Boundary control with integral action for hyperbolic systems of conservation laws: stability and experiments

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Abstract

A strict Lyapunov function for boundary control with integral actions of hyperbolic systems of conservation laws that can be diagonalised with Riemann invariants, is presented. The time derivative of this Lyapunov function can be made strictly negative definite by an appropriate choice of the boundary conditions and the integral control gains. Previous stability results are extended to guarantee the local convergence of the state towards a desired set point. Furthermore, the control can be implemented as a feedback of the state only measured at the boundaries. The control design method is illustrated with an hydraulic application, namely the level and flow regulation in a reach of the Sambre river and in the micro-channel of Valence, respectively through simulations and experimentations.

Key words: Lyapunov stability, Saint-Venant equations, Systems of conservation laws, Riemann invariants.

1 Introduction

In this paper, we are concerned with two-by-two systems of conservation laws that are described by hyperbolic quasi-linear partial differential equations, with one independent time variable $t \in [0, \infty)$ and one independent space variable on a finite interval $x \in [0, L]$. Such systems are used to model many physical situations and engineering problems. A famous example is that of Saint-Venant (or shallow water) equations which describe the flow of water in irrigation channels and waterways. This example will be presented in Section 4. Other typical examples include gas and fluid transportation networks, packed bed and plug-flow reactors, drawing processes in glass and polymer industries, road traffic etc. For such systems, the considered boundary control problem is the problem of designing feedback control actions at the boundaries (i.e. at $x = 0$ and $x = L$) in order to ensure that the smooth solution of the Cauchy problem converges to a desired steady-state.

This problem has been previously considered in the literature ([7] e.g.). Initial results of asymptotic stability were presented by Greenberg and Li-Tatsien [5] and Slemrod [9]. Later on they have been generalized and applied to the control of networks of open channels in our previous papers [1]-[3] and in Leugering and Schmitt [6].

The present paper is in the direct continuation of our previous paper [2] where a static proportional feedback control law was presented and the closed-loop stability analyzed with an appropriate Lyapunov function. But obviously, a static control law may be subject to steady-state regulation errors in case of constant disturbances or model inaccuracies. In the present paper we show how additional integral actions can be introduced in the control law in order to cancel the static errors and how the Lyapunov function can be modified in order to prove

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the asymptotic stability of the closed-loop system. The statement of the control law and the Lyapunov stability analysis are developed in Sections 2 and 3 for a generic *homogeneous* system of two *linear* conservation laws. In Section 4, we consider the practical application to open channels described by Saint-Venant equations that form a set of two *nonhomogeneous* and *nonlinear* conservation laws. We show that, in the case where the friction effects and the channel slope are neglected, the approximate linearized system is in the linear homogeneous form considered in the theoretical analysis. This clearly motivates the use of a control with integral actions in order to cope with steady state errors that come from modelling uncertainties associated to small but unknown slope and friction. Moreover, the connection with the classical PI control (as implemented in finite dimensional system) is emphasized. Finally in Section 5 we present simulation results on a realistic example of a pool of the Sambre river (length 11km, width 40m) and an experimental validation on a small laboratory plant (length 7m, width 10cm). These results clearly show, not only the wide range of potential hydraulic applications, but also the control robustness when implemented on physical systems with unmodelled nonlinearities.

2 Boundary Control of Hyperbolic Systems of Conservation Laws

2.1 Statement of the Problem

An hyperbolic system of two linear conservation laws of the following general form is considered:

$$\partial_t h(t,x) + \partial_x q(t,x) = 0, \quad (1)$$
$$\partial_t q(t,x) + cd\partial_x h(t,x) + (c-d)\partial_x q(t,x) = 0, \quad (2)$$

where:

* t and x are the two independent variables: a time variable $t \in [0,+\infty)$ and a space variable $x \in [0,L]$ on a finite interval;
* (h,q); $[0,+\infty) \times [0,L] \to \Omega \subset I\!R^2$ is the 2-vector of the state variables $h(t,x)$ and $q(t,x)$ of the system;
* c and d are two real positive constants.

The first equation (1) can be interpreted as a mass conservation law with h the density and q the flux. The second equation can then be interpreted as a momentum conservation law.

We are concerned with the solutions of the Cauchy problem for the system (1)-(2) over $[0,+\infty) \times [0,L]$ under an initial condition:

$$h(0,x) = h^0(x), \qquad q(0,x) = q^0(x), \qquad x \in [0,L]$$

where $h^0(x)$ and $q^0(x)$ are two given functions, and two boundary conditions of the form:

$$g_0(h(t,0), q(t,0), u_0(t)) = 0, \qquad t \in [0,+\infty), \quad (3)$$
$$g_L(h(t,L), q(t,L), u_L(t)) = 0, \qquad t \in [0,+\infty), \quad (4)$$

with $g_0, g_L : \Omega \times I\!R \to I\!R$ and where $u_0, u_L : [0,+\infty) \to I\!R$ are the control actions.

The *boundary control problem* is then the problem of finding control actions $u_0(t)$ and $u_L(t)$ such that, for any smooth enough initial condition $(h^0(x), q^0(x))$, the Cauchy problem has a unique smooth solution converging towards 0 for all x in $[0,L]$.

2.2 Riemann Coordinates

In order to solve this boundary control problem, the *Riemann coordinates* (see e.g. [8] p. 79) defined by the following change of coordinates are introduced:

$$a(t,x) = q(t,x) + dh(t,x), \quad (5)$$
$$b(t,x) = q(t,x) - ch(t,x), \quad (6)$$

With these coordinates, the system (1)-(2) is written under the following diagonal form:

$$\partial_t a(t,x) + c\partial_x a(t,x) = 0, \quad (7)$$
$$\partial_t b(t,x) - d\partial_x b(t,x) = 0, \quad (8)$$

The change of coordinates (5)-(6) is inverted as follows:

$$h(t,x) = \frac{a(t,x) - b(t,x)}{c+d}, \quad (9)$$
$$q(t,x) = \frac{ca(t,x) + db(t,x)}{c+d}. \quad (10)$$

In the Riemann coordinates, the control problem can be restated as the problem of determining the control actions in such a way that the solutions $a(t,x)$, $b(t,x)$ converge towards zero.
In our previous paper [2], we have shown that this problem can be solved by selecting $u_0(t)$ and $u_L(t)$ such that the Riemann coordinates $a(t,x)$, $b(t,x)$ satisfy linear boundary conditions of the following form:

$$a(t,0) + k_0 b(t,0) = 0, \quad (11)$$
$$b(t,L) + k_L a(t,L) = 0, \quad (12)$$

with k_0 and k_L real constants to be tuned. The Lyapunov function

$$U(t) = U_1(t) + U_2(t) \quad (13)$$

where:

$$U_1(t) = \frac{A}{c}\int_0^L a^2(t,x)e^{-(\mu/c)x}dx,$$
$$U_2(t) = \frac{B}{d}\int_0^L b^2(t,x)e^{+(\mu/d)x}dx,$$

(A, B and μ are positive constant coefficients) then allows to prove the exponential convergence of the system trajectories towards 0 if $|k_0 k_L| < 1$. Remark that system (7)-(8) with boundary conditions (11)-(12) consist of two delay lines connected in feedback, with gains k_0 and k_L, which makes the stability condition $|k_0 k_L| < 1$ intuitive.

In the present paper, our contribution is to extend this Lyapunov stability analysis to the case where integral terms are introduced in the control law and to illustrate the methodology with experimental results.

3 Integral Actions and Lyapunov Stability Analysis

In order to cope with static errors, integral terms will be added to the control laws defined by (11)-(12) and the Lyapunov function (13) will be modified accordingly. Moreover, in order to simplify the notations in the Lyapunov stability analysis, the following notation is used $h_0(t)$=$h(t,0)$ and similar notations h_L, q_0, q_L, a_0, a_L, b_0, b_L for all variables at the two boundaries.

The boundary control laws $u_0(t)$ and $u_L(t)$ are defined such that the boundary conditions (3)-(4) expressed in the Riemann coordinates satisfy the linear relations (11)-(12) augmented with appropriate integrals as follows:

$$a_0(t) + k_0 b_0(t) + m_0 y_0(t) = 0, \tag{14}$$
$$b_L(t) + k_L a_L(t) + m_L y_L(t) = 0, \tag{15}$$

where k_0, k_L and m_0, m_L are constant design parameters that have to be tuned to guarantee the stability. The integral y_0 on the flow q at the boundary $x = 0$ and the integral y_L on the other state h at the boundary $x = L$ are defined as:

$$y_0(t) = \int_0^t q_0(s)ds = \int_0^t \frac{ca_0(s) + db_0(s)}{c+d}ds,$$
$$y_L(t) = \int_0^t h_L(s)ds = \int_0^t \frac{a_L(s) - b_L(s)}{c+d}ds.$$

The goal of this Section is to prove the following theorem

Theorem 1 *Let m_0, m_L and k_0, k_L be four constants such that the six following inequalities hold:*

$$m_0 > 0, \tag{16}$$
$$m_L < 0, \tag{17}$$
$$\frac{d}{c} < 1, \tag{18}$$
$$|k_0| < 1, \tag{19}$$
$$|k_L| < \frac{c}{d}, \tag{20}$$
$$|k_0 k_L| < 1. \tag{21}$$

Then there exist five positive constants A, B, μ, N_0 and N_L such that, for every solution $(a(t,x), b(t,x))$, $t \geq 0$, $x \in [0,L]$, of (7), (8), (14) and (15) the following function:

$$U(t) = \frac{A}{c}\int_0^L a^2(t,x)e^{-\mu x/c}dx + \frac{B}{d}\int_0^L b^2(t,x)e^{\mu x/d}dx + \frac{c+d}{2}N_0 y_0^2(t) + \frac{c+d}{2}N_L y_L^2(t)$$

satisfies:

$$\dot{U} \leq -\mu U.$$

In particular, there exists $C > 0$, independent of a, b, y_0 and y_L, such that

$$\psi(t) \leq C\psi(0)\exp(-\mu t), \quad \forall t \geq 0$$

with

$$\psi(t) \hat{=} \int_0^L (a^2(t,x) + b^2(t,x))dx + |y_0(t)|^2 + |y_L(t)|^2.$$

Remark 1 *As it has been mentioned above, in our previous paper [2] the special case with $m_0 = m_L = 0$ in the boundary conditions (14)- (15) and $N_0 = 0$, $N_L = 0$ has been treated. We have shown that inequality $|k_0 k_L| < 1$ is sufficient to have $\dot{U} \leq -\mu U$ for some $\mu > 0$ along the system trajectories and ensure the convergence of $a(t,x)$ and $b(t,x)$ to zero.*

Proof

The function $U(t)$ is clearly definite positive. The time derivative of $U(t)$ along the trajectories of the linear system (7)-(8) is

$$\begin{aligned}\dot{U} &= -\mu U - A\left(e^{-\mu L/c}a_L^2 - a_0^2\right) - B\left(b_0^2 - e^{\mu L/d}b_L^2\right) \\ &+\mu\frac{c+d}{2}\left[N_0 y_0^2(t) + N_L y_L^2(t)\right] \\ &+N_0 y_0(t)(ca_0(t) + db_0(t)) + N_L y_L(t)(a_L(t) - b_L(t)),\end{aligned}$$

or:

$$\dot{U} = -\mu U + \dot{U}_0 + \dot{U}_L,$$
$$\dot{U}_0 = Aa_0^2 - Bb_0^2 + N_0 y_0 (ca_0 + db_0) + \mu \frac{c+d}{2} N_0 y_0^2(t),$$
$$\dot{U}_L = -\tilde{A} a_L^2 + \tilde{B} b_L^2 + N_L y_L (a_L - b_L) + \mu \frac{c+d}{2} N_L y_L^2(t)$$
with $\tilde{A} = Ae^{-\mu L/c}, \tilde{B} = Be^{\mu L/d}$.

The last two terms $\dot{U}_0$ and $\dot{U}_L$ depend only on the Riemann coordinates at the two boundaries, i.e. at $x = 0$ and at $x = L$.

The analysis of $\dot{U}_0$ gives (using (14)-(15)):

$$\begin{aligned}\dot{U}_0 &= Aa_0^2 - Bb_0^2 + N_0 y_0 (ca_0 + db_0) + \mu N_0 y_0^2 \\ &= \left[Ak_0^2 - B\right] b_0^2 + \left[2Ak_0 m_0 + N_0(d - ck_0)\right] b_0 y_0 \\ &\quad + \left[Am_0^2 + \mu N_0 \frac{c+d}{2} - N_0 m_0 c\right] y_0^2.\end{aligned}$$

Hence $-\dot{U}_0$ is a positive definite quadratic form of the variables b_0, y_0 if:

$$(i): \quad \left[Ak_0^2 - B\right] b_0^2 + \left[2Ak_0 m_0 + N_0(d - ck_0)\right] b_0 y_0 + \left[Am_0^2 + \mu N_0 \frac{c+d}{2} - N_0 m_0 c\right] y_0^2 < 0,$$

which is equivalent to:

$$* \, Ak_0^2 - B < 0, \tag{22}$$
$$* \, (ii): \Delta_i = 4ABm_0^2 + 4N_0(Ak_0 d - cB)m_0 + N_0^2(d - ck_0)^2 + 4(B - Ak_0^2)\mu N_0 \frac{c+d}{2} < 0, \tag{23}$$

where Δ_i is a polynomial in μ, m_0 and N_0. Δ_i considered as a polynomial of degree 2 in N_0 takes negative values only if its discriminant Δ_{ii},

$$\Delta_{ii} = 16(Ak_0^2 - B)\left[m_0^2(Ad^2 - Bc^2) - 2m_0\mu\frac{c+d}{2}(Ak_0 d - Bc) + \left(\frac{c+d}{2}\right)^2 \mu^2 (Ak_0^2 - B)\right],$$

viewed as a polynomial of degree 2 in μ and m_0, is positive which is equivalent to

$$(iii): \quad \left[m_0^2(Ad^2 - Bc^2) - 2m_0\mu\frac{c+d}{2}(Ak_0 d - Bc) + \left(\frac{c+d}{2}\right)^2 \mu^2 (Ak_0^2 - B)\right] < 0.$$

The discriminant of the quadratic form (iii) in μ and m_0 is

$$\Delta_{iii} = 4\left(\frac{c+d}{2}\right)^2 \left[AB(ck_0 - d)^2\right]$$

and therefore is always nonnegative. Hence the roots of the left-hand side polynomial of (iii) are real and expressed as:

$$\mu_{0,1} = \frac{m_0(Ak_0 d - cB) + |m_0|\sqrt{AB}|ck_0 - d|}{(Ak_0^2 - B)\frac{c+d}{2}},$$

$$\mu_{0,2} = \frac{m_0(Ak_0 d - cB) - |m_0|\sqrt{AB}|ck_0 - d|}{(Ak_0^2 - B)\frac{c+d}{2}}.$$

In order to have $0 < \mu_{0,1} < \mu_{0,2}$, because of (22) and since $m_0 > 0$ (see (16)), we require that

$$Ak_0 d - cB < 0. \tag{24}$$

In addition, since from (iii) we have $\mu_{0,1}\mu_{0,2} = -m_0^2(Ad^2 - Bc^2)$, we also require that

$$Ad^2 - Bc^2 < 0. \tag{25}$$

From now on we thus assume that the parameters A and B are chosen such that inequalities (22) and (25) hold as (24) is deduced from (22), (25). This implies that $0 < \mu_{0,1} < \mu_{0,2}$ and that inequality (iii) is satisfied if $\mu \in (0, \mu_{0,1})$.

Furthermore inequality (ii) is satisfied if $N_0 \in (N_{0,1}, N_{0,2})$ with:

$$N_{0,1} = \frac{-4\left[\mu\frac{c+d}{2}(B - Ak_0^2) + (Ak_0 d - cB)m_0\right] - \sqrt{\Delta_{ii}}}{2(d - ck_0)^2},$$

$$N_{0,2} = \frac{-4\left[\mu\frac{c+d}{2}(B - Ak_0^2) + (Ak_0 d - cB)m_0\right] + \sqrt{\Delta_{ii}}}{2(d - ck_0)^2}.$$

$N_{0,2}$ is positive if

$$0 < \mu < \mu_{0,1} < \frac{(Ak_0 d - cB)m_0}{\frac{c+d}{2}(Ak_0^2 - B)}.$$

Hence it exists $N_0 > 0$ such that inequality (i) is satisfied.

The analysis of $\dot{U}_L$ is performed in the same way:

$$\begin{aligned}\dot{U}_L &= -\tilde{A} a_L^2 + \tilde{B} b_L^2 + N_L y_L (a_L - b_L) + \mu\frac{c+d}{2} N_L y_L^2 \\ &= \left[\tilde{B} k_L^2 - \tilde{A}\right] a_L^2 + \left[2\tilde{B} k_L m_L + N_L(1 + k_L)\right] a_L y_L \\ &\quad + \left[\tilde{B} m_L^2 + N_L m_L + \mu\frac{c+d}{2} N_L\right] y_L^2\end{aligned}$$

and $-\dot{U}_L$ is a positive definite quadratic form of the variables a_L, y_L if:

$$(iv): \left[\tilde{B} k_L^2 - \tilde{A}\right] a_L^2 + \left[2\tilde{B} k_L m_L + N_L(1 + k_L)\right] a_L y_L + \left[\tilde{B} m_L^2 + N_L m_L + \mu\frac{c+d}{2} N_L\right] y_L^2 < 0.$$

The same arguments as for $\dot{U}_0$ show that there exists $N_L > 0$ such that $\dot{U}_L \leq -\mu U_L$ if

$$\tilde{B}k_L^2 - \tilde{A} < 0, \quad (26)$$
$$\tilde{B} - \tilde{A} < 0, \quad (27)$$

and $\mu \in (0, \mu_{L,1})$ with

$$\mu_{L,1} = \frac{-m_L(A + k_L B) - |m_L|\sqrt{AB}|k_L + 1|}{(A - Bk_L^2)\frac{c+d}{2}}.$$

Conditions (18)-(19) and (20)-(21) allow to choose the positive constants A and B such that:

$$\inf\left(\frac{c^2}{d^2}, \frac{1}{k_0^2}\right) > \frac{A}{B} > \max\left(1, k_L^2\right). \quad (28)$$

Then μ can be chosen small enough $(\mu \in (0, \mu_{0,1}) \cap (0, \mu_{L,1}))$ such that inequalities (22)-(25)-(24) and (26)-(27) are satisfied simultaneously, i.e.:

$$\inf\left(\frac{c^2}{d^2}, \frac{1}{k_0^2}\right) > \frac{A}{B} > \frac{1}{\sigma}\max\left(1, k_L^2\right), \quad (29)$$

with $\sigma = e^{\mu L(\frac{1}{c}+\frac{1}{d})}$.

So, $\dot{U}(t) \leq -\mu U(t)$ along the trajectories of the linear system (7)-(8). ■

Remark 2 *The converse is true, i.e. if there exist five positive constants A, B, μ, N_0 and N_L such that, for every solution $(a(t,x), b(t,x))$, $t \geq 0$, $x \in [0, L]$, of (7), (8), (14) and (15) the following function is asymptotically stable:*

$$U(t) = \frac{A}{c}\int_0^L a^2(t,x)e^{-\mu x/c}dx + \frac{B}{d}\int_0^L b^2(t,x)e^{\mu x/d}dx + \frac{c+d}{2}\left(N_0 y_0^2(t) + N_L y_L^2(t)\right),$$

then the four constants m_0, m_L and k_0, k_L verify:

$$m_L < 0, \qquad m_0 > 0, \quad (30)$$
$$|k_0| < 1, \qquad \frac{d}{c} < 1, \quad (31)$$
$$|k_L| < \frac{c}{d}, \qquad |k_0 k_L| < 1. \quad (32)$$

Remark 3 *Obviously, there is no difference in switching h and q in the definitions of the integrals (14)-(15):*

$$y_0(t) = \int_0^t h_0(s)ds = \int_0^t \frac{a_0(s) - b_0(s)}{c+d}ds,$$
$$y_L(t) = \int_0^t q_L(s)ds = \int_0^t \frac{ca_L(s) + db_L(s)}{c+d}ds.$$

In this case, one obtains the following conditions:

$$m_0 < 0, \qquad m_L > 0,$$
$$\frac{c}{d} < 1, \qquad |k_L| < 1,$$
$$|k_0| < \frac{d}{c}, \qquad |k_0 k_L| < 1.$$

4 Application to the Saint-Venant Linearized System

4.1 Non Linear System

A prismatic open channel with a constant rectangular section and a constant slope is considered. The flow dynamics are described by the Saint-Venant equations [10], [4]:

$$\partial_t H + \partial_x(Q/\hat{b}) = 0, \quad (33)$$
$$\partial_t Q + \partial_x\left(\frac{Q^2}{\hat{b}H} + \frac{1}{2}g\hat{b}H^2\right) = g\hat{b}H(I - J), \quad (34)$$

where $H(t,x)$ represents the water level and $Q(t,x)$ the water flow rate, $\hat{b}$ the channel width and g the gravitation constant. I is the bottom slope and J is the friction slope expressed with the Manning-Strickler expression:

$$J(H,Q) = \frac{n_M^2 Q^2}{[S(H)]^2[R(H)]^{4/3}},$$

with n_M the Manning coefficient while $S(H) = \hat{b}H$ is the wet surface and $R(H)$ is the hydraulic radius given by:

$$R(H) = \frac{S(H)}{P(H)},\ P(H) = \hat{b} + 2H := \text{wet perimeter}.$$

4.2 Linearized system

An equilibrium (H_e, Q_e) is a constant solution of equations (33)-(34), i.e. $H(t,x) = H_e$, $Q(t,x) = Q_e$ $\forall t$ and $\forall x$ which satisfies the relation:

$$J(H_e, Q_e) = I. \quad (35)$$

A linearized model is used to describe the variations around this equilibrium. The following notations are introduced:

$$h(t,x) \hat{=} H(t,x) - H_e(x),\ q(t,x) \hat{=} Q(t,x) - Q_e(x).$$

The linearized model around the equilibrium (H_e, Q_e) is then written as

$$\partial_t \hat{b} h(t,x) + \partial_x q(t,x) = 0, \tag{36}$$

$$\partial_t q(t,x) + cd\partial_x \hat{b} h(t,x) + (c-d)\partial_x q(t,x) = -\gamma h(t,x) - \delta q(t,x), \tag{37}$$

with:

$$c = \sqrt{gH_e} + \frac{Q_e}{H_e \hat{b}},\ d = \sqrt{gH_e} - \frac{Q_e}{H_e \hat{b}},$$

$$\gamma = g\hat{b}H_e \frac{\partial J}{\partial H}(H_e, Q_e),\ \delta = g\hat{b}H_e \frac{\partial J}{\partial Q}(H_e, Q_e).$$

In the special case where the channel is horizontal ($I = 0$) and the friction slope is negligible ($n_M \approx 0$), we observe that $\gamma = \delta = 0$ and that this linearized system is exactly in the form of the linear hyperbolic system (1)-(2) that we have handled in Section 2. It is therefore legitimate to apply the control with integral actions that has been analyzed above to open channels having small bottom and friction slopes.

4.3 Connection with classical PI control

We have seen above that the feedback control laws must be defined in order that the boundary conditions (14)-(15) hold. The derivation of an explicit expression of the control laws obviously requires an explicit formulation of the boundary conditions (3)-(4). In this Section, we illustrate how the control laws can be derived and we clarify the connection with classical PI control.
In Section 5, we shall present practical simulations and experimental results for channels that are bounded by either overflow spillways or underflow gates.

The gate characteristics of overflow gates are expressed as:

$$Q(t,0) = (c_0\hat{b})^3 \left[2g(H_{up} - U_0(t))\right]^{(3/2)}, \tag{38a}$$

$$Q(t,L) = (c_L\hat{b})^3 \left[2g(H(t,L) - U_L(t))\right]^{(3/2)}, \tag{38b}$$

while for underflow gates, the gate characteristics are expressed as:

$$Q(t,0) = c_0 U_0(t)\hat{b}\sqrt{2g(H_{up} - H(t,0))}, \tag{39a}$$

$$Q(t,L) = c_L U_L(t)\hat{b}\sqrt{2g(H(t,L) - H_{do})}, \tag{39b}$$

where c_0 and c_L are the gate water flow coefficients, while U_0 and U_L denote the control signals at the upstream and downstream gates respectively. H_{up} is the water level at the upstream of the upstream gate, H_{do} is the water level at the downstream of the downstream gate.

In order to explicit the control laws, the gate characteristics (39) are linearized about the steady-state (H_e, Q_e):

$$q(t,0) = K_0' h(t,0) + K_0 u_0(t), \tag{40a}$$

$$q(t,L) = K_L' h(t,L) + K_L u_L(t), \tag{40b}$$

with, for the spillway gates

$$K_0 = -3g(c_0\hat{b})^2 Q_e^{1/3}, \qquad K_0' = 0, \tag{41}$$

$$K_L = -3g(c_L\hat{b})^2 Q_e^{1/3}, \qquad K_L' = 3g(c_L\hat{b})^2 Q_e 1/3, \tag{42}$$

and for the underflow gates

$$K_0 = c_0\hat{b}\sqrt{2g(H_{up} - H_e)}, \qquad K_0' = -\frac{Q_e}{2(H_{up} - H_e)}, \tag{43}$$

$$K_L = c_L\hat{b}\sqrt{2g(H_e - H_{do})}, \qquad K_L' = \frac{Q_e}{2(H_e - H_{do})}. \tag{44}$$

Moreover, using the definition of the Riemann coordinates (5)-(6), the boundary conditions (14)-(15) are rewritten as

$$q(t,0) + \lambda_0 h(t,0) + \mu_0 \int_0^t q(s,0)ds = 0, \tag{45a}$$

$$q(t,L) + \lambda_L h(t,L) + \mu_L \int_0^t h(s,L)ds = 0, \tag{45b}$$

with:

$$\lambda_0 = \frac{(d - k_0 c)}{1 + k_0}, \qquad \lambda_L = \frac{(k_L d - c)}{1 + k_L},$$

$$\mu_0 = \frac{m_0}{1 + k_0}, \qquad \mu_L = \frac{m_L}{1 + k_L}.$$

Then, by eliminating $h(t,0)$ between (40a) and (45a), we get the following PI control law for u_0:

$$u_0(t) = K_{po} q(t,0) + K_{io} \int_0^t q(s,0)ds$$

with

$$K_{po} = \frac{\lambda_0 + K_0'}{\lambda_0 K_0}, \qquad K_{io} = \frac{\mu_0 K_0'}{\lambda_0 K_0}.$$

Similarly, by eliminating $q(t,L)$ between (40b) and (45b), we get the following PI control law for u_L:

$$u_L(t) = -K_{pL} h(t,L) - K_{iL} \int_0^t h(s,L)ds$$

with

$$K_{pL} = \frac{\lambda_L + K_L'}{K_L}, \qquad K_{iL} = \frac{\mu_L}{K_L}.$$

Hence the control law u_0 is a PI dynamic feedback of the flow rate $q(t,0) = Q(t,0) - Q_e$ and the control law u_L is a PI dynamic feedback of the water depth $h(t,L) = (H(t,L) - H_e)$. These control laws are implemented with direct on-line measurements of the water levels H_{up}, H_{do}, $H(t,0)$, $H(t,L)$.

5 Simulations and experimental results

5.1 Simulations

Various simulations have been carried out with the data of the Sambre river located in Belgium.
Two simulation results are described here, the first one showing the impact of the integral terms m_0 and m_L, the second one the efficiency against constant perturbations.
A pool of the Sambre river is considered, it is bounded by two mobile spillway gates as illustrated in Fig. (1). The characteristic parameters of the pool are given in Table (1).
The angular positions of the two mobile gates are the control actions (see (38)). More precisely, these two controls aim at regulating the upstream flow rate at a prescribed set point Q_e and the downstream water level at a prescribed level H_e.

Fig. 1. A mobile spillway gate on the Sambre river

parameters	$\hat{b}$ (m)	L (m)	slope $(m.m^{-1})$	n_M^{-1} $(m^{1/3}.s^{-1})$
values	40	11239	$7.92.10^{-5}$	33

Table 1
Parameters of one reach of the Sambre river

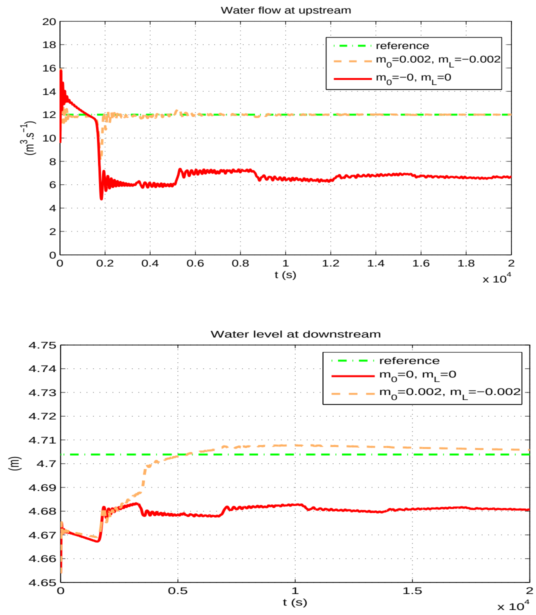
The set points are:

$$Q_e = 12m^3.s^{-1},\ H_e(L) = 4.7m.$$

The initial condition is assumed to be another steady state with the following values:

$$Q(0,x) = 10m^3.s^{-1},\ H(0,L) = 4.65m.$$

The simulation results are presented in Fig. (2) and (3). The control parameters are $k_0 = -0.0837$, and $k_L = -0.0384$ while the values m_0, m_L are given in the figure captions. In Fig. (2), a first simulation is done without integral actions ($m_0 = m_L = 0$). In this case the closed loop is stable (since $|k_0 k_L| < 1$) but, as expected, there is a significant static error resulting from the bottom and the frictions slopes. A second simulation with integral actions ($m_0 = -m_L = 0.002$) gives a fully satisfactory result since the closed loop is stable and the static errors of the two regulated variables are cancelled.



a)

b)

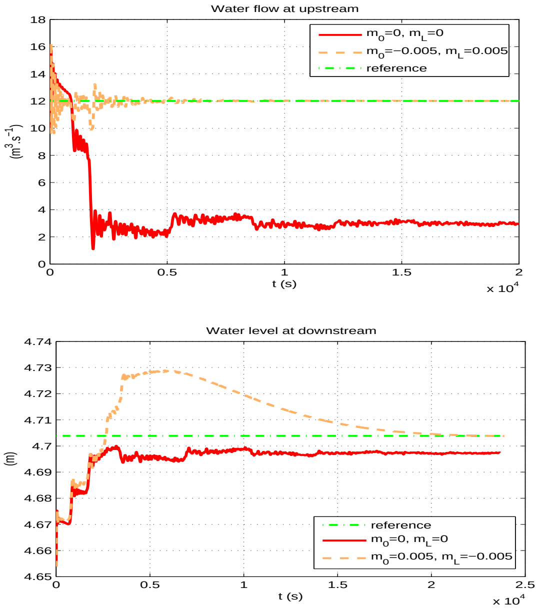
Fig. 2. Water flows at upstream (a) and water levels at downstream(b) for different values of the integral terms

The simulations presented in Fig. (3) allow to assess the efficiency of the control against a constant unknown disturbance. The disturbance is a constant positive side flow rate of $1.12m^3.s^{-1}$ uniformly distributed along the pool (i.e. a disturbance of about 10% of the flow rate). There is also a simulation without integral actions given in order to have an idea of the effect of the disturbance. A controller with integral actions ($m_0 = -m_L = 0.005$) totally compensates the unknown constant disturbance. Remark that this latter simulation is done with greater integral gains than in Fig. (2). However, it should be mentioned that other simulation experiments, that are not shown here, have indicated that the closed loop becomes unstable when the integral gains reach a value of the order of 0.008.

5.2 Experimentations

An experimental validation has been performed on the Valence micro-channel, Fig. (1) & (6), Tab.2. This pilot channel is located at ESISAR [4] /INPG [5] engineering

[4] École Supérieure d'Ingénieurs en Systèmes industriels Avancés Rhône-Alpes
[5] Institut National Polytechnique de Grenoble



a)

b)

Fig. 3. Water flows at upstream (a) and water levels at downstream (b) for different values of the integral terms

school in Valence (France). It is operated under the responsibility of the LCIS [6] laboratory. This experimental channel (total length=8 meters) has an adjustable slope and a rectangular cross-section (width=0.1 meter). The channel is ended at downstream by a variable overflow spillway and furnished with three underflow control gates (Fig. (4) and Fig. (6)). Ultrasound sensors provide water level measurements at different locations of the channel (Fig. (5)).

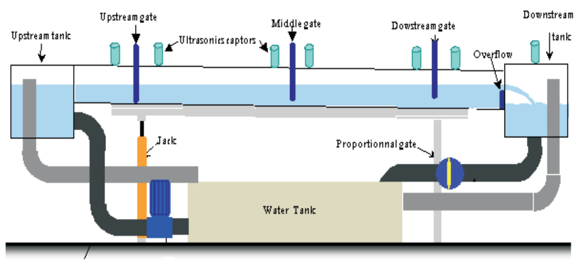


Fig. 4. Pilot channel of Valence

parameters	$\hat{b}(m)$	$L\ (m)$	$K\ (m^{1/3}.s^{-1})$
values	0.1	7	97
parameters	c_0	c_L	slope $(m.m^{-1})$
values	0.6	0.73	1.6‰

Table 2
Parameters of the channel of Valence

For the experimentation reported here, the middle gate is completely open and we have a single pool (length=7

[6] Laboratoire de Conception et d'Intégration des Systèmes

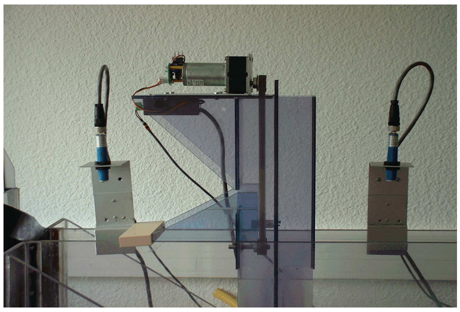


Fig. 5. Pilot channel of Valence: gate and ultrasound sensors

Fig. 6. Pilot channel of Valence

meters) bounded by two underflow gates. The flow rate at the gate is not directly measured but calculated from the gate characteristics (39).

The water levels H_{up} at the upstream of the upstream gate and H_{do} at the downstream of the downstream gate are controlled on-line to stand at the following values:

$$H_{up} = 1.72dm, H_{do} = 0.85dm.$$

In order to satisfy the stability condition (29), parameters k_0 and k_L are set to:

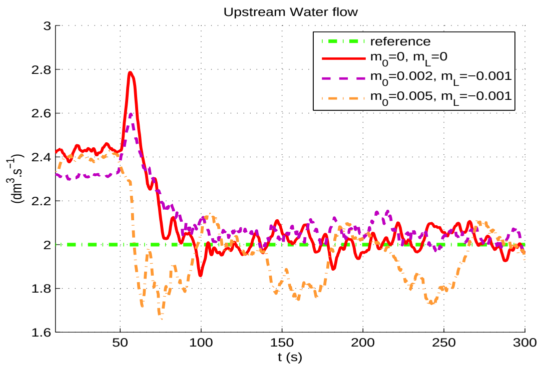
$$k_0 = -0.213,\ k_L = -1.157,\ k_0 k_L = 0.247.$$

Fig. (7) illustrate the efficiency of the control. Three experiments are shown with increasing values of the integral gains m_0 and m_L indicated in the figure captions. In the experiment, the system is initially in open loop at a steady state:

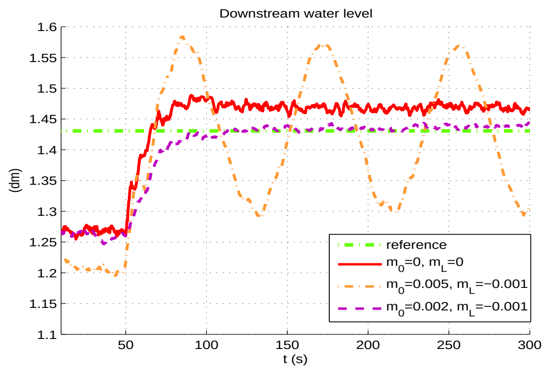
$$Q(0,x) \approx 2.35dm^3.s^{-1},\ H(0,L) \approx 1.25dm.$$

The loop is closed at time $t = 50sec$ with a new set point given by:

$$Q_e(0) = 2dm^3.s^{-1},\ H_e(L) = 1.43dm.$$



a)



b)

Fig. 7. Water flows at upstream (a) and levels at downstream (b)

Without integral action ($m_0 = m_L = 0$), there is clearly an offset of about 4cm on the level $H(t, L)$. But this static error is efficiently cancelled by the integral actions ($m_0 = 0.002$, $m_L = -0.001$). The experiments also illustrate the sensitivity of the transient behavior with respect to the choice of the gain values. For the largest tested values ($m_0 = 0.005$, $m_L = -0.001$), the closed loop system starts to oscillate (Fig. 7) and becomes unstable for still larger values of m_0.

6 Conclusion

This paper was concerned with the boundary control of hyperbolic systems of conservation laws. We have shown how integral actions can be added to the static control law previously proposed in [2] in order to cope with constant disturbances. The main contribution of the paper is a Lyapunov stability analysis of the proposed feedback control system. In Theorem 1, we have given sufficient conditions on the values of the control parameters to guarantee the exponential convergence for linear homogeneous systems. Although it is not a trivial task, the Lyapunov analysis can be extended to the linearized nonhomogeneous system (36)-(37) and even, following the method of [2] to **nonlinear** two-by-two systems of quasi linear hyperbolic equations. The efficiency of the approach has been illustrated with simulations on a realistic waterway model and experimental validations on a small laboratory pilot canal.

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