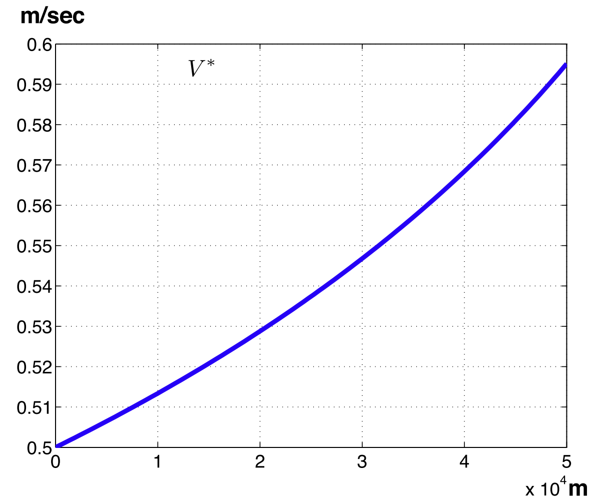
$$-\lambda_2(x) < 0 < \lambda_1(x) \quad \forall x.$$

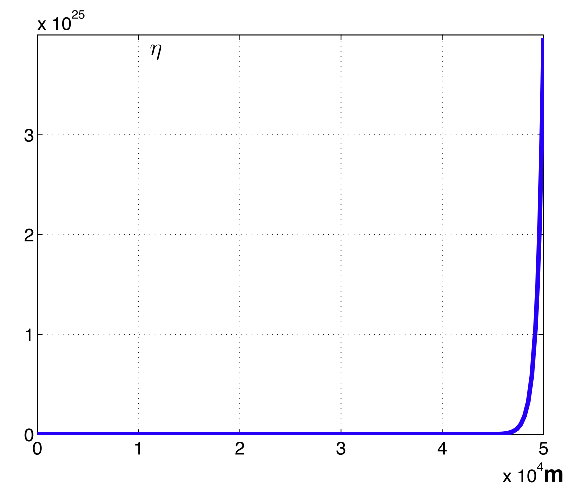
According to our analysis in Section 3, in order to check the condition for the existence of a basic quadratic control Lyapunov function, we need to solve the following third-order differential system on $[0, L]$ (with $H^*(x) = Q^*/V^*(x)$):

$$\frac{dV^*}{dx} = \frac{gC}{Q^*}\left(\frac{(V^*(x))^5}{gQ^* - (V^*(x))^3}\right) \qquad V^*(0) = V_o^*,$$
$$\frac{d\psi}{dx} = \frac{\gamma_1(x)}{\lambda_1(x)} + \frac{\delta_2(x)}{\lambda_2(x)} \qquad \psi(0) = 0,$$
$$\frac{d\eta}{dx} = \frac{e^{\psi(x)}\delta_1(x)}{\lambda_1(x)} + \frac{\gamma_2(x)}{e^{\psi(x)}\lambda_2(x)}\eta^2(x) \qquad \eta(0) = 0.$$

The first equation computes the steady-state profile $V^*(x)$. It is obtained from (61) and $Q^* = H^*V^*$. The solution ψ of the second equation is such that $\varphi = \exp(\psi)$ involved in the computation of $a(x)$ and $b(x)$ (see (32)). The third equation is the ODE (40) in the statement of Theorem 1.

As a matter of illustration, we compute the solution of this system with the following parameter values: $L = 50$ km, $g = 9.81$ m/s^2, $C = 0.002$ s^2/m, $Q^* = 1$ m^3/s and the initial condition $V_o^* = 0.5$ m/s. The function η exists over the interval $[0, L]$ and is shown in the following figure. However, it must be mentioned that η ceases to exist for a length slightly larger than 50 km as it can be seen from the figure. But this numerical illustration clearly shows that the function η should exist for most reasonable real life applications regarding irrigation systems or navigable rivers.





Let us now impose a boundary condition of the form

$$y_1(t,0) = k_1 y_2(t,0) \tag{64}$$

with

$$k_1^2 \leqslant \frac{\lambda_2(0)q_2(0)}{\lambda_1(0)q_1(0)}$$

to the system (32). Then, using the definition (31) of the y_i coordinates, the definition (62) of the z_i Riemann coordinates and the physical boundary condition (60), it is a matter of few calculations to get the physical stabilizing control law which implements the boundary condition (64)

$$Q_1(t) = \frac{H(t,0)}{H^*(0)} \times \left[Q^* - \frac{\varphi_1(0) + k_1\varphi_2(0)}{\varphi_1(0) - k_1\varphi_2(0)}\sqrt{gH^*(0)}(H(t,0) - H^*(0))\right]$$

for the open channel represented by the Saint–Venant equations. We remark that this control law is a non-linear feedback function of the water depth $H(t, 0)$ although it is derived on the basis of a linearized model. Obviously, a similar derivation leads to a control law for $Q_2(t)$ at the other side of the channel.

5. The case of uniform systems

In this section, we consider the special case of a uniform system of the following form

$$\partial_t z_1 + \lambda_1 \partial_x z_1 + \gamma z_1 + \delta z_2 = 0,$$
$$\partial_t z_2 - \lambda_2 \partial_x z_2 + \gamma z_1 + \delta z_2 = 0.$$

Here the parameters λ_i, γ, δ are supposed to be positive constants that do not depend on x. The interest of looking at that special case is twofold:

1. As it is shown in [3], this system is the linearization of the Saint–Venant equations for a channel with a non-zero slope and a uniform steady-state.
2. As we shall see hereafter, it allows to explicitly apprehend the exact limitation of the Lyapunov approach for analyzing the boundary feedback stabilizability.

We introduce the following notations:

$$c := \frac{\gamma}{\lambda_1} + \frac{\delta}{\lambda_2} > 0, \tag{65}$$

$$\alpha := \frac{\delta}{\lambda_1} > 0, \tag{66}$$

$$\beta := \frac{\gamma}{\lambda_2} > 0. \tag{67}$$

Then we have $a(x) = \alpha e^{cx}$, $b(x) = \beta e^{-cx}$ and (40) becomes

$$\eta' = \alpha e^{cx} + \beta e^{-cx}\eta^2, \qquad \eta(0) = 0. \tag{68}$$

We let $\theta = \beta e^{-cx}\eta$. Then (68) becomes

$$\theta' + c\theta = \alpha\beta + \theta^2, \qquad \theta(0) = 0. \tag{69}$$

The Cauchy problem (69) has a solution defined on $[0, L]$ for every $L > 0$ if and only if the polynomial $X^2 - cX + \alpha\beta$ vanishes on $(0, +\infty)$, i.e. if and only if

$$\Delta \geqslant 0 \quad \text{with } \Delta := c^2 - 4\alpha\beta. \tag{70}$$

From (65)–(67) and (70), we observe that

$$\Delta = \left(\frac{\gamma}{\lambda_1} + \frac{\delta}{\lambda_2}\right)^2 - 4\frac{\delta}{\lambda_1}\frac{\gamma}{\lambda_2} = \left(\frac{\gamma}{\lambda_1} - \frac{\delta}{\lambda_2}\right)^2 \geqslant 0 \tag{71}$$

and we conclude that the solution η of (68) is defined on $[0, L]$ for all $L > 0$. Then, if the two controls u_1 and u_2 are available, we see that the stabilization of the system may be achieved for any L with the control laws (47) provided that the tuning parameters k_1 and k_2 satisfy inequalities (48). One recovers that the Lyapunov approach works whatever L be, if we control both sides.

In contrast, if only the control u_1 is available and if the second boundary condition is of the form $u_2(t) = My_1(t, L)$ where M is a fixed given constant, there is a general limitation to the Lyapunov approach although the solution of (68) is defined for all L. Assuming that (70) holds, the two zeros of the polynomial $X^2 - cX + \alpha\beta$ are

$$\theta_1 := \frac{c - \sqrt{\Delta}}{2}, \qquad \theta_2 := \frac{c + \sqrt{\Delta}}{2} \tag{72}$$

and (69) is equivalent to

$$\theta'\left(\frac{1}{\theta - \theta_2} - \frac{1}{\theta - \theta_1}\right) = \theta_2 - \theta_1, \qquad \theta(0) = 0. \tag{73}$$

This gives

$$\theta(L) = \alpha\beta \frac{e^{\theta_2 L} - e^{\theta_1 L}}{\theta_2 e^{\theta_2 L} - \theta_1 e^{\theta_1 L}}, \tag{74}$$

$$\eta(L) = \alpha \frac{e^{\theta_2 L} - e^{\theta_1 L}}{\theta_2 e^{\theta_2 L} - \theta_1 e^{\theta_1 L}} e^{cL}, \tag{75}$$

and the Lyapunov approach works if and only if (57) holds. Note that, with (65)–(67), (70) and (72), one has (possibly up to a permutation of θ_1 and θ_2; this permutation does not induce any problem since the right hand side of (74) is invariant if we switch θ_1 and θ_2),

$$\theta_1 := \frac{\gamma}{\lambda_1}, \qquad \theta_2 := \frac{\delta}{\lambda_2}.$$

6. Conclusion and final remark

Conditions for boundary feedback stabilizability of linear hyperbolic systems in canonical form have been established. The main result was to show that the existence of a basic quadratic control Lyapunov function requires that the solution of an associated ODE is defined on the considered interval. This result has been used to give explicit conditions for the existence of linear boundary feedback control laws. The analysis is illustrated with an application to the boundary feedback stabilization of open channels represented by Saint–Venant equations with non-uniform steady-states.

An interesting final remark is that we could believe that more general stabilizability conditions could be obtained by considering a more general Lyapunov function candidate (with an additional cross-term) of the form

$$V(y) := \int_0^L \left(q_1(x)y_1^2 + q_2(x)y_2^2 + q_3(x)y_1y_2\right)dx. \tag{76}$$

In fact, this not true because it can be shown that, for the control system (32)–(33), if (76) is a control Lyapunov function then $q_3(x)$ must be zero. The proof of this assertion is given in Appendix.

Appendix

Proposition 3. *Let q_1, q_2, and q_3 in $C^1([0, L])$ be such that*

$$V(y) := \int_0^L \left(q_1(x)y_1^2 + q_2(x)y_2^2 + q_3(x)y_1y_2\right)dx \tag{77}$$

is a control Lyapunov function for the control system (32)–(33). Then $q_3 = 0$.

Proof of Proposition 3. Since V is a control Lyapunov function, we have

$$\begin{aligned} &q_1(x) > 0, \qquad q_2(x) > 0, \\ &4q_1(x)q_3(x) > q_3(x)^2, \quad \forall x \in [0, L]. \end{aligned} \tag{78}$$

The time derivative $\dot{V}$ of V along the trajectories of (32) is

$$\begin{aligned} \dot{V} &= \int_0^L 2q_1y_1\partial_t y_1 + 2q_2y_2\partial_t y_2 + q_3y_2\partial_t y_1 + q_3y_1\partial_t y_2 dx \\ &= -\int_0^L 2q_1y_1(\lambda_1\partial_x y_1 + ay_2) + 2q_2y_2(-\lambda_2\partial_x y_2 + by_1) \\ &\quad + q_3y_2(\lambda_1\partial_x y_1 + ay_2) + q_3y_1(-\lambda_2\partial_x y_2 + by_1)dx \\ &= -B - \int_0^L I dx - X, \end{aligned} \tag{79}$$

with

$$\begin{aligned} B := {} & \lambda_1(L)q_1(L)y_1(L)^2 - \lambda_1(0)q_1(0)y_1(0)^2 \\ & - \lambda_2(L)q_2(L)y_2(L)^2 + \lambda_2(0)q_2(0)y_2(0)^2, \end{aligned} \tag{80}$$

$$\begin{aligned} I := {} & y_1^2(-\partial_x(\lambda_1 q_1) + q_3 b) + 2y_1y_2(q_1 a + q_2 b) \\ & + y_2^2(\partial_x(\lambda_2 q_2) + q_3 a), \end{aligned} \tag{81}$$

$$X := \int_0^L q_3(\lambda_1 y_2\partial_x y_1 - \lambda_2 y_1\partial_x y_2)dx. \tag{82}$$

Using (78), we have

$$B \leqslant \lambda_1(L)q_1(L)y_1(L)^2 + \lambda_2(0)q_2(0)y_2(0)^2. \tag{83}$$

Let us recall that, if V is a control Lyapunov function,

$$\forall y_1 \in H^1(0,L), \qquad \forall y_2 \in H^1(0,L),$$
$$\exists u_1 \in \mathbb{R}, \qquad \exists u_2 \in \mathbb{R} \text{ such that } \dot{V} \leqslant 0. \tag{84}$$

Let $H_0^1(0,L) := \{\xi \in H^1(0,L); \xi(0) = 0; \xi(L) = 0\}$. Proposition 3 follows from (79), (81), (82), (83), (84) and the following lemma. □

Lemma 1. *Assume that there exists $C > 0$ such that, for every $y_1 \in H_0^1(0,L)$ and for every $y_2 \in H_0^1(0,L)$,*

$$\int_0^L q_3(\lambda_1 y_2 \partial_x y_1 - \lambda_2 y_1 \partial_x y_2)\mathrm{d}x \geqslant -C \int_0^L (y_1^2 + y_2^2)\mathrm{d}x, \tag{85}$$

then

$$q_3 = 0. \tag{86}$$

Proof of Lemma 1. Let $\chi \in C^\infty(\mathbb{R})$ be such that

$$\chi = 0 \quad \text{on } (-\infty, -1] \quad \text{and} \quad \chi = 1 \quad \text{on } [1, +\infty).$$

Let a_1, a_2, a_3 and a_4 be four real numbers such that

$$0 < a_1 < a_2 < a_3 < a_4 < L.$$

For $n \in \mathbb{N}$, let $y_1^n \in C^\infty([0,L])$ and $y_2^n \in C^\infty([0,L])$ be defined by

$$y_1^n(x) := \chi\left(n(x - a_1)\right)\chi\left(n(a_3 - x)\right), \quad \forall x \in [0,L], \tag{87}$$
$$y_2^n(x) := \chi\left(n(x - a_2)\right)\chi\left(n(a_4 - x)\right), \quad \forall x \in [0,L]. \tag{88}$$

Clearly, there exists $n_0 \in \mathbb{N}$ such that

$$y_1^n(0) = y_1^n(L) = y_2^n(0) = y_2^n(L) = 0, \quad \forall n \geqslant n_0. \tag{89}$$

Moreover, straightforward computations lead to

$$\lim_{n\to+\infty} \int_0^L q_3(\lambda_1 y_2^n \partial_x y_1^n - \lambda_2 y_1^n \partial_x y_2^n)\mathrm{d}x$$
$$= -q_3(a_3)\lambda_1(a_3) - q_3(a_2)\lambda_2(a_2), \tag{90}$$

$$\lim_{n\to+\infty} \int_0^L (y_1^n)^2 + (y_2^n)^2 \mathrm{d}x = (a_3 - a_1) + (a_4 - a_2). \tag{91}$$

From (85), (89)–(91), we get that

$$q_3(a_3)\lambda_1(a_3) + q_3(a_2)\lambda_2(a_2) \leqslant C((a_3 - a_1) + (a_4 - a_2)). \tag{92}$$

We now move a_1, a_2, a_3 and a_4 so that they converge to some given $a \in [0,L]$. Then, using (92), we get

$$q_3(a)(\lambda_1(a) + \lambda_2(a)) \leqslant 0. \tag{93}$$

From (93) and our assumption that $\lambda_1 > 0$ and $\lambda_2 > 0$, we have

$$q_3(a) \leqslant 0, \quad \forall a \in [0,L]. \tag{94}$$

We now exchange y_1^n and y_2^n, i.e. we replace (87) and (88) by

$$y_1^n(x) := \chi\left(n(x - a_2)\right)\chi\left(n(a_4 - x)\right), \quad \forall x \in [0,L], \tag{95}$$
$$y_2^n(x) := \chi\left(n(x - a_1)\right)\chi\left(n(a_3 - x)\right), \quad \forall x \in [0,L], \tag{96}$$

respectively. Following the same arguments as above we now get that

$$q_3(a) \geqslant 0, \quad \forall a \in [0,L]. \tag{97}$$

Lemma 1 follows from (94) and (97). □

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