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A SECOND ORDER MODEL OF ROAD JUNCTIONS IN FLUID MODELS OF TRAFFIC NETWORKS

Bertrand Haut and Georges Bastin

CESAME, Avenue G. Lemaître, 4
1348 Louvain-la-Neuve, Belgium

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Abstract. This article deals with the modeling of junctions in a road network from a macroscopic point of view. After reviewing the Aw & Rascle second order model, a compatible junction model is proposed. The properties of this model and particularly the stability are analyzed. It turns out that this model presents physically acceptable solutions, is able to represent the capacity drop phenomenon and can be used to simulate the traffic evolution on a network.

1. **Introduction.** In the fluid paradigm for road traffic modeling, the traffic is described in terms of two basic macroscopic variables: the density and the speed of the vehicles at position x along the road at time t (denoted $\rho(x,t)$ and $v(x,t)$). A usual way to describe a network traffic model is as follows:

- First, the equations binding the values of ρ and v to the initial conditions on an infinite single road are considered. These equations are usually a set of partial differential equations (PDE). A traditional problem studied for such systems is the Riemann problem which is an initial value problem where the initial condition consists of two constant values:
$$(\rho(x,0), v(x,0)) = \begin{cases} (\rho_l, v_l) & \text{if } x < 0 \\ (\rho_r, v_r) & \text{if } x \geq 0. \end{cases} \tag{1}$$
The Riemann problem is important, not only since it allows an explicit solution but also because the solution of any initial value problem with arbitrary initial conditions can be constructed from a set of appropriate Riemann problems (see e.g. [3]).
- Then, the junctions at the nodes of the network are introduced. The junctions represent the connections between different roads, for example the merging of two roads in one or the fork of one road in two. An appropriate description of the behaviour of the drivers at the junction must then be provided. One way to do this is to describe the solution of the Riemann problem at the junction.
If we consider a junction with some incoming and some outgoing roads, the initial state is
$$(\rho_i(x,0), v_i(x,0)) = (\rho_{i,0}, v_{i,0}) \qquad \forall x,\ \forall i \tag{2}$$

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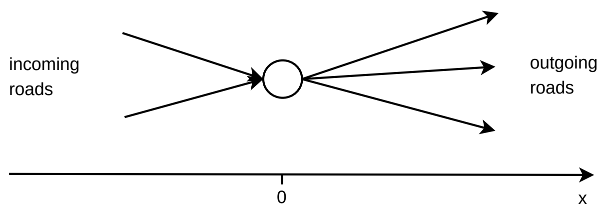


where subscript i refers to road i. As we shall see later in this paper, the Riemann problem (2) is underdetermined. Additional constraints are needed to get a unique solution. The establishment of these additional conditions constitutes one of the main parts of this article.

The first time and space continuous fluid flow models that were developed in the literature were based on the LWR model (see [15], [17], [12], [5], [9] and [13]). The LWR model is a first order model which means that there is only one PDE describing the evolution of the traffic state and that the solution to the Riemann problem (1) consists of one *wave* connecting the two initial states.

In this paper we intend to establish a second order traffic model for **road networks** that is based on the Aw and Rascle second order model for **single roads** (see [2]). In Section 2, we briefly review the Aw & Rascle model. In Section 3 the model of the junction is presented. This model expresses which solution of the Riemann problem at the junction is selected. In Section 4 the different properties of the model are analyzed. The existence of a solution in the BV (bounded variation) space is demonstrated in Section 5. The model is compared with the existing models and some conclusions are drawn in Section 6.

2. **The Aw & Rascle model for single roads.** The Aw and Rascle model (see [2]) for a single road is described by two equations:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0 &(3)\\ (\partial_t + v\partial_x)v + (\partial_t + v\partial_x)p(\rho) &= 0. &(4) \end{aligned}$$

where $p(\rho)$ is a smooth increasing function of the density such as $\frac{d^2}{d\rho^2}(\rho p(\rho)) > 0$. All the drivers move in the direction of positive x ($v > 0$). The first equation represents the conservation of the flow while the second equation describes the evolution of the speed of the drivers in a function of the surrounding traffic state. One additional term may be added to the right part of (4), the relaxation term $\frac{V(\rho)-v}{\tau}$. This term expresses the fact that the drivers tend to adopt a preferential speed V function of the surrounding density ρ. The dynamics induced by this term are usually slow and are negligible when dealing with very short time such as the crossing of a junction. This term will thus be omitted in this article.

Multiplying (3) by $(v + p(\rho))$ and (4) by ρ and adding up these two equations, we obtain

$$\partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0.$$

Therefore the system is composed of two conserved quantities: ρ and $\rho(v + p(\rho))$.

The system (3)–(4) may have discontinuous solutions. The meaning of the differential equations in presence of discontinuities and the admissibility of these discontinuities are explained in [3]. Because the system is expressed by two equations of conservation, the solution of a Riemann problem

$$(\rho(x,0), v(x,0)) = \begin{cases} U_l = (\rho_l, v_l) & \text{if } x < 0 \\ U_r = (\rho_r, v_r) & \text{if } x \geq 0 \end{cases}$$

consists of the connection of the left state U_l to an intermediate state U_c by a first wave and the connection of this intermediate state to the right state U_r by a second wave. There are two waves because there are two conservation laws. The two waves are different:

- the first one may be a shock or a rarefaction wave. A shock wave is a discontinuity in ρ and/or in v travelling at a constant speed. A rarefaction wave is a self-similar solution, i.e. it depends only on x/t.
- the second one must be a contact discontinuity. The contact discontinuity separates two constant states with the same speed but different densities. This contact discontinuity travels at the same speed as the vehicles.

We will not present here the complete and rigorous description of the Riemann problem (see [2]) but only the two most simple and common cases. In the first two graphs of Figure 1, we have represented in the $(\rho, \rho v)$ plane:

- the two initial state U_l and U_r;
- the straight line $(\rho, \rho v_r)$ passing through 0 and U_r;
- the curve $\Upsilon = (\rho, \rho K - \rho p(\rho))$ passing through U_l. There is only one value of K such that this curve passes through U_l, this value is equal to $v_l + p(\rho_l)$;
- the intermediate state U_c which is at the intersection of the straight line and Υ.

Two cases must be considered:

- If $v_l > v_r$, then the solution consists of a shock wave connecting U_l to U_c followed by a contact discontinuity connecting U_c to U_r. See Figure 1 a). The first graph represents the three states in the $(\rho, \rho v)$ plane, the second graph the initial state in the (ρ, x) plane and the third graph the state on the road after some time. The speed of the shock wave is equal to the slope of the line connecting U_l to U_c.
- If $v_l < v_r$, then the solution consists of a rarefaction wave connecting U_l to U_c followed by a contact discontinuity connecting U_c to U_r. See Figure 1 b). The space interval occupied by the rarefaction wave is

$$\left[\left.\frac{d(\rho K - \rho p(\rho))}{d\rho}\right|_{\rho=\rho_l} t,\ \left.\frac{d(\rho K - \rho p(\rho))}{d\rho}\right|_{\rho=\rho_c} t \right]$$

which correspond to t times the slope of the curve Υ evaluated at U_l and U_c.

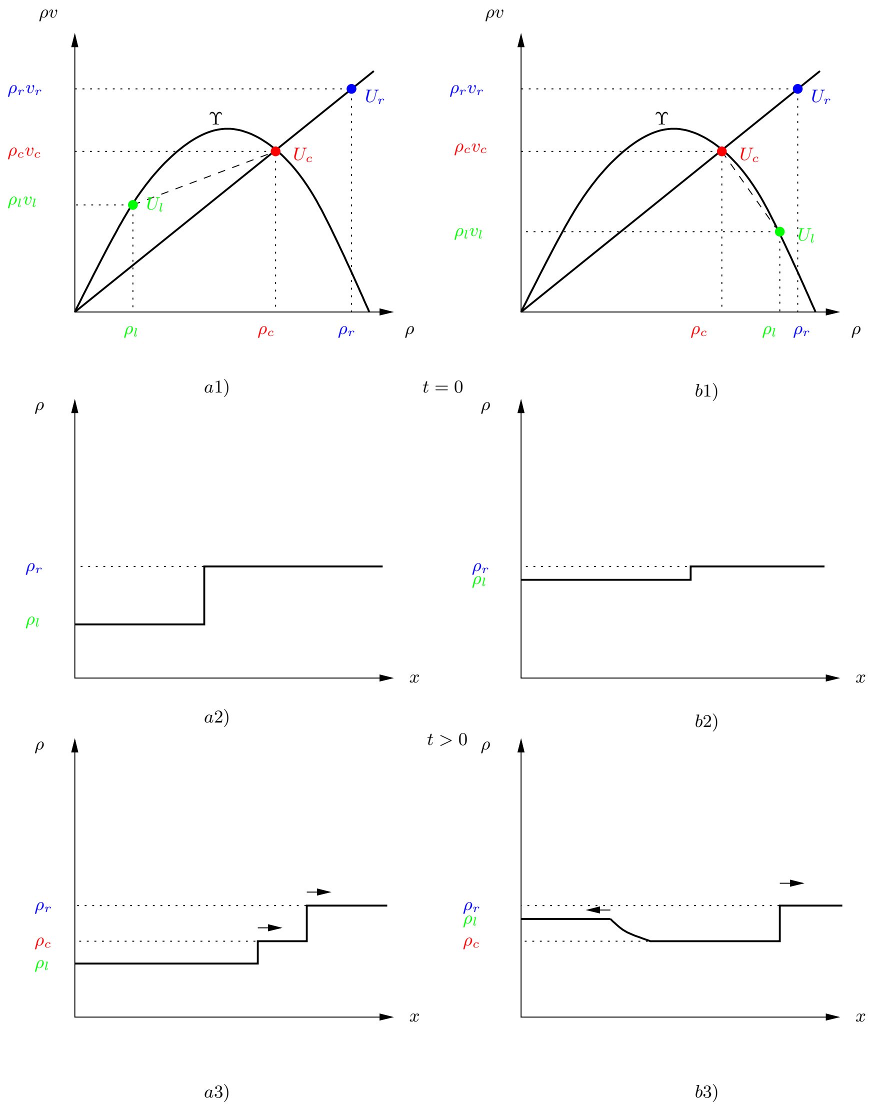


FIGURE 1. Two cases are represented in this figure. The first case (a) brings into play a shock wave and a contact discontinuity. In the second case (b) a rarefaction wave and a contact discontinuity are present. For each case, three graphs are represented. In the first row, the two initial states U_l, U_r and the intermediate state U_c are represented. The intermediate state is at the intersection of the curve Υ and the straight line passing trough 0 and U_r. The second row represents the corresponding initial states in the (x, ρ) plane. The states of the densities after some time are represented in the last row.

3. **The Riemann problem at the junction.** Having described the solution of a Riemann problem on a single road, it remains to depict the solution of a Riemann problem at junctions in order to be able to model the complete evolution of the traffic state on a network.

If we consider a Riemann problem at a junction, the initial state is

$$(\rho_i(x,0), v_i(x,0)) = (\rho_{i,0}, v_{i,0}) \qquad \forall x,\ \forall i \tag{5}$$

where subscript i refers to road i. See Figure 2 a) for an example of a junction with two incoming and one outgoing roads. Since a constant state is an equilibrium for

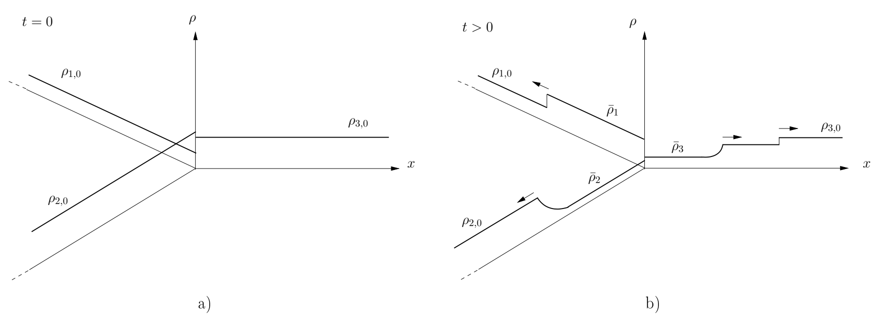


FIGURE 2. A Riemann problem for a junction with two incoming and one outgoing roads: the initial density and the density after some times.

the single road model, a modification of the state may only appear initially at the junction. We may therefore consider distinct Riemann problems on each road:

$$(\rho_i(x,0), v_i(x,0)) = \begin{cases} (\bar\rho_i, \bar v_i) & \text{if } x = 0 \\ (\rho_{i,0}, v_{i,0}) & \text{if } x \neq 0. \end{cases} \tag{6}$$

where $(\bar\rho_i, \bar v_i)$ is the new state on the road i after the first interaction between the drivers. The set of solutions of these Riemann problems on each road (6) will provide the solution of the junction Riemann problem (see Fig. 2 b)). The junction modeling problem then implies that we select appropriate values $\bar\rho_i, \bar v_i$ such that the collection of solutions of the Riemann problem (6) provides a global consistent solutions to the overall Riemann problem (5) at the junction.

Of course, the waves produced on the incoming (resp. outgoing) roads must have a negative (resp. positive) velocity to go away from the junction in order to get a sensible model. To take this constraint into account in the model, we will restrict the set of possible values of $(\bar\rho_i, \bar v_i)$ to a subset of $\mathbb{R}^2$ called the *admissible region* such that all waves produced by the Riemann problems (6) have negative (resp. positive) speed if i corresponds to an incoming (resp. outgoing) road.

This section is structured as follow: first the admissible regions for incoming and outgoing road are described and then an additional condition is presented in order to have a unique physically acceptable solution to the problem of selecting $(\bar\rho_i, \bar v_i)$.

3.1. **The admissible regions for the junction models.** As mentioned in the introduction, in order to formulate a junction model we first need to explicit for each road the admissible regions for the values of $(\bar\rho_i, \bar v_i)$. The shape of this admissible region will be different if we consider an incoming or an outgoing road.

3.1.1. *Incoming road.* On incoming roads, only the negative waves produced by the Riemann problem

$$(\rho_i(x,0), v_i(x,0)) = \begin{cases} (\bar{\rho}_i, \bar{v}_i) & \text{if } x = 0 \\ (\rho_{i,0}, v_{i,0}) & \text{if } x < 0 \end{cases}$$

are obviously admissible. Since the speed of the second type wave (contact discontinuity with a speed equal to the speed of the drivers) is necessarily positive, the only admissible wave is a wave of the first type (shock or rarefaction wave). Hence $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ must be on the curve Υ passing through $(\rho_{i,0}, \rho_{i,0} v_{i,0})$ where Υ is defined as

$$\Upsilon = (\rho, \gamma(\rho)) = (\rho, \rho K - \rho p(\rho))$$

with

$$K = v_{i,0} + p(\rho_{i,0}).$$

In this case, the intermediate state (U_c in Figure 1) of the Riemann problem $(\rho_{i,0}, \rho_{i,0} v_{i,0})$–$(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ is $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ and there is no second wave with positive speed. Defining σ as

$$\sigma = \arg\max_{\rho} \gamma(\rho),$$

two cases can be considered:

1. If $\rho_{i,0} < \sigma$ then the only possibility to have a wave with negative speed, is to have $\bar{\rho}_i > \tau(\rho_{i,0})$ (see Fig. 3(a)) where, for each $\rho \neq \sigma$, $\tau(\rho)$ is the unique number $\tau(\rho) \neq \rho$ such that

   $$\gamma(\rho) = \gamma(\tau(\rho)).$$

   In that case, the wave on the incoming road is a shock wave with a negative speed equal to the slope of the curve connecting $U_{i,0}$ to $\bar{U}_i$.
2. If $\rho_{i,0} \geq \sigma$ then two sub-cases must be distinguished:
   - if $\bar{\rho}_i \geq \rho_{i,0}$ the solution consists of a shock wave with a negative speed;
   - if $\bar{\rho}_i \leq \rho_{i,0}$ the solution will be a rarefaction wave. In order that the right limit of this rarefaction wave has a negative speed, we need that $\bar{\rho}_i \geq \sigma$ (see Fig. 3(b)).

The admissible region for an incoming road is thus composed of the part of the curve Υ represented in Fig. 3 and, of course, $(\rho_{i,0}, v_{i,0})$ for which there is no wave.

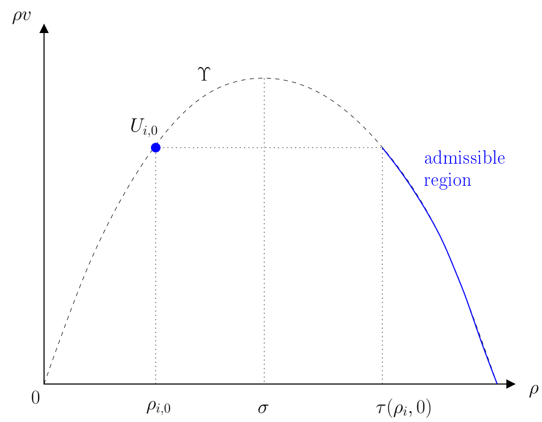
To be complete, we can specify the admissible region case of an empty road ($\rho_{i,0} = 0$). In this case, the variable $v_{i,0}$ has no physical meaning but only a mathematical sense. We can consider two cases:

- If $v_{i,0} \neq 0$, one can show that the state $(0, v_{i,0})$ cannot be connected to another state with a non-positive speed. The admissible region is thus $(\rho_{i,0}, v_{i,0})$.
- If $v_{i,0} = 0$, the state $(0,0)$ can be connected to any state $(\bar{\rho}_i, 0)$. The choice of the density $\bar{\rho}_i$ is arbitrary since all the solutions of the Riemann problem

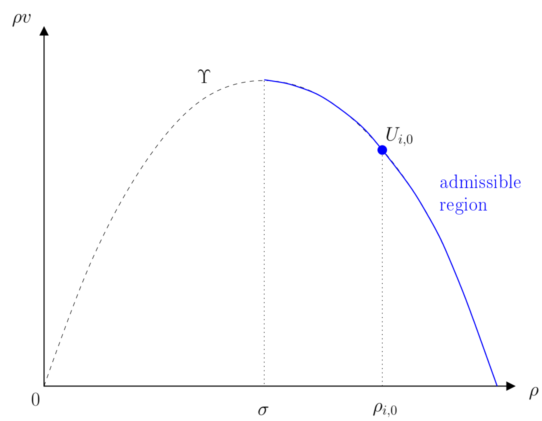
  $$(\rho(x,0), v(x,0)) = \begin{cases} (0,0) & \text{if } x < 0 \\ (\bar{\rho}_i, 0) & \text{if } x = 0 \end{cases}$$

  are identical in $L^1[-\infty, 0]$. Even if all values are possible for $\bar{\rho}_i$, the most meaningful is zero like the density on the incoming road.

We can notice a great similarity with the LWR first order models for which a value called the "sending capacity" or the "traffic demand" was introduced by



(a) $\rho_{i,0} \leq \sigma$



(b) $\rho_{i,0} > \sigma$

FIGURE 3. The admissible regions for an incoming road.

Daganzo (see [6] for details). This value represents the greatest possible outflow of a road segment and is equal to

$$\text{sending capacity} = \begin{cases} Q(\rho) & \text{if } \rho \leq \arg\max_\rho Q(\rho) \\ \max Q(\rho) & \text{if } \rho > \arg\max_\rho Q(\rho) \end{cases}$$

where $Q(\rho)$ represents the flow associated to the density ρ. In our second order model, the greatest possible outflow of a road segment is the maximal flow for a point on the admissible region and is equal to

$$\text{sending capacity} = \begin{cases} \gamma(\rho) & \text{if } \rho \leq \arg\max_\rho \gamma(\rho) \\ \max \gamma(\rho) & \text{if } \rho > \arg\max_\rho \gamma(\rho). \end{cases}$$

The similarity is obvious with the replacement of $Q(\rho)$ by $\gamma(\rho)$. In the second order model, the sending capacity is a function of the density but also of the speed (via the value of $v_{i,0} + p(\rho_{i,0})$ which is used in the definition of γ).
The admissible region satisfies some intuitive ideas:

- if there is nobody on the incoming road ($\rho_{i,0} = 0$), the maximal flow allowed to leave the road is zero;
- if there are few vehicles on the incoming road ($\rho_{i,0} \ll \sigma$), the flow allowed to leave the road is low (less than $\rho_{i,0} v_{i,0}$);
- if there is a lot of vehicles on the incoming road ($\rho_{i,0} \gg \sigma$), the flow allowed to leave the road may be high (up to $\gamma(\sigma)$).

3.1.2. *Outgoing road.* On outgoing roads, only the positive waves produced by the Riemann problem

$$(\rho_i(x,0), v_i(x,0)) = \begin{cases} (\bar{\rho}_i, \bar{v}_i) & \text{if } x = 0 \\ (\rho_{i,0}, v_{i,0}) & \text{if } x > 0 \end{cases}$$

are obviously admissible. This case is more complex than the previous one since we may have here the simultaneous presence of the two waves. The second wave has always a positive speed, we can thus connect any intermediate state U_c on the curve $(\rho, \rho v_{i,0})$ to $U_{i,0} = (\rho_{i,0}, \rho_{i,0} v_{i,0})$. The admissible region is thus the $(\rho, \rho v_{i,0})$ curve and the region of the $(\rho, \rho v)$–plane which can be connected to the curve $(\rho, \rho v_{i,0})$ with any rarefaction or shock wave with positive speed.

In order to characterize the admissible region all the possible curves Υ are considered. Υ has been defined as a curve passing by the left state of the Riemann problem ($(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ in this case). Since Υ only depends on $K = \bar{v}_i + p(\bar{\rho}_i)$, considering all curves Υ means considering all the possible values of $K = \bar{v}_i + p(\bar{\rho}_i) \in]0, \infty[$. To emphasize the fact that the curve Υ depends on the value K, it will be denoted Υ_K from now on.

Three cases must be considered (see Fig. 4):

a: If $0 \leq K \leq v_{i,0}$ then the curve Υ_K is necessarily located under the curve $(\rho, \rho v_{i,0})$. The left state $(\bar{\rho}_i, \bar{v}_i)$ is connected to the intermediate state U_c which is the vacuum ($\rho = 0$) by a rarefaction wave. In order that the left limit of this rarefaction wave has a positive speed, we need that

$$\left. \frac{d(\rho K - \rho p(\rho))}{d\rho} \right|_{\rho = \bar{\rho}_i} \geq 0.$$

It implies that $\bar{\rho}_i$ must be lesser than σ_K (see Fig. 4(a)).

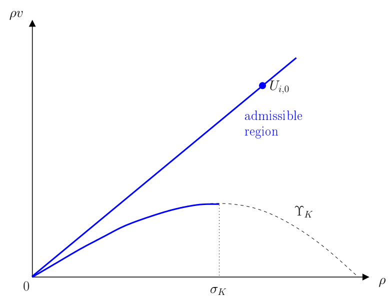
b: If $v_{i,0} < K$ and $\gamma_K(\sigma_K) \leq \sigma_K v_{i,0}$ then the curve Υ_K is greater than the curve $(\rho, \rho v_{i,0})$ at the beginning but crosses the line before its maximum. If $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ is on the part of Υ_K above the curve $(\rho, \rho v_{i,0})$, it is connected to the intermediate state U_c by a shock wave with positive speed. If $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ is on the part of Υ_K under the curve $(\rho, \rho v_{i,0})$, it is connected to U_c by a rarefaction wave. In order that the left limit of this rarefaction wave has a positive speed, we must have $\bar{\rho}_i$ lesser than σ_K (see Fig. 4(b)).

c: If $v_{i,0} < K$ and $\gamma_K(\sigma_K) > \sigma_K v_{i,0}$ then the maximum of Υ_K is above the curve $(\rho, \rho v_{i,0})$. If $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ is on the part of Υ_K under the curve $(\rho, \rho v_{i,0})$, it should be connected to U_c by a rarefaction wave with a negative speed, which is impossible.

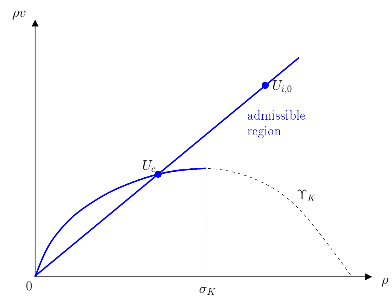
If $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ is above the straight line, it is connected to U_c by a shock wave whose speed is the slope of the curve connecting $(\bar{\rho}_i, \bar{\rho}_i \bar{v}_i)$ to U_c. In order to have a positive slope, we must have that

$$\bar{\rho}_i \leq \tau_K(\rho_c)$$

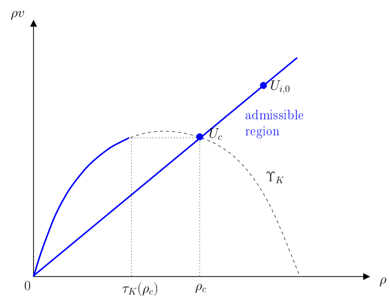
where ρ_c is the density of the intermediate state U_c (see Fig. 4(c)).



(a) $0 \leq K \leq v_{i,0}$



(b) $v_{i,0} < K$ & $\gamma_K(\sigma_K) \leq \sigma_K v_{i,0}$



(c) $v_{i,0} < K$ & $\gamma_K(\sigma_K) > \sigma_K v_{i,0}$

FIGURE 4. Some parts of the admissible region for an outgoing road.

If we combine the admissible regions for all the values of K, we obtain the global admissible region (AR) for an outgoing road depicted in Figure 5.

In fact, the AR only depends on the speed of the vehicles on the road. If $v_1 \leq v_2$, the AR associated with v_1 will be included in the AR associated with v_2. We can also consider the two limit cases:

- If $v_{i,0} = 0$, then the AR is reduced to the states $(\bar{\rho}_i, 0)$ for any $\bar{\rho}_i$. The choice of the density $\bar{\rho}_i$ is arbitrary since all the solutions of the Riemann problem on the outgoing road are identical in $L^1[0, \infty]$. Even if all values are possible for $\bar{\rho}_i$, the most meaningful is $\rho_{i,0}$ like the density on the incoming road.
- If $\rho_{i,0} = 0$, then the AR is simply a function of the speed $v_{i,0}$ as described above. Of course, the variable $v_{i,0}$ no longer has any physical meaning. To remove the inconsistence of drivers acting in a function of a non physical variable, it may be useful to add a relaxation term $\frac{V(\rho)-v}{\tau}$ at the right of (4). This relaxation term tends to remove such situations where $\rho_{i,0}$ and $v_{i,0}$ are both equal to zero.

This AR satisfies some intuitive ideas:

- if there is nearly nobody on the outgoing road ($\rho_{i,0} \approx 0$, $v_{i,0} \gg 0$), the AR is quite large;
- if there are many vehicles on the outgoing road ($\rho_{i,0} \gg 0$, $v_{i,0} \approx 0$), the AR is smaller.

3.2. **Additional conditions.** The description of the admissible regions has shown that many states are admissible. It is clear that additional conditions are needed in order to have a unique solution to the Riemann problem. The choice of these conditions is the main part of the establishment of a new road junction model.

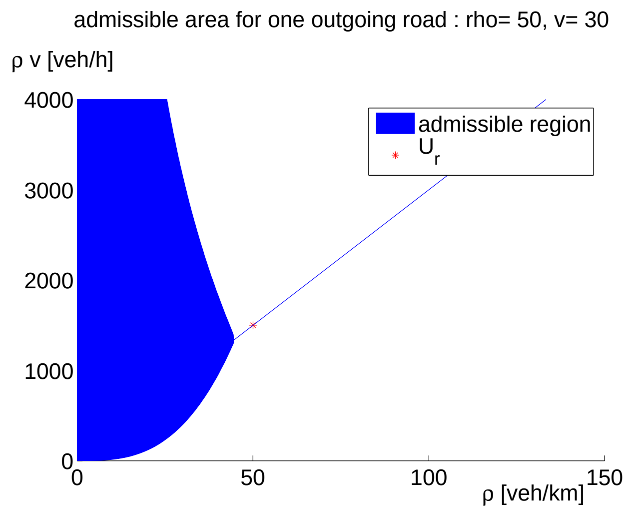


FIGURE 5. The admissible region for an outgoing road.

1. A first condition is indisputable: the conservation of flow. The sum of the entering flows must be equal to the sum of the leaving flows at the junction.
2. The second equation of the Aw and Rascle model (4) describes the behaviour of the drivers. It says that the Lagrangian derivative of the speed is equal to the Lagrangian derivative of $-p(\rho)$. It means that a driver will adapt his speed if the quantity $p(\rho)$ is modified.

   The most natural extension of this behaviour at the junction is

$$v_a - v_b = p(\rho_a) - p(\rho_b)$$

   where the subscripts a and b means "after" and "before" the junction. In other words, the quantity $v + p(\rho)$, which describes the behaviour of the drivers, is "conserved" through the junction by the drivers. Here, the meaning of conservation is not the same as in "conservation of the flow": the total flow of the quantity $v + p(\rho)$ is not necessarily the same before and after the junction but each driver tends to conserve his quantity $v + p(\rho)$ which describes his behaviour. If there is only one incoming road, all the drivers have the same behaviour, and thus:

$$v_a + p(\rho_a) = v_b + p(\rho_b).$$

   In the case where there are several incoming roads, it is natural to consider that the behaviour of the drivers after the junction is an average of the driver behaviours from the incoming roads. The additional condition used in our model to represent this fact is:

$$v_a + p(\rho_a) = \sum_i \beta_i (v_i + p(\rho_i)) \tag{7}$$

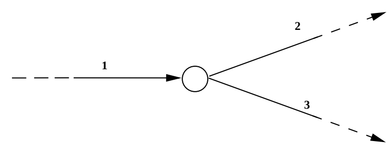
   where β_i is the proportion of the drivers coming from the incoming road i.

   It must be noticed that (7) is an additional *assumption* which is not motivated by microscopic considerations. The plausibility of this assumption is confirmed by the properties of the resulting model (see Section 4).

Another approach could be to return to the microscopic model underlying the Aw & Rascle model (see [1]) and analyze what happens at the junction. This approach is taken in [10, 11] and leads to a homogenization problem which needs to introduce a new function $p(\rho)$ after the junction.

3. The two previous additional conditions are not sufficient to have a unique solution to the Riemann problem at the junction. In order to get a unique solution, like for the first order model (see [12], [5], [14] and [13]), we may assume that the drivers act such that the flow entering the outgoing roads is maximized with respect to the previous restrictions.
4. In some case, there still exists two solutions and we may choose to privilege the one with maximal speed. This is reasonable since the drivers tend to act in order to maximize their speed. In fact, it does not matter since the solution will be the same in the L^2 norm whatever the choice.

3.2.1. *The diverging junction.*



It is reasonable to assume that, the drivers having a fixed destination intention, the proportions of the total flow entering into road 2 and 3 are fixed (α_2 and α_3).

With the additional conditions presented in section 3.2, the optimization problem at the junction can be expressed as

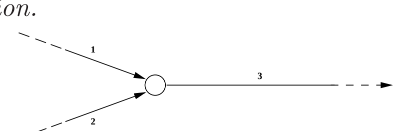
$$\max_{\bar\rho_i,\bar v_i} \bar\rho_1 \bar v_1$$

subject to

$$\left\{\begin{array}{rl} \bar\rho_1\bar v_1 = & \frac{\bar\rho_2\bar v_2}{\alpha_2} = \frac{\bar\rho_3\bar v_3}{\alpha_3} \\ \bar v_1 + p(\bar\rho_1) = & \bar v_2 + p(\bar\rho_2) = \bar v_3 + p(\bar\rho_3) \\ (\bar\rho_i,\bar v_i) \in & AR_i \end{array}\right.$$

where AR_i is the admissible region associated to road i with initial state $(\rho_{i,0}, v_{i,0})$. The solution of this optimization problem produces the new values of (ρ_i, v_i) at the junctions which allow us to solve the Riemann problems on each road.

3.2.2. *The merge junction.*



For the merge junction, we need to introduce a coefficient describing how the available space on the outgoing road is spread out between the incoming roads in case of congestion. A simple way to introduce this coefficient is to make it depend on the flows wishing to enter the outgoing road (see [13]):

$$\begin{aligned} \beta_1 &= \frac{f_1^*}{f_1^* + f_2^*} \\ \beta_2 &= 1 - \beta_1 \end{aligned} \tag{8}$$

where f_i^* is the maximal flow able to leave the road i (the "sending capacity"). Other coefficients may also be envisaged. With the additional conditions introduced above, the optimization problem at the junction can be expressed as

$$\max_{\bar{\rho}_i, \bar{v}_i} \bar{\rho}_1 \bar{v}_1 + \bar{\rho}_2 \bar{v}_2$$

subject to

$$\begin{cases} \frac{\bar{\rho}_1 \bar{v}_1}{\beta_1} = & \frac{\bar{\rho}_2 \bar{v}_2}{\beta_2} = \bar{\rho}_3 \bar{v}_3 \\ \bar{v}_3 + p(\bar{\rho}_3) = & \beta_1(\bar{v}_1 + p(\bar{\rho}_1)) + \beta_2(\bar{v}_2 + p(\bar{\rho}_2)) \\ (\bar{\rho}_i, \bar{v}_i) \in & AR_i. \end{cases}$$

Solving this optimization problem produces the new values of (ρ_i, v_i) at the junctions. These new values are used as initial conditions for the Riemann problems on each road.

4. **Properties of the junction model.**

The solutions of the Riemann problem at the junctions have some good properties that will be described in this section. We shall successively show that the solution of the optimization exists and is unique (taking into account Condition 4 of Section 3.2), that the solutions are physically acceptable and finally that they are able to represent some interesting phenomena such as the *capacity drop phenomenon*.

4.1. **Existence and uniqueness.**

The existence and uniqueness is proved by the following line of reasoning:

1. Given the initial states on the incoming roads, it is possible to compute the values $(v_{i,0} + p(\rho_{i,0}))$, the curves Υ_{K_i} and thus the flow which may leave the incoming roads $[0, f_i^*]$.
2. Given the values $(v_{i,0} + p(\rho_{i,0}))$ on the incoming roads, we can compute the curves Υ_{K_j} on which the new states of the outgoing roads must lie.
3. On an outgoing road j, there always exists an intersection between the curve Υ_{K_j} and the AR_j (at least the state $(0,0)$). Once the left part of the curve Υ_{K_j} enters the AR, it stays inside the AR for all lower density/flow on Υ_{K_j}. So there exists a maximal flow f_j^* such that all flows in $[0, f_j^*]$ are feasible with a state in the AR and on Υ_{K_j}.
4. The problem on a merge junction in terms of flow

$$\max \bar{f}_3$$

subject to

$$\begin{aligned} \bar{f}_1 + \bar{f}_2 &= \bar{f}_3 \\ \bar{f}_1 &= \beta_1 \bar{f}_3 \\ \bar{f}_2 &= \beta_2 \bar{f}_3 \\ \bar{f}_1 &\in [0, f_1^*] \\ \bar{f}_2 &\in [0, f_2^*] \\ \bar{f}_3 &\in [0, f_3^*] \end{aligned}$$

has a unique solution.
5. The problem on an diverging junction in terms of flow

$$\max \bar{f}_1$$

$$\begin{array}{rcl} \bar{f}_1 & = & \bar{f}_2 + \bar{f}_3 \\ \alpha_2 \bar{f}_1 & = & \bar{f}_2 \\ \alpha_3 \bar{f}_1 & = & \bar{f}_3 \\ \bar{f}_1 & \in & [0, f_1^*] \\ \bar{f}_2 & \in & [0, f_2^*] \\ \bar{f}_3 & \in & [0, f_3^*] \end{array}$$

has a unique solution.

6. There is a unique correspondence between the solution in terms of flow and a solution in terms of density/speed. In general, for an incoming road there is only one point of the AR which corresponds to a specified flow and for an outgoing road only one point of Υ_{K_j} inside the AR corresponding to a specified flow.

   In some particular cases, there may be two possible points on the admissible region. This may happen on an incoming road if $\bar{f}_1 = f_1^*$ and $\rho_{1,0} < \sigma_K$ and on an outgoing road if the point at the intersection between Υ_{K_j} and the line $\rho v = \rho v_{j,0}$ correspond to the specified flow. For simplicity one can take as a final solution the states corresponding to the maximal speed. This choice between the two possible points has no real consequence since the two lead to the same solution in the L^2 space.

4.2. **Physically acceptable.** The solutions of this junction model are intuitively acceptable.

A first important property of this junction model is its coherence with the Aw & Rascle single road model. For the diverging junction if all the incoming drivers take the same outgoing road ($\alpha_2 = 0, \alpha_3 = 1$) or for the merge junction if one of the incoming roads is empty ($\rho = 0$), the solutions are the same as the classical Aw & Rascle model for an infinite single road.

A second important point is that numerical simulations produce plausible results. Consider the merge junction case. If there is not too much traffic on the roads, all the flow on the incoming roads may pass (see Table 1). In this situation the total outflow in the final situation (2880 veh/h) is equal to the sum of the initial inflows (2×1440 veh/h). In the second simulation (see Table 2) the traffic on the incoming roads is more important. In this case only a part (2165 veh/h) of the total inflow flow (3300 veh/h) may pass.

The numerical results presented in the tables were performed with the $p(\rho)$ function defined as

$$p(\rho) = \frac{v_{ref}}{\gamma} \left(\frac{\rho}{\rho_{max}} \right)^{\gamma}$$

with $v_{ref} = 120$ [km/h], $\rho_{max} = 90$ [veh/km] and $\gamma = 2$. This is a plausible $p(\rho)$ function derived from microscopic consideration (see [1]).

A last point is the analysis of the situations where no flow can pass through the junction. For a merge junction, this situation only happens when there is no traffic on the incoming road or when the speed on the outgoing road is equal to zero (the AR is reduced to curve $(\rho, 0)$ which is associated to a zero flow). For the diverging junction, the only situation where no flow can leave the incoming road is when the speed on one of the outgoing roads is equal to zero (the AR is reduced to curve $(\rho, 0)$ which is associated to a zero flow).

	Incoming roads			Outgoing road		
	ρ	v	f	ρ	v	f
Initial states	20	72	1440	51.4	58.36	3000
	20	72	1440			
Final states	20	72	1440	53.9	53.43	2880
	20	72	1440			

TABLE 1. There is not too much traffic on the two incoming roads for the capacity of the outgoing road. The whole flow may pass.

	Incoming roads			Outgoing road		
	ρ	v	f	ρ	v	f
Initial states	30	55	1650	51.4	58.36	3000
	30	55	1650			
Final states	80.7	13.42	1082	52	41	2165
	80.7	13.42	1082			

TABLE 2. There is too much traffic on the two incoming roads for the capacity of the outgoing road. Only a fraction of the flow may pass.

It implies that, if only one of the outgoing roads is jammed up, the outflow of the other road is equal to zero. This is a rather hard constraint on the outflows. The reason is that the Aw & Rascle model, used to describe the evolution on one road, does not make any distinction between the different lanes. In a single lane road, if a driver stops because he is unable to turn left, he also blocks all the drivers wanting to turn right. To remove this hard constraint, we would need to adapt the Aw & Rascle model to take the different lanes into account.

4.3. **Invariant region.** It can be shown (see [2]) that the single road model (3)–(4) admits an invariant region $\mathcal{R}$

$$\mathcal{R} := \{(\rho, v) \mid 0 \leq v + p(\rho) \leq v_m, \quad \rho \geq 0,\ 0 \leq v \leq v_m\}. \tag{9}$$

It is easy to see that the new proposed model for the junction keeps this invariant region. The first inequalities are satisfied since a new value of $v + p(\rho)$ may only appear on an outgoing road and this new value is either equal to the value on an incoming road or a convex combination of the values on two incoming roads.

The solution of the maximization problem always exists with $\rho \geq 0$ and $v \geq 0$ and, since $p(\rho) \geq 0$, the last inequalities are trivial.

4.4. **Capacity drop phenomenon.** The proposed junction model might be able to naturally represent the *capacity drop phenomenon*. The *capacity drop phenomenon* is a critical phenomenon which represents the fact that the outflow of a traffic jam is significantly lower than the maximum achievable flow at the same location. We can easily understand this phenomenon at a junction where two roads merge in one: if there are too many vehicles trying to access the same road, there is a sort of mutual embarrassment between the drivers which results in an outgoing flow lower than the optimal possible flow. This phenomenon has been experimentally observed (see [4] and [8]). The flow decrease, which may range up to 15 %, has a considerable influence when considering traffic control ([16]). To have a model describing this phenomenon is thus a critical feature in the establishment of a traffic state regulation strategy.

The reason why our junction model might be able to represent the capacity drop phenomenon is illustrated in Figure 6. In this figure, we have represented the graphs relative to one of the two incoming roads and the outgoing one. We consider an initial equilibrium (states 1 of Fig. 6) where the state of the outgoing road is on the border of the Admissible Region. Starting from the initial state 1, we consider on the incoming road an increase of flow ($\rho v \nearrow$) with an increase of density and a decrease of speed ($\rho \nearrow, v \searrow$) such that the quantity $v + p(\rho)$ decreases (state 2). We must now solve a new Riemann problem at the junction. Since the quantity $v + p(\rho)$ has decreased on one of the incoming road, the new state on the outgoing road must lie on a curve $v+p(\rho) = C_2^{out}$ lower than the curve $v+p(\rho) = C_1^{out}$. Since the total flow able to leave the incoming road is greater than the initial flow (f_1^{in}), the new state on the outgoing road will be state 3 (the state with the maximum flow on the curve $\Upsilon_{C_2^{out}}$). Assuming that nothing has changed on the second incoming road, a decrease of the flow on the outgoing road implies a decrease of the flow on the incoming roads. The new flow of the incoming road must be less than f_1^{in} (for example state 3).

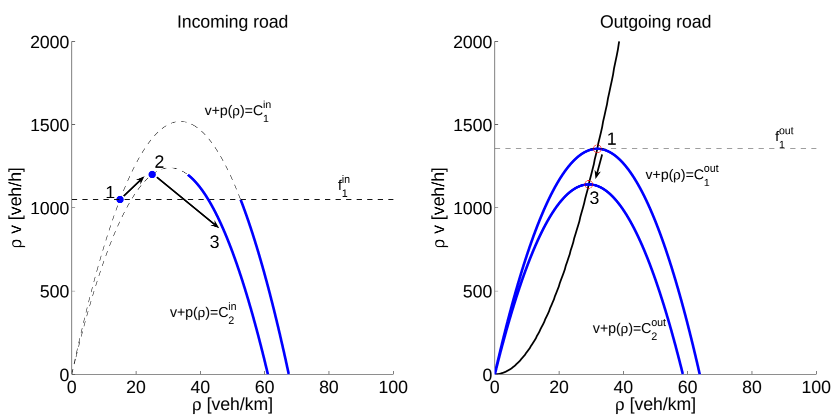


FIGURE 6. Illustration of the capacity drop phenomenon with an increase of flow at one incoming road.

To illustrate this phenomenon, we can carry out a series of simulations whose solutions are presented in Table 3. To simplify, we consider the case where the states on road one and road two are the same. We start from an equilibrium (row 1) with a total passing flow of 3000 veh/h. If we apply a perturbation (row 2) consisting of an increase of the incoming flow (3300 veh/h) we obtain the new equilibrium represented in row 3. To this new equilibrium corresponds a total passing flow of 2165 veh/h. This illustrates the capacity drop phenomenon: the incoming flows are higher but the outgoing flow drops down. Moreover, if we try to return to the initial state by an opposite perturbation on the incoming roads (row 4), we do not recover the same total passing flow but a lower one.

To summarize, this increase of inflows leading to a decrease of outflow can only appear if:

	state	ρ_1	v_1	f_1	ρ_3	v_3	f_3
1	equilibrium	20	75	1500	51.4	58.36	3000
2	perturbation	30	55	1650	51.4	58.36	3000
3	equilibrium	80.7	13.42	1082.5	52	41.6	2165
4	perturbation	20	75	1500	52	41.6	2165
5	equilibrium	91.5	15.9	1458.5	47.8	61	2917

TABLE 3. The capacity drop phenomenon illustrated in a particular case.

- the state on the outgoing road remains on the boundary of the admissible region;
- the inflows are such that the mean "behaviour" of the drivers ($\sum_i \beta_i(v_i + p(\rho_i))$) decreases.

The total decrease of the outflow is linked to the value of the decrease of $v + p(\rho)$ for a fixed increase of the flow ($\rho v \nearrow, \rho \nearrow, v \searrow$). So it can be claimed that the importance of the capacity drop phenomenon is linked to the value of $p'(\rho)$. If the derivative is small then the phenomenon may be important. To the author's knowledge, this possibility of increasing inflows leading to a decreasing outflow is not present in the junction models developed so far for the LWR model. These models are thus unable to represent the capacity drop phenomenon. It would be interesting to analyze experimental observations at a junction to see if the capacity drop phenomenon corresponds to the situation described by our model.

4.5. **Stability of equilibria.** In this section, we intend to investigate the stability of the equilibria according to the notion of stability as defined in [7]. An equilibrium at a junction is defined as a solution of the Riemann problem (i.e. the (ρ_i, v_i) values at the junction satisfying the conditions of Section 3.2). If a wave arrives at the junction, it will perturb the initial state and a new equilibrium will appear. Stability means that small perturbations produce small variations of the equilibrium. More precisely, an equilibrium (ρ_i^*, v_i^*) is *stable* if there exists two constants $C > 0$ and $\delta > 0$ such that:

$$\forall (d\rho, dv) \text{ s.a. } |d\rho| + |dv| < \delta \Rightarrow \sum_i |\rho_i^* - \bar{\rho}_i| + |v_i^* - \bar{v}_i| \leq C(|d\rho| + |dv|) \qquad (10)$$

where $(d\rho, dv)$ is the perturbation on one of the roads and $(\bar{\rho}_i, \bar{v}_i)$ is the new equilibrium on road i. The equilibrium will be *unstable* if it is impossible to find $C > 0$ and $\delta > 0$ such that (10) is true.

To analyze the different possibilities, we introduce the curve Γ which is the curve of the maxima of Υ_K. This curve separates the (ρ–ρv) plane in two regions D_1 and D_2. In Figure 7, are represented the different possibilities for the position of the equilibrium states. For an incoming road, we can decompose the possible region for the equilibria as $\mathring{D}_1 \bigcup \Gamma \bigcup \mathring{D}_2$ and for an outgoing road as $\mathring{D}_1 \bigcup \Gamma$ where the superscript $^\circ$ refers to the interior. In this study of the stability, we exclude the particular cases where ρ or v equal zero.

The three following claims will be proven:

1. If one of the states on the incoming roads belongs to $\mathring{D}_1$ and one of the states on the outgoing roads belongs to Γ then the equilibrium is unstable.
2. If all the states on the outgoing roads belong to $\mathring{D}_1$ then the equilibrium is stable.

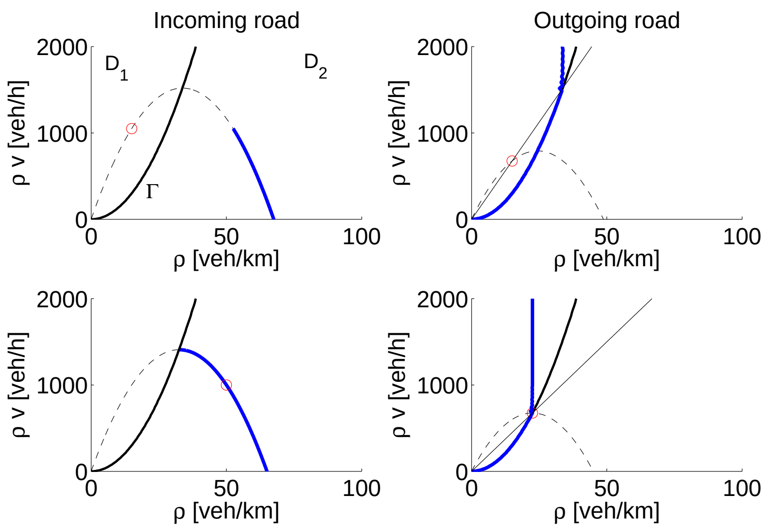


FIGURE 7. The possibilities for the position of the states at equilibrium.

3. If all the states on the incoming roads belong to $\overset{\circ}{D}_2$ and at most one state on the outgoing roads belongs to Γ then the equilibrium is stable.

During the proofs, the following conventions are taken:

- the superscripts * and $^-$ refer to the initial states before and the final states after the perturbation;
- the subscript i is dropped, ρ^* and $\bar{\rho}$ always refer to the density on a specific road depending on the context;
- $d\rho, df$ and dv refer to the variations of density, flow and speed on the road associated to the perturbation;
- a quantity is said to be bounded when it is always less than $|d\rho| + |dv|$ multiplied by a constant.

The proofs will be showing that, on each road, there exists a constant $C > 0$ (which may be different for each road and function of the equilibrium under consideration) such that $|\rho_i^* - \bar{\rho}_i| + |v_i^* - \bar{v}_i| \leq C(|d\rho| + |dv|)$.

4.5.1. *First claim.*

> If one of the states on the incoming roads belongs to $\overset{\circ}{D}_1$ and one of the states on the outgoing roads belongs to Γ then the equilibrium is unstable.

An example of instability has already been presented in Figure 6 for the illustration of the capacity drop phenomenon. In this situation, it is possible, with an arbitrary small change in ρ and ρv on the incoming road, to make the state on this road jump to the region D_2. The variation of the density in the final solution will thus be of the order of $\tau_K(\rho_{i,0}) - \rho_{i,0}$ which is not linked to the amplitude of the perturbation. This equilibrium is unstable.

4.5.2. *Second claim.*

> If all the states on the outgoing roads belong to $\overset{\circ}{D}_1$ then the equilibrium is stable.

The proof is divided in two parts. First the influence of a perturbation occurring on an outgoing road is analyzed and then of a perturbation on an incoming road.

Perturbation on an outgoing road.

If the perturbation occurs on an outgoing road, there is a lower bound on the amplitude of this perturbation. This bound comes from the fact that only shock waves can reach the junction from an outgoing road and this shock wave should have a large amplitude in order to make the state on the outgoing road jump from $\mathring{D}_1$ to $\mathring{D}_2$. If we impose that δ in (10) is smaller than this bound, it is impossible to have a perturbation coming from an outgoing road.

Perturbation on an incoming road.

If the perturbation occurs on an incoming road and is small enough, the flow associated with this new state may pass. This state becomes the new equilibrium state on the incoming road in the final solution. The states on the outgoing roads may change but the variation can be bounded by the following four steps:

1. The amplitude of the flow variation can be bounded in function of the density–speed perturbation. Indeed we have on the incoming road that

$$\begin{aligned} f &= \rho v \\ f + df &= (\rho + d\rho)(v + dv) \\ df &= v d\rho + \rho dv + d\rho dv. \end{aligned}$$

   Based on the invariant region $\mathcal{R}$ defined in (9), we introduce $V = \max(v_m, \rho_m)$ where $\rho_m = p^{-1}(v_m)$. Denoting A the amplitude of the initial perturbation

$$A = |d\rho| + |dv|,$$

   if $A \leq 1$ the amplitude of the flow perturbation may be bounded

$$|df| \leq (2V + 1)A = C^1 A.$$

   Since the variation of the flow on the outgoing road is lesser than or equal to $|df|$, the flow variation on the outgoing road is bounded. The flow variation is lesser if this is a diverging junction. In this case, the variation of flow on the incoming road has been shared between the two outgoing roads.
2. The variation of the curve $\Upsilon_{K_{out}}$ to which belongs the state on the outgoing road can be bounded. Indeed, on the incoming road we had previously

$$v + p(\rho) = K_{in},$$

   and after the change

$$v + dv + p(\rho + d\rho) = K'_{in}$$

   with

$$\begin{aligned} |K_{in} - K'_{in}| &\leq |dv| + \max(p'(\rho))\,|d\rho| \\ &\leq (1 + \max(p'(\rho)))A = C^2 A. \end{aligned}$$

   On the outgoing road the state which was lying previously on the curve $\Upsilon_{K_{out}}$ will now belong to the curve $\Upsilon_{K'_{out}}$ with $K'_{out} = K_{out} + dK_{out}$. Since K'_{out}

is a convex combination of the values of K'_{in} on the incoming roads, we have that

$$|dK_{out}| \leq |K_{in} - K'_{in}| \leq C^2 A.$$

The new state on the outgoing road belongs to the curve

$$\Upsilon_{K'_{out}} = \{(\rho, \rho v) \mid \rho v = K_{out}\rho - \rho p(\rho) + dK_{out}\rho\}.$$

On Figure 8, the new state will be between the two lines ($\rho v = f^* + |df|$ and $\rho v = f^* - |df|$) and on the curve $\Upsilon_{K_{out}}$ shifted by $dK_{out}\rho$.

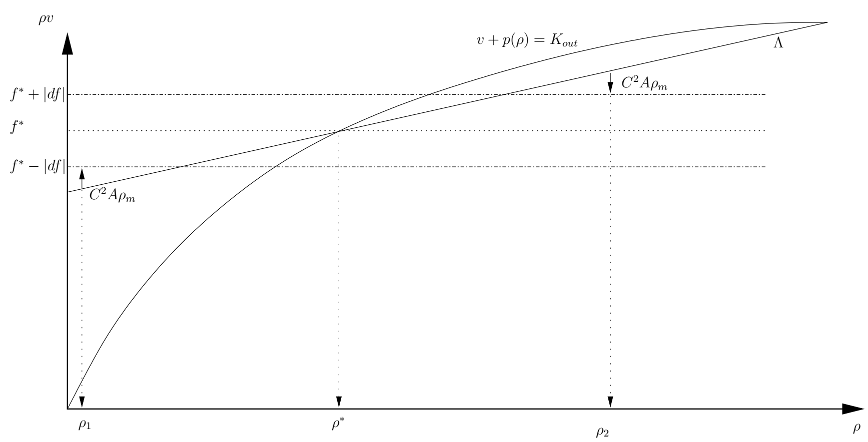


FIGURE 8. Bounds on the variation of ρ and f on the outgoing road. The state (ρ^*, f^*) is the original state before the perturbation.

3. It is possible to find a bound on the new value $\bar{\rho}$ for the outgoing road. Consider the line Λ passing through the original state (ρ^*, f^*) and the maximum of the curve $\Upsilon_{K_{out}}$. Since $(\rho^*, f^*) \in \overset{\circ}{D}_1$ and $(\rho p(\rho))'' > 0$, this line has a derivative $a > 0$. If we consider that the new curve $\Upsilon_{K'_{out}}$ will be $\Upsilon_{K_{out}}$ shifted by at most $C^2 A\rho_m$ and that A is small enough such that the cross of the lines occur such as represented in Figure 8 then:
   - on the right of ρ^* the curve $\Upsilon_{K_{out}}$ is above Λ and is shifted less than $C^2\rho_m$ below. The possible intersection between $\Upsilon_{K'_{out}}$ and the line $f^* + |df|$ occurs on the left of ρ_2 with
   $$|\rho_2 - \rho^*| = \frac{df + C^2 A\rho_m}{a} \leq \frac{C^1 + C^2\rho_m}{a} A.$$
   - on the left of ρ^* the curve $\Upsilon_{K_{out}}$ is below Λ and is shifted less than $C^2\rho_m$ above. The possible intersection between $\Upsilon_{K'_{out}}$ and the line $f^* + df$ occurs thus on the right of ρ_1 with
   $$|\rho_1 - \rho^*| = \frac{df + C^2 A\rho_m}{a} \leq \frac{C^1 + C^2\rho_m}{a} A.$$

The new state on the outgoing road being between the lines $\rho v = f^* - |df|$ and $\rho v = f^* + |df|$ and on the curve $\Upsilon_{K'_{out}}$, the new density $\bar{\rho}$ is between ρ_1 and ρ_2. We have thus

$$|\rho^* - \bar{\rho}| \leq \frac{C^1 + C^2\rho_m}{a} A = C^3 A.$$

4. The total variation on the outgoing road can thus be bounded

$$\begin{aligned} |\rho^* - \bar{\rho}| &\le C^3 A \\ |v^* - \bar{v}| &\le \frac{f^* + df}{\rho^* - C^3 A} - \frac{f^*}{\rho^*} \\ &\le \frac{f^* + C^1 A}{\rho^* - C^3 A} - \frac{f^*}{\rho^*} = \frac{C^1 A}{\rho^* - C^3 A} + \left(\frac{f^*}{\rho^* - C^3 A} - \frac{f^*}{\rho^*} \right). \end{aligned} \tag{11}$$

Taking A small enough such that $C^3 A \le \frac{\rho^*}{2}$,

$$\begin{aligned} |v^* - \bar{v}| &\le \frac{2C^1}{\rho^*} A + \frac{C^3 f^* A}{\rho^* (\rho^* - C^3 A)} \\ &\le \frac{2C^1}{\rho^*} A + \frac{2C^3 f^*}{{\rho^*}^2} A. \end{aligned}$$

It can be concluded that there exists a constant C such that

$$|\rho^* - \bar{\rho}| + |v^* - \bar{v}| \le C(|d\rho| + |dv|)$$

and this equilibrium is stable.

4.5.3. *Third claim.*

> If all the states on the incoming roads belong to $\overset{\circ}{D}_2$ and at most one state on the outgoing roads belongs to Γ then the equilibrium is stable.

The proof is divided in two parts. First the influence of a perturbation occurring on an outgoing road is analyzed and then of a perturbation on an incoming road. Perturbation on an outgoing road.

As explained in the first point of Section 4.5.2, only perturbations coming from an outgoing road whose state belongs to Γ need to be analyzed. If the perturbation occurs on such a road, this perturbation may only be a shock wave since it is the only wave with a negative speed.

In Figure 9 the initial state 1 and the new state are represented after the perturbation (state 2). Nothing having changed on the incoming roads, the solution will belong to the same curve $\Upsilon_{K_{out}}$ and the total flow able to leave the incoming road is at least f^*. The final state on the outgoing road will be the state 3 which is the state on the curve $\Upsilon_{K_{out}}$ and on the new admissible region (function of the speed $v^* + dv$) with the maximal flow (see Figure 4(c)).

For our analysis, we adopt the following notations

$$\begin{aligned} f(d\rho) &= K(\rho^* + d\rho) - (\rho^* + d\rho)p(\rho^* + d\rho) - f^* \\ d\bar{\rho} &= \bar{\rho} - \rho^* \end{aligned}$$

where $f \in C^\infty$.

The proof is done in three steps:

1. A bound on the final variation of ρ and v on the outgoing road can be obtained. This is done by obtaining first a bound on the flow, then on the density and finally on the speed.

   For the flow, the final variation is equal to the initial variation which can be bounded in function of $d\rho$:

$$df \le \|f'\|_\infty \, d\rho.$$

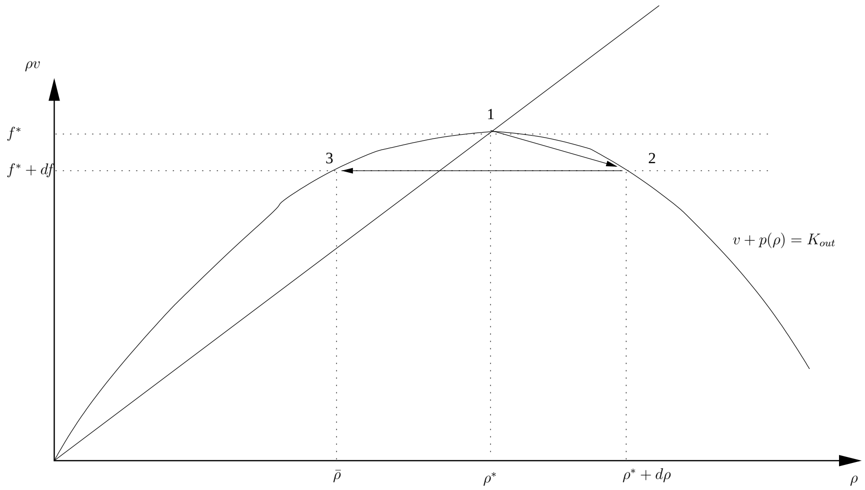


FIGURE 9. Bounds on the variation of ρ and f on the outgoing road. The state (ρ^*, f^*) is the original state before the perturbation.

To find the bound on the density, a Taylor series expansion can be made around the origin of the function f:

$$f(\rho) = -a\rho^2 + \frac{f'''(\rho')}{3!}\rho^3 \qquad \text{with } 0 \leq \rho' \leq \rho,\ a > 0.$$

The goal is to show that

$$\exists \epsilon, C' > 0 \quad \text{such as} \quad \forall \left|d\rho\right| \leq \epsilon,\ \left|d\bar{\rho}\right| \leq C' \left|d\rho\right|.$$

In the worst case of the Taylor development, i.e. when the smallest perturbation of $d\rho$ produces the biggest change $d\bar{\rho}$, we have

$$f(\rho) = -a\rho^2 - \frac{\|f'''\|_\infty}{3!}\rho^3.$$

In this case, a perturbation $d\rho$ implies

$$df = -ad\rho^2 - \frac{\|f'''\|_\infty}{3!}d\rho^3.$$

With any $C' > 1$ and $d\rho$ small enough,

$$f(-C'd\rho) = -a(-C'd\rho)^2 - \frac{\|f'''\|_\infty}{3!}(-C'd\rho)^3 < df \tag{12}$$

thus

$$|d\bar{\rho}| \leq C'd\rho$$

since $d\bar{\rho}$ is defined such that $f(d\bar{\rho}) = f(d\rho)$. To prove (12), simply check that

$$-a(-C'd\rho)^2 - \frac{\|f'''\|_\infty}{3!}(-C'd\rho)^3 < -ad\rho^2 - \frac{\|f'''\|_\infty}{3!}d\rho^3$$

is equivalent to

$$a(C'^2 - 1)d\rho^2 > \frac{\|f'''\|_\infty}{3!}(1 + C'^3)d\rho^3$$

which is true for $d\rho$ small enough since $cx^2 > x^3 \quad \forall x \in [0, c)$.

On the basis of the $|d\rho|$ and $|dv|$ bounds, the final variation of speed on the outgoing road can be bounded using the same reasoning as in (11).

It can be concluded that, on the outgoing road, there exists a constant C such that

$$|\rho^* - \bar{\rho}| + |v^* - \bar{v}| \leq C(|d\rho| + |dv|)$$

for $|d\rho|$ and $|dv|$ small enough.

2. Due to the variation of the admissible region on the outgoing road, the states on the incoming roads may also have changed. The final variation on one of the incoming roads can be bounded as follow:
   - On the incoming roads, there is a decrease of flow less than df.
   - In term of density, we have (see Fig. 10)

     $$|\bar{\rho} - \rho^*| \leq \frac{1}{a} df \tag{13}$$

     where $a > 0$ is the slope of the line passing through (ρ^*, f^*) and $(\sigma_K, \gamma_K(\sigma_K))$.
   - Since $\left|\bar{f} - f^*\right|$ and $|\bar{\rho} - \rho^*|$ are bounded in function of $d\rho$ and dv, by the same reasoning as in (11), this is also true for $|\bar{v} - v^*|$.

   It can be concluded that there exists a constant C such that

   $$|\rho^* - \bar{\rho}| + |v^* - \bar{v}| \leq C(|d\rho| + |dv|)$$

   on the incoming road.

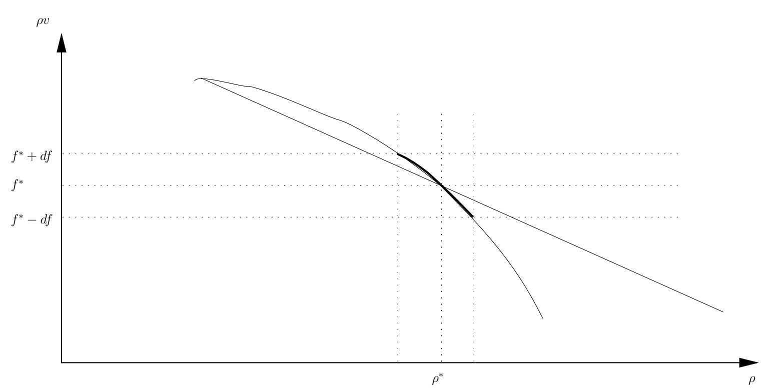


FIGURE 10. Maximum variation of the density for a fixed maximal variation of the flow on an incoming road.

3. Eventually this decrease of the flow leaving the incoming road may influence the state on a second outgoing road (which must belong by assumption to $\overset{\circ}{D}_1$). This variation can be bounded as in Section 4.5.2.

Perturbation on an incoming road.

If the perturbation occurs on the incoming road and is small enough, this perturbation may only be a contact discontinuity since this is the only wave with a positive speed. This case is represented in Figure 11. This perturbation will influence the states of the outgoing roads. Since the variation on the outgoing road belonging to $\overset{\circ}{D}_1$ can be bounded like in Section 4.5.2, it is only necessary to consider the case where the outgoing road belongs to Γ.

Two cases must be considered: when the density on the incoming road decreases and when it increases.

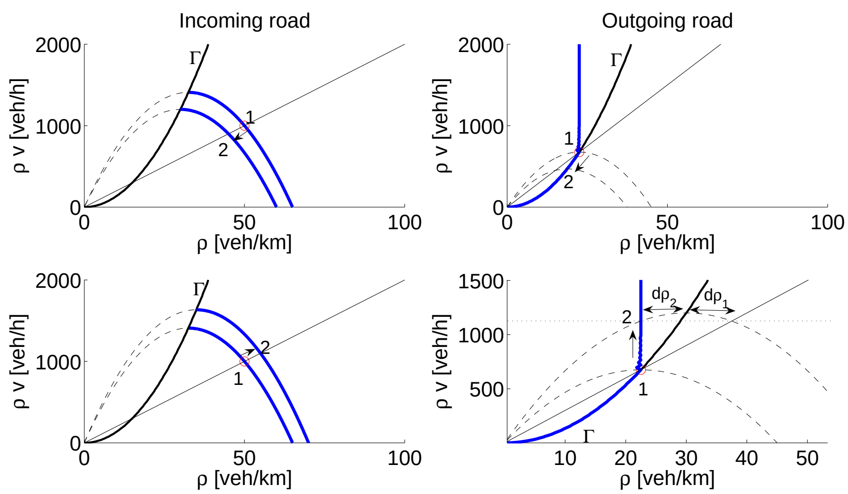


FIGURE 11. The two top figures correspond to a decrease of density on the incoming road while the two bottom figures correspond to an increase of density.

When the density on the incoming road decreases, the new state on this incoming road is on a curve $\Upsilon_{K'_{in}}$ with $K'_{in} \leq K_{in}$. This decrease of K_{in} can easily be bounded in function of $d\rho$:

$$|dK_{in}| \leq \|p'\|_\infty |d\rho| . \tag{14}$$

On the outgoing road, the state will belong to a new curve $\Upsilon_{K'_{out}}$ with

$$|dK_{out}| \leq |dK_{in}| . \tag{15}$$

Since too much flow is trying to pass from the incoming roads to the outgoing road (this was already the case before the perturbation), the new state on the outgoing road will be the maximum of the curve $\Upsilon_{K'_{out}}$. To prove that the variation of the traffic variable ρ and v on the outgoing road can be bounded in function on $d\rho$, from (14) and (15), it is sufficient to prove that the variation of $(\sigma_K, \gamma_K(\sigma_K))$ may be bounded by the variation of dK.

Let $g(\rho)$ be defined as $((\rho p(\rho))')^{-1}$, we have that $g' = \frac{1}{(\rho p(\rho))''} \in C^0$ and thus $g \in C^1$. It is easy to see that

$$\sigma_K = g(K)$$

and thus

$$|d\sigma_K| \leq \|g'\|_\infty |dK| .$$

We have also that

$$\gamma_K(\sigma_K) = Kg(K) - g(K)p(g(K)) = h(K).$$

Since g and p are C^1, $h(K)$ is C^1 and

$$d\gamma_K(\sigma_K) \leq \|h'\|_\infty dK.$$

Having a bound on the variation of the flow and the density in function of $d\rho$, a bound on the variation of the speed can be deduced. It can thus be concluded that there exists C such that

$$|\rho^* - \bar{\rho}| + |v^* - \bar{v}| \leq C(|d\rho| + |dv|)$$

on the outgoing road.

On the incoming road, we know that the final state is on the curve $\Upsilon_{K'_{in}}$. Moreover the variation of flow is linked to the variation of flow on the outgoing road and thus can be bounded in function of $d\rho$. Using the same reasoning as in (13), we can bound the total variation of the density and speed on the incoming road in function of $d\rho$.

When the density on the incoming road increases, we are in the situation at the bottom of Fig. 11. The proof is a combination of the previous explanations. On the outgoing road :

- the new state belongs to the curve $\Upsilon_{K'_{out}}$ with dK_{out} bounded in function of $d\rho$;
- the state with the maximum flow belonging to $\Upsilon_{K'_{out}}$ inside the admissible region is the state 2 of Figure 11;
- since dK_{out} is bounded, the difference of the flow df' associated to the maximums of $\Upsilon_{K_{out}}$ and $\Upsilon_{K'_{out}}$ is bounded;
- the variation of flow between state 2 and state 1 is bounded by df';
- for the density, $d\rho_1$ is bounded by df'/v^*;
- $d\rho_2$ may be bounded in function of $d\rho_1$;
- the final variation of density on the outgoing road is less than $d\rho_2$;
- since the variation of flow and density may be bounded, the variation of speed may also be bounded.

On the incoming road, since the variation of flow is bounded, this is also true for the density and the speed.

Since for all small variations and on all roads there exists C such that

$$|\rho^* - \bar{\rho}| + |v^* - \bar{v}| \leq C(|d\rho| + |dv|)$$

then the equilibrium is stable.

5. **Existence of a solution in the BV space.** In the previous sections, only solutions to Riemann problems at the junction were considered. Using the previous stability results, it is possible to prove the existence of a solution for a larger class of initial conditions (see [7]).

If we consider a network with only one junction and an initial state $((\rho_{1,0}(x), v_{1,0}(x)), (\rho_{2,0}(x), v_{2,0}(x)), (\rho_{3,0}(x), v_{3,0}(x)))$ which is a stable equilibrium s.a

$$\|(\rho_{i,0}(x), v_{i,0}(x))\|_{BV} \leq \epsilon$$

and

$$\sup_x |\rho_{i,0}(x) - \rho_{i,0}| + \sup_x |v_{i,0}(x) - v_{i,0}| \leq \epsilon,$$

then for ϵ small enough, there exists an entropic solution satisfying the initial conditions and the equilibrium conditions at the junction.

In this equation $\| \ \|_{BV}$ denotes the total variation defined by

$$\|u\|_{BV} = \sup\{v(u, P) : P \text{ is a partition of } [a, b]\}$$

where

$$v(u,P) = \sum_{k=1}^{n} d(u(t_k), u(t_{k-1}))$$

with $P = \{x_0 < t_1 < \cdots < x_n\}$ a partition of the domain of u.

Proof. For the complete proof see [7] and [3]. ◻

6. **Conclusions.** To the author's knowledge, the only other junction models developed on the basis of the Aw & Rascle single road model are presented in [10, 11] and [7].

The idea developed in [10] is similar to the idea presented in the present paper: each driver tries to conserve his behaviour and the resulting behaviour on the outgoing road is a mean of the different incoming behaviours. The main difference is that [10] analyzes the mixing problem on the basis of an underlying microscopic model whereas our approach stays at a macroscopic scale with the formulation (7). The approach of [10] leads to a homogenization problem which may need to introduce a new function $p(\rho)$ after the junction. The model presented in this paper can therefore be seen as an approximation of this homogenization problem. The main advantage of (7) is its simpler formulation leading to easier computations.

It is worth mentioning that the two approach are close and even identical for a diverging junction. Indeed, in this case, the two models predict that the quantities $v+p(\rho)$ on the outgoing roads are the same as the quantity $v+p(\rho)$ on the incoming road. Since the two models select the new values of (ρ_i, v_i) near the junction by performing a maximization of the total passing flow, they lead to the same values.

For a merge junction, the situation is more complex since [10] may introduce a new function $p(\rho)$ after the junction. It would be interesting to perform simulations using the two models and compare the results.

There are some important differences between the model presented in this paper and the model of [7]:

1. In our model the existence of the invariant region $\mathcal{R}$ is a natural consequence of the choice made for the additional criterion ("conservation" of $v + p(\rho)$). In the model [7], the same invariant region exists but is a consequence of the introduction of a maximal speed in the description of the admissible region of an outgoing road.
2. Our model is able to represent the capacity drop phenomenon while this is not the case for the model [7]. The main reason is the choice made for the order in which some variables are fixed or maximized on the outgoing road:
    - In our model, the order is
        (a) fix the admissible region
        (b) fix the $v + p(\rho)$ value
        (c) maximize the flow
    - In the model [7], the order is
        (a) fix the admissible region
        (b) maximize the flow and then take it as a constraint
        (c) maximize the speed/density or minimize the total variation of ρ along the solution on the outgoing roads.

    In our model, the maximization of the flow is the final criterion used to chose the solution while in the model [7] this maximization occurs before any other additional criterion.

3. One limitation of our model is that it is developed only for 2-incoming-1-outgoing and 1-incoming-2-outgoing roads junction. The model [7] is developed for n-incoming-m-outgoing roads junction but also has an important limitation, i.e. $m \geq n$.
4. The two models have more or less the same properties when dealing with stability of equilibrium and existence of solution around a stable equilibrium.

Adding only one new assumption ("conservation" of the quantity $v + p(\rho)$ representing the behaviour of the drivers) to some commonly admitted assumptions (conservation of the flow, sharing of the available space on the outgoing road based on coefficients function of the sending capacities, drivers acting in order to maximize the passing flow), we obtain a coherent and realistic model for the junction able to represent the capacity drop phenomenon. This junction model, which doesn't add any new parameters compared to those introduced by the Aw & Rascle single road model (the function $p(\rho)$), combined with this single road model, provides a complete description of the traffic evolution on a road network.

The extension of the junction model to n-incoming-1-outgoing and 1-incoming-n-outgoing roads is straightforward. It does not need to introduce additional assumptions. If we consider multiple incoming and multiple outgoing roads, the extension is more difficult. Additional criteria must be added in order to describe how the different flows interact.

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E-mail address: haut@inma.ucl.ac.be
E-mail address: bastin@inma.ucl.ac.be