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TRAFFIC REGULATION OF AN UNDERGROUND RAILWAY TRANSPORTATION SYSTEM BY STATE FEEDBACK

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SUMMARY

The commonly used state-space models for the traffic description of urban underground railway lines are not suitable for practical on-line control. On the other hand, the proposed orginal linear formulation is very convenient for optimal state feedback control implementation. Results of simulations relative to a theoretical line as well as to the Brussels Railway lines show the benefit to be expected using this new formulation for on-line traffic control.

KEY WORDS Traffic regulation Underground railway system Transportation

1. INTRODUCTION

The aim of this paper is to discuss three possible models of an underground public transportation system and to propose an original traffic control method, with particular application to the Brussels Underground Railway Line.

An underground transportation system is known to have an unstable behaviour. Consider, for example, a delayed train. Because of this delay the time interval since the last train is increased and more people have to enter the vehicle, with a resulting increasing delay. In order to restore a disturbed traffic to an acceptable situation regulation is therefore necessary.

If we consider an intra-city railway, with high traffic density and passengers arriving randomly at the stations and getting on the first available train regardless of the nominal schedule, it is necessary, from the passengers' viewpoint, to control the traffic in order to minimize the waiting time at the station, i.e. to keep the time intervals between successive trains as close as possible to their nominal values. On the other hand, if we consider connections with other transportation systems it is necessary to control the traffic according to the nominal schedule. For better service to the passengers the control has therefore to respect the trade-off between the two objectives.

The control action consists of instructions (speed during the running time between stations, waiting time at the station, ...) given by the centralized traffic controller, on the basis of the available information, consisting mainly of the situation of each train in the system. Constraints have of course to be imposed on the control, e.g. maximum speed, minimum waiting time at the station, security rules imposed by traffic lights.

In sections 2 and 3 a mathematical model for the traffic dynamics is introduced; three state-space representations are described and compared with respect to their simplicity, their stability and the possibility of implementation in on-line traffic regulation. In section 4 a regulation

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policy is proposed taking into account the two above objectives as well as the control limitations. In section 5, we present numerical results for a simplified theoretical subway line, whereas section 6 shows how to implement the proposed regulation policy for a real system, taking into account the physical characteristics and constraints and gives simulation results relative to Brussels Railways Line. In the Appendix a more sophisticated model is proposed.

2. THE MATHEMATICAL MODEL OF SASAMA AND OKHAWA

In this section we shall briefly summarize the mathematical model for traffic dynamics proposed by Sasama and Okhawa.[1]

Consider I trains (upper index $i = 1, \ldots, I$) and a line with $K+1$ stations (lower index $k = 0, \ldots, K$). Defining

t_k^i as the departure time of the ith train from the kth station
r_k^i as the running time of the ith train from the kth to the $(k+1)$th station
s_k^i as the staying time of the ith train at the kth station
it follows immediately that

$$t_{k+1}^i = t_k^i + r_k^i + s_{k+1}^i \tag{1}$$

The running time r_k^i can be modelled further as

$$r_k^i = R_k + u_k^i \tag{2}$$

where R_k is the nominal running time from the kth to the $(k+1)$th station and u_k^i is the control action applied to the ith train between the kth and $(k+1)$th stations, in order to increase $(u_k^i > 0)$ or decrease $(u_k^i < 0)$ the running time.

The staying time s_k^i depends on the time interval between the departure of the preceding train and the arrival of the present train. Sophisticated models of this staying time have been proposed (e.g. by Rice[2]), on the basis of queuing theory. Nevertheless the assumption of a linear relationship seems acceptable: the staying time increases proportionally to the number of passengers getting on the train, i.e. to the time elapsed since the departure of the last train, under the hypothesis of randomly arriving passengers.

The staying time s_k^i is therefore modelled as

$$s_k^i = a_k^i(t_k^i - s_k^i - t_k^{i-1}) + D + w_k^i \tag{3}$$

where D is the minimal staying time at a station, when no passenger gets on the train and the driver closes the doors as soon as possible, where a_k^i is the delay rate representing the effect of the time interval since the departure of the last train on the staying time and where w_k^i is a disturbance term.

We propose in the appendix a more sophisticated linear model where the staying time modification is proportional also to the number of passengers leaving the train.

Defining

$$c_k^i = \frac{a_k^i}{1 + a_k^i},\ b_k^i = \frac{1}{1 + a_k^i} \text{ and } v_k^i = b_k^i w_k^i$$

relation (3) can be written

$$s_k^i = c_k^i(t_k^i - t_k^{i-1}) + b_k^i D + v_k^i \tag{4}$$

Defining the nominal schedule representing the operating state without any control or disturbance ($u_k^i = 0$ and $v_k^i = 0$), i.e.

$$T_{k+1}^i = T_k^i + R_k + c_{k+1}^i(T_{k+1}^i - T_{k+1}^{i-1}) + b_{k+1}^i D \tag{5}$$

and x_k^i as the deviation of the actual departure time t_k^i from its nominal value T_k^i, i.e. $x_k^i = t_k^i - T_k^i$ it follows that

$$(1 - c_{k+1}^i)x_{k+1}^i + c_{k+1}^i x_{k+1}^{i-1} = x_k^i + u_k^i + w_k^i \tag{6}$$

Equation (6) constitutes the basic equation for the transfer of the ith train from the kth to the $(k+1)$th station.

In these relations D, R_k, c_k^i and b_k^i are characteristic parameters of the line, to be estimated.

The delay rate a_k^i (and therefore c_k^i) can be estimated by linear regression on a large number of observations (t_k^i, s_k^i) at each station k, according to equation (4) under the following conditions:

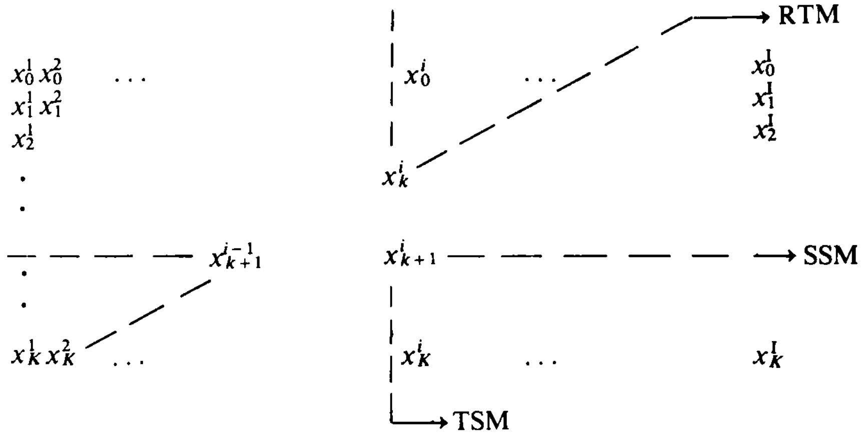
(a) no regulation at all (e.g. no traffic lights!)
(b) no nominal schedule
(c) the doors are closed and the train starts as soon as possible
(d) nominal speed between stations.

As it is rather unrealistic to operate the system under these conditions in order to identify the parameters c_k^i, another method consists of measuring the passenger flow arriving at the station and the riding time.

Usual values of c_k^i are, for Brussels Railways, in the range of 0·01 to 0·05. These parameters depend of course on th station (k) but also on the hour (through i). In the following we omit the upper index i in c_k^i, without loss of generality.

3. STATE-SPACE REPRESENTATIONS

For I trains and $(K+1)$ stations consider the following array of deviations from the nominal schedule



(7)

For a given sequence of control actions and disturbances (u_k^i and v_k^i) this array is generated according to equation (6).

State-space formulations of the traffic dynamics would be very convenient to apply the classical control theory. Two formulations are proposed in Reference[1]:

(a) 'The stations sequential model'

This can be written in matrix form as

$$\mathbf{X}_{k+1} = \mathbf{A}_k\mathbf{X}_k + \mathbf{A}_k\mathbf{U}_k \tag{8}$$

where

$\mathbf{X}_k$ is the vector of the deviations of all trains at the kth station (i.e. the kth row of array (7))
$\mathbf{U}_k$ is the vector of the controls applied at the kth station
$\mathbf{V}_k$ is the vector of disturbances.

More precisely, for $k = 1, \ldots, K$

$$\mathbf{X}_k = \begin{bmatrix} x_k^1 \\ x_k^2 \\ \cdot \\ \cdot \\ x_k^I \end{bmatrix}, \mathbf{U}_k = \begin{bmatrix} u_k^1 \\ u_k^2 \\ \cdot \\ \cdot \\ u_k^I \end{bmatrix}, \mathbf{V}_k = \begin{bmatrix} v_k^1 \\ v_k^2 \\ \cdot \\ \cdot \\ v_k^I \end{bmatrix} \tag{9}$$

and $\mathbf{A}_k = \mathbf{C}_{k+1}^{-1}$ where $\mathbf{C}_{k+1}$ is the following bidiagonal matrix:

$$\mathbf{C}_{k+1} = \begin{bmatrix} 1 - c_{k+1} & & & \\ c_{k+1} & 1 - c_{k+1} & & 0 \\ & \ddots & \ddots & \\ 0 & & c_{k+1} & 1 - c_{k+1} \end{bmatrix} \tag{10}$$

The matrix equation (8) describes the transfer of all trains from the kth to the $(k + 1)$th station, i.e. it generates array (7) row by row, with as initial condition the first row, i.e. the deviations of all trains at the first station. The order of the representation is the number of trains (generally about 100!). The dynamical matrix $\mathbf{A}_k$ is lower triangular with eigenvalues equal to $1/(1 - c_{k+1})$, which, for usual values of c_{k+1} are greater than one.

This means that, without control, the norm of the state vector increases from one row of the array to the next: this is the well-known intrinsically unstable behaviour of a high density public transportation system.

(b) The train sequential model (TSM)

This can be written in matrix form as

$$\mathbf{X}_{i+1} = \tilde{\mathbf{A}}\mathbf{X}_i + \tilde{\mathbf{B}}\mathbf{U}_i + \tilde{\mathbf{B}}\mathbf{V}_i,\ i \geqslant 2 \tag{11}$$

where

$\mathbf{X}_i$ is the vector of the perturbations of the ith train for the $(k + 1)$th station (i.e. the ith column of array (7))
$\mathbf{U}_i$ is the vector of controls applied to the ith train
$\mathbf{V}_i$ is the perturbation vector.

More precisely

$$\mathbf{X}_i = \begin{bmatrix} x_1^i \\ x_2^i \\ \cdot \\ \cdot \\ x_K^i \end{bmatrix}, \mathbf{U}_i = \begin{bmatrix} u_0^i + x_0^i \\ \vdots \\ u_{K-1}^i \end{bmatrix}, \mathbf{V}_i = \begin{bmatrix} v_0^i \\ \vdots \\ v_{K-1}^i \end{bmatrix} \tag{12}$$

and

$$\tilde{\mathbf{B}} = \begin{bmatrix} 1-c_1 & & & 0 \\ -1 & 1-c_2 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1-c_k \end{bmatrix}^{-1} \tag{13a}$$

$$\tilde{\mathbf{A}} = \tilde{\mathbf{B}} \begin{bmatrix} -c_1 & & 0 \\ & \ddots & \\ 0 & & -c_K \end{bmatrix} \tag{13b}$$

Equation (11) describes the evolution of the time deviations at all stations from the i th train to the $(i+1)$th train, with, as initial condition, the deviations of the first train. The order of this representation is the number of stations (for Brussels Railways about 25). The dynamical matrix $\tilde{\mathbf{A}}$ is lower triangular with negative eigenvalues $-c_k/(1-c_k)$ having absolute values less than one. This means that the norm of the state vector decreases from one column of array (7) to the next.

(c) The real time model (RTM)

We now propose an original state-space formulation based on information propagation considerations and which seems to be more promising than the two above formulations. Consider equation (6), generating array (7). A particular x^i_{k+1} is generated, in addition to the control u^i_k and the disturbance w^i_k, by the two deviations x^{i-1}_{k+1} and x^i_k. This generation process can be generalized. If we consider successively the perturbations belonging to the diagonal characterized by $i+k=j$, the elements of diagonal $(j+1)$ can be computed from the elements of diagonal j (in addition to the control and disturbance terms). It is therefore natural to define as state vector the elements of such a diagonal. The corresponding state-space formulation becomes, in matrix form:

$$\mathbf{X}_{j+1} = \mathbf{A}\mathbf{X}_j + \mathbf{B}\mathbf{U}_j + \mathbf{B}\mathbf{V}_j \tag{14}$$

where

$$\mathbf{X}_j = \begin{bmatrix} x_1^{j-1} \\ \vdots \\ x_K^{j-K} \end{bmatrix}, \quad \mathbf{U}_j = \begin{bmatrix} u_0^j \\ \vdots \\ u_{K-1}^{j-K+1} \end{bmatrix}, \quad \mathbf{V}_j = \begin{bmatrix} v_0^j \\ \vdots \\ v_{K-1}^{j-K+1} \end{bmatrix} \tag{15}$$

The dynamical matrices $\mathbf{A}$ and $\mathbf{B}$ take particularly simple forms (bidiagonal and diagonal, respectively), requiring no matrix inversion:

$$\mathbf{A} = \begin{bmatrix} \frac{-c_1}{1-c_1} & & 0 \\ \frac{1}{1-c_2} & \frac{-c_2}{1-c_2} & \\ & \ddots & \ddots \\ 0 & \frac{1}{1-c_k} & \frac{-c_k}{1-c_K} \end{bmatrix} \mathbf{B} = \begin{bmatrix} \frac{1}{1-c_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{1-c_k} \end{bmatrix} \tag{16}$$

Equation (14) describes the evolution of the deviations from one diagonal (characterized by a sum of indices equal to j) to the next (sum $= j + 1$). The order of the representation is the number of stations. The dynamical matrix is bidiagonal and its eigenvalues are $-c_k/(1-c_k)$, the same as for TSM.

An interesting property of this last formulation is that the components of the state vector are known nearly simultaneously. Because of the traffic security requirements (e.g. no more than one train at a time in a line section between two successive stations) the deviations $\mathbf{x}_i^j$ (jth train at the ith station) and x_{i+1}^{j-1} (preceding train at the next station), are known in a short time. It will be seen in the next section that this property makes possible real time practical implementation of a state feedback control policy. It is the reason why we call this model the 'real time model' (RTM).

4. TRAFFIC REGULATION

(a) Performance index

A wide variety of optimization criteria can be considered depending on the control purposes. The following criterion takes into account the objectives discussed in the introduction: regularity of the intervals between successive trains and regularity with respect to the nominal schedule

$$J = \sum_{i,k} p_k (x_k^i)^2 + \sum_{i,k} q_k (x_k^i - x_k^{i-1})^2 + \sum_{i,k} (u_k^i)^2 \tag{17}$$

The first term penalizes the deviations from the nominal schedule; the second term penalizes the deviations of the time intervals between trains and is therefore related to the average waiting time for the passengers and the congestion of trains. The third term is a measure of the amplitude of the control actions, which are zero for the nominal schedule. The values of the coefficients p_k and q_k depend on the control purpose and reflect the trade-off between the regulation objectives.

(b) Theoretical optimal control

The control minimizing the quadratic criterion (17) under linear constraints (the dynamical equations (8), (11) or (14)) is known to be a linear function of the state vector (linear state feedback control), with a gain matrix obtained recursively from the backward evolution of the Riccati equation (see for example Reference[3]).

For a given set of initial conditions the array (7) can be generated optimally by use of this state feedback control.

For SSM the initial condition consists of the first row of array (7) (i.e. the deviations of all the trains at the first station), for TSM it is the first column (i.e. the deviations of the first train at all the stations) whereas for RTM the initial condition consists only of the deviation of the first train at the first station. As less information is involved in the initial condition for RTM than for SSM and TSM the performance index takes a better value. In fact RTM allows one to elaborate the optimal choice of the first row and the first column of array (7), as opposed to both other formulations where these initial conditions are imposed *a priori* with corresponding loss of degrees of freedom. The optimal control based on RTM produces therefore a better value of the performance index then the policies based on TSM and SSM.

(c) Optimal control implementation

The practical implementation of the optimal state feedback control is impossible for SSM and TSM, because this control is a function of future deviations and would imply long-term prediction of these deviations. For instance the control to apply to the ith train at the kth station is a linear combination of all the deviations relative to this station for the complete operation period (SSM) or to this train for its transfer from the first station to the last one (TSM). For SSM the control to be applied at 8 a.m. at a given station could be a function of the deviations of trains passing through the station 12 hours later! A simplified implementable procedure proposed in Reference[1] consists of computing the complete optimal feedback gain and applying a truncated control involving only two known deviations. This solution is absolutely not optimal but requires nevertheless the computation of the complete gain matrix (and therefore the solution of a large dimension Riccati equation).

For RTM however the numerical values of the state vector components can be known nearly simultaneously by the centralized traffic controller. Because of the traffic security requirements (e.g. at most one train at a time in each direction in a section between two successive stations) the deviations x_k^i with $i+k=j$ (i.e. x_k^i = deviation of a train i at the kth station, x_{k-1}^{i+1} = deviation of the next train at the preceding station, $x_{k-2}^{i+2}\ldots$) are known in a short time interval, so the index j characterizing the state vector in RTM reflects the time evolution of the system. On-line state feedback control is therefore implementable: the control to be applied to the ith train between the kth and $(k+1)$th stations is a linear combination of deviations x_β^α with $\alpha+\beta=i+k$. These deviations are known or can be easily predicted by a short-term predictor (at most a two step predictor). *Among the three discussed formulations it is therefore the only one allowing complete on-line state feedback control.*

(d) One-step optimization

The control law can be significantly simplified by restriction of the problem to a one-step optimal control problem. Consider, for the real time model the state vector $\mathbf{X}_j$ corresponding to deviations x_k^i with $i+k=j$. The restricted one-step criterion is defined as

$$J(j)=X_{j+1}^{\mathrm{T}}\mathbf{P}\mathbf{X}_{j+1}+(\mathbf{X}_{j+1}-\mathbf{X}_j)^{\mathrm{T}}\mathbf{Q}(\mathbf{X}_{j+1}-\mathbf{X}_j)+\mathbf{U}_j^{\mathrm{T}}U_j \tag{18}$$

where **P** and **Q** are diagonal matrices characterized, by the p_k and q_k respectively,

$$\mathbf{P}=\begin{bmatrix} P_1 & & 0\\ & \ddots & \\ 0 & & P_k\end{bmatrix} \text{ and } \mathbf{Q}=\begin{bmatrix} q_1 & & 0\\ & \ddots & \\ 0 & & q_k\end{bmatrix} \tag{19}$$

The original performance criterion J defined in (17) is the sum on all diagonals (index j) of the partial criterion $J(j)$. The restricted problem consists of minimizing, for a given j, the partial criterion $J(j)$, subject to the constraint (14). This restriction to a one-step criterion means that, given a set of perturbations on a diagonal j, we want to restore an acceptable situation at step $(j+1)$.

Because of the particular forms of the dynamical matrices **A** and **B** which characterize the RTM, optimal one-step control takes the very simple form

$$U_j=\begin{bmatrix} f_1 & & & 0\\ g_2 & f_2 & & \\ & \ddots & \ddots & \\ 0 & & g_k & f_k\end{bmatrix}\mathbf{X}_j \tag{20a}$$

or, in scalar notation,

$$u_k^i = g_{k+1} x_k^i + f_{k+1} x_{k+1}^{i-1} \tag{20b}$$

where

$$f_{k+1} = \frac{q_k + p_k c_k}{(1-c_k)^2 + p_k + q_k} \text{ and } g_{k+1} = -\frac{p_k + q_k}{(1-c_k)^2 + p_k + q_k} \tag{21}$$

The control u_k^i is therefore a weighted sum of the deviation of the ith train at the kth station and of the preceding train at the next station. When u_k^i is calculated x_k^i is known, x_{k+1}^{i-1} is either known or is to be predicted (one or at most two steps predictor).

It must be noted that this simple form of the one-step optimal control is directly related to the particular form of matrices **A** and **B** and appears therefore only for the RTM. For both other models no benefit can be expected from the restriction to the one-step optimization criterion.

It is interesting to introduce the control (20) into the dynamical equations in order to investigate the closed-loop behaviour for the three models. The closed loop eigenvalues are

for the SSM:

$$\frac{1-g_k}{1-c_k} = \frac{1-c_k}{(1-c_k)^2 + p_k + q_k} \tag{22}$$

for the TSM and the RTM:

$$\frac{f_k - c_k}{1-c_k} = \frac{q_k - c_k(1-c_k)}{(1-c_k)^2 + p_k + q_k} \tag{23}$$

For the SSM, the eigenvalues are always positive and depend on the sum $p_k + q_k$. For $(p_k + q_k)$ larger than $c_k(1-c_k)$ they are less than one (stability) For the TSM and the RTM the eigenvalues are zero for $q_k = c_k(1-c_k)$, positive for $q_k \geqslant c_k(1-c_k)$ and negative for $q_k \leqslant c_k(1-c_k)$

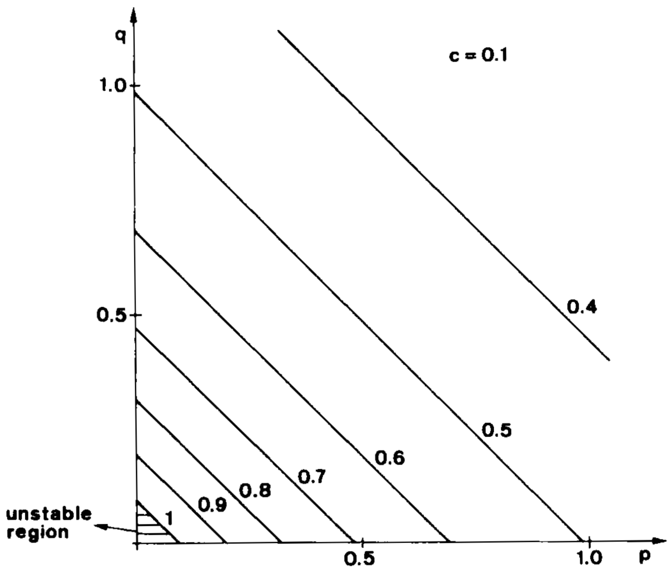


Figure 1. Cuves of iso-eigenvalues for SSM (closed loop)

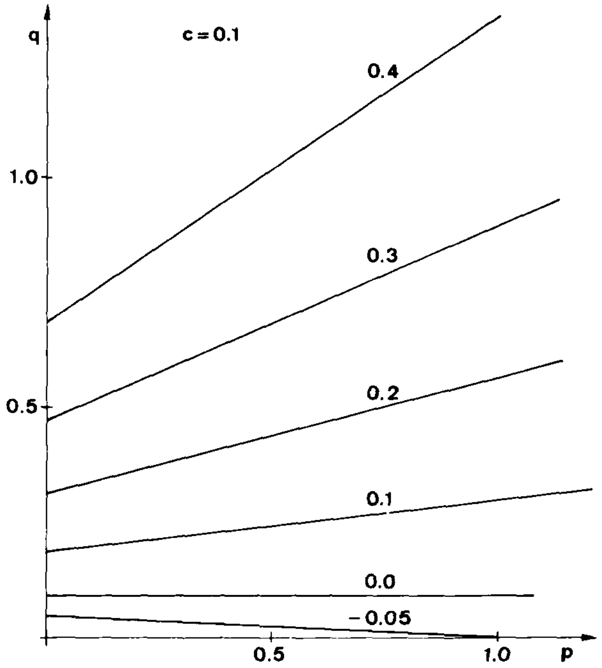


Figure 2. Curves of iso-eigenvalues for TSM and RTM (closed loop)

Figures 1 and 2 give the iso-value curves of the eigenvalues in the (p, q) plane for a given c_k ($c_k = 0 \cdot 1$). These curves are useful for the choice of the weighting coefficients p_k and q_k with respect to the desired transient behaviour of the traffic regulation.

5. THEORETICAL SIMULATIONS

A first set of simulations consists of the generation of the deviations array (7), on the basis of the RTM (equation (14) to (16)), with implementation of the one-step optimal control policy (equations (20) and (21)), without any reference to the nominal time schedule. Clearly the generated array is not realistic: when compared to the nominal time schedule physical impossibilities immediately occur, such as violation of security requirements, shorter transfer times between stations, crossing of trains .... The only interest of these simulations is to compare the effects of several control policies. Simulation results corresponding to the actual Brussels transportation system, taking into account the physical and security constraints, are given in the next section.

The following basic assumptions are adopted:

(a) We consider 15 trains ($I = 15$) and 7 stations ($K = 6$).

(b) The c_k^i are taken equal and constant ($c_k^i = 0 \cdot 1$).

(c) An initial delay of 60 s is imposed on the first train at the first station.

The simulation results are visualized by diagrams giving the deviations of all the trains at the third and seventh stations (a positive deviation means a delay). With these diagrams it is easy to compare the actual waiting times at these stations to their nominal values, according to

$$x_k^i - x_k^{i-1} = (t_k^i - t_k^{i-1}) - (T_k^i - T_k^{i-1})$$

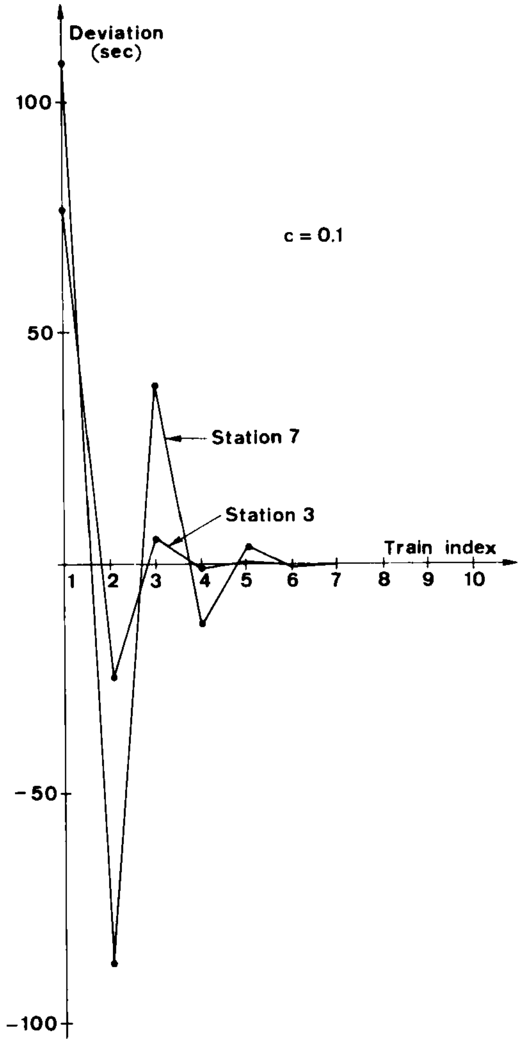


Figure 3. Free system

where $(t_k^i - t_k^{i-1})$ and $(T_k^i - T_k^{i-1})$ are, respectively, the actual and the nominal waiting times at station k, between trains $(i-1)$ and i. The slope of the segment $[x_k^{i-1}, x_k^i]$ is therefore proportional to this difference (a positive slope means an increased waiting time). Several control policies are implemented.

Case 1. Free system: no control is applied. It can be seen in Figure 3 that the deviation of the first train increases dramatically (from 60 s at the first station to 112 s at the seventh station) and that important deviations are induced for the other trains. The regularity of the waiting times is very bad: several waiting times are increased (for instance between trains 2 and 3), others are reduced (for instance between trains 1 and 2). This unstable behaviour is quite uncomfortable, from the passenger's point of view.

Case 2. Optimal control policy (equations (20), (21)) with $p = 1$ and $q = 0$ (Figure 4).

Case 3. Optimal control policy with $p = 0$ and $q = 5$ (Figure 5).

Case 4. Optimal control policy with $p = 1$ and $q = 5$ (Figure 6).

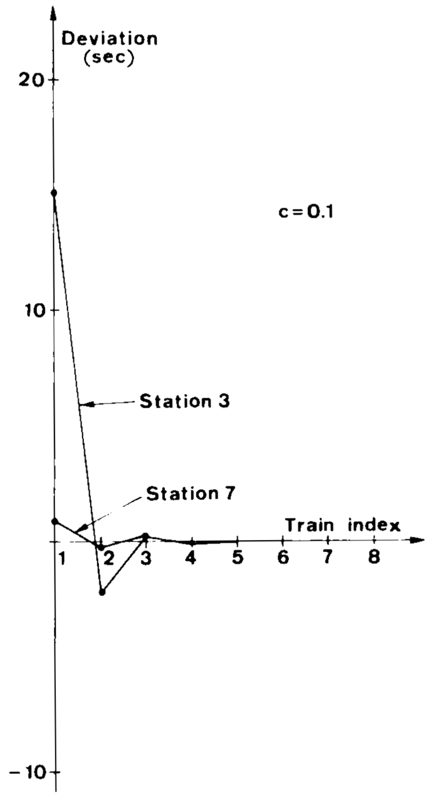


Figure 4. Controlled system with $p = 1$ and $q = 0$

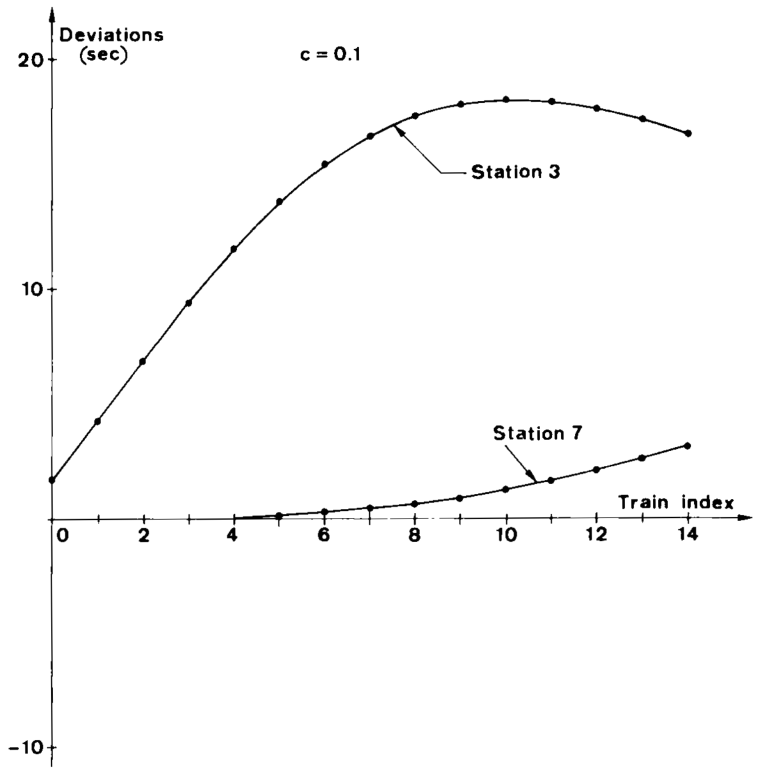


Figure 5. Controlled system with $p = 0$ and $q = 5$

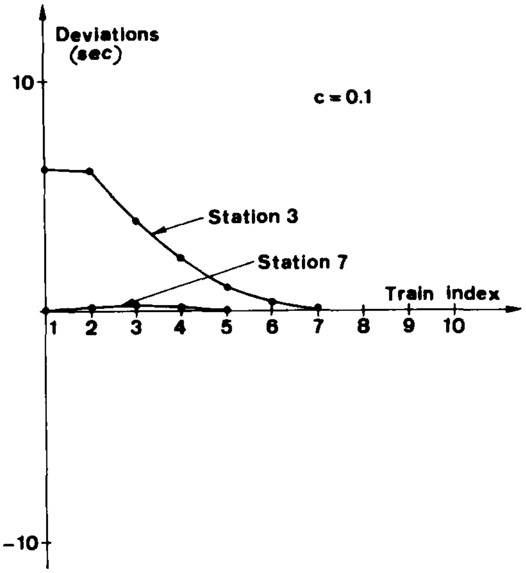


Figure 6. Controlled system with $p = 1$ and $q = 1$

The vertical scale is not the same for Figures 4, 5 and 6 as for Figure 3. The difference between these policies consists of the choice of the weighting coefficients p and q characterizing the performance index (18).

The effects of p and q are to ensure a better regularity with respect to the nominal schedule or with respect to the nominal waiting times respectively. It can be observed that the regularity with respect to the nominal schedule is better for case 2 than for case 3, but that the regularity of the waiting times is better for case 3 (smooth curves). Case 4 is an acceptable trade-off between both objectives. In any case the system behaviour is much better than for the free system. Table I gives for the four policies the maximum train deviations and the maximum waiting time deviations at the third and the seventh stations.

Table I.

	Station	Case 1 free	Case 2 $p = 1$, $q = 0$	Case 3 $p = 0$, $q = 5$	Case 4 $p = 1$, $q = 1$
Maximum train deviation	3	74·1	14·8	18·2	6·2
	7	112·9	0·9	3·2	0·2
Maximum waiting time deviation	3	94·8	17	2·7	2·1
	7	200·7	1·2	0·5	0·1

6. PRACTICAL IMPLEMENTATION AND SIMULATION RESULTS

We now study the problem of implementation of the proposed traffic control policy for an existing transportation system with its physical characteristics and constraints.

The implementation of the policy needs a preliminary analysis of the system, providing:

(a) the topological structure of the line and the values of the parameters of the linear model (R^i_k, D and c^i_k) obtained from a statistical analysis as discussed in section 2
(b) the physical constraints: maximum speed, security constraints (minimal distance between successive trains, ...) imposed by use of traffic lights,
(c) the nominal time schedule, which of course has to be coherent with the linear model (equation (5)).

Consider now the system under operating conditions. The theoretical control u^i_k to be applied to the kth train at the ith station is calculated by the centralized controller according to equations (20) and (21). The necessary information is the deviation of the considered train at this station and that of the preceding train at the next station. Practically this control can be considered as an instruction given to the driver, in order to modify the staying time at the

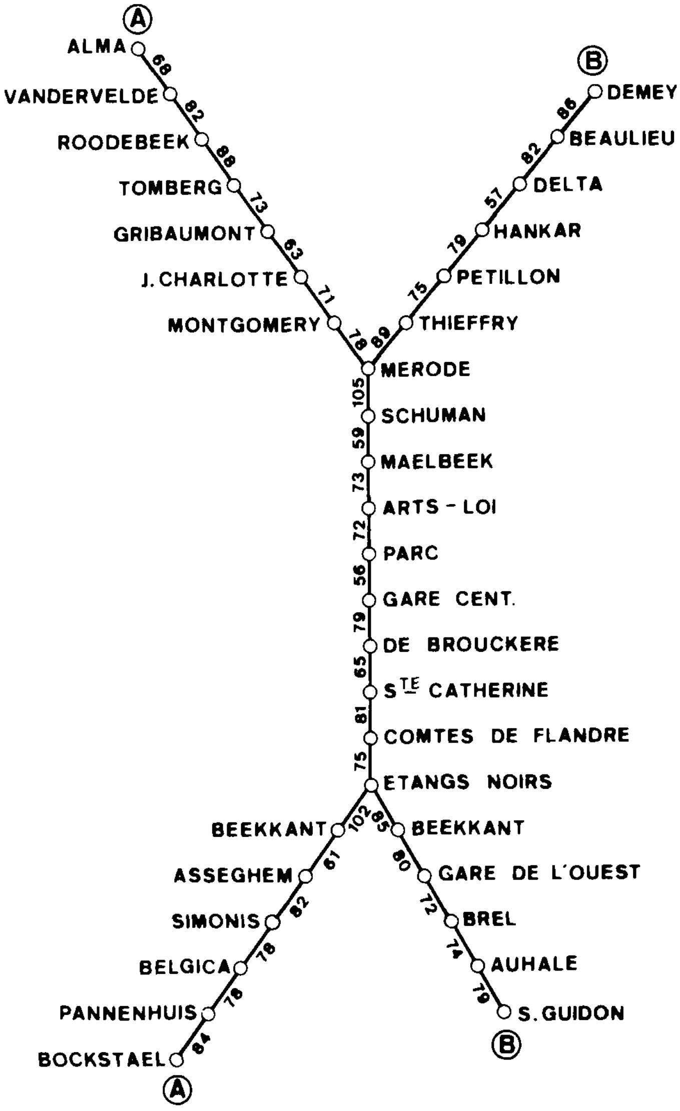


Figure 7. Brussels Underground configuration (standard running times)

station and the running time to the next station. The driver has to follow this theoretical instruction, but conforming himself to the other security requirements. The modification of the staying time can be imposed by use of the traffic light at the station. The modification of the running time can be realized by the choice between three nominal running speeds (low, normal or high).

The implementation of this traffic control policy has been investigated for the Brussels Underground Transportation System. This system consists of two lines (A and B) with a long common section where trains of both lines are operated alternately. The structure of the line as well as the standard running times are given in Figure 7. The minimum waiting time D is 7 s and the delay rate parameters (c_k^i) have been estimated from statistical data. As the passenger flow is not high, compared with the maximum embarkment rate, even during rushhours, the values of c_k^i are small (about 0·03). A nominal time schedule has been generated on basis of these values of the parameters.

A program has been implemented for the simulation of the complete system, taking into account branching, traffic security requirements, minimal distances between trains, ... This program generates the absolute times t_k^i, in connection with the nominal schedule (T_k^i). It allows the introduction of control terms (u_k^i) as well as disturbance terms for any train at any station. The proposed traffic policy has been tested on the basis of the simulation program. Several results were obtained under the following conditions:

(a) For the simulations the c_k^i are considered as Gaussian random variables with mean 0·03 and standard deviation 0·01. The array of c_k^i is the same for all simulations.
(b) The constraints on the control are the following: in addition to the respect of the security requirements the staying time may not be reduced by more than 5 s and the running time modifications may not exceed 1/10 of the standard running time.
(c) For all simulations a delay of 240 s is imposed on the first train at the first station of the common section (Merode).

The results are summarized in diagrams giving the deviations of the trains at the first and last

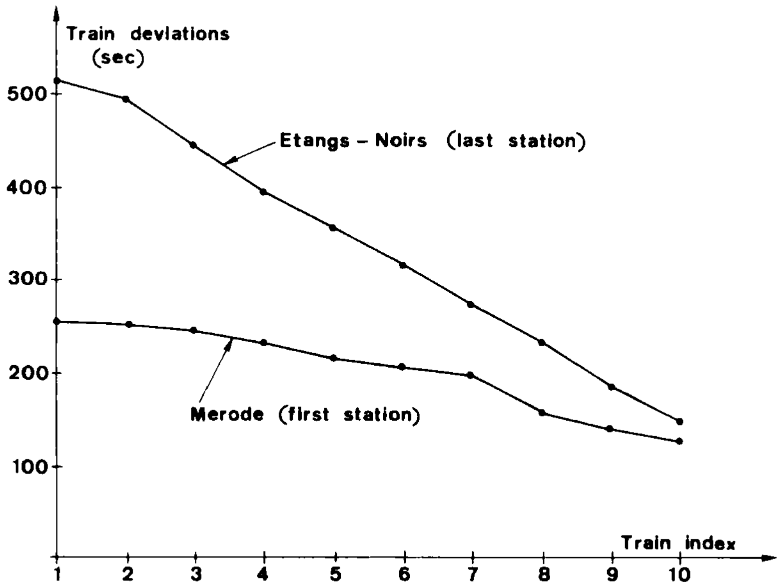


Figure 8. Free system

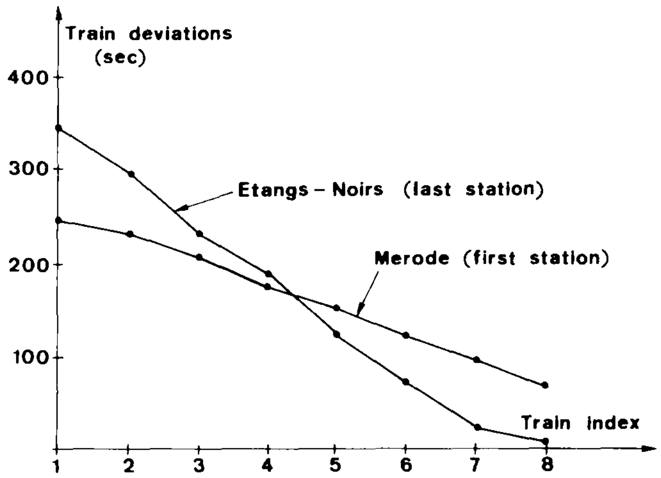


Figure 9. Controlled system with $p = 5$, $q = 0$

stations of the common section (Merode and Etangs-Noirs, respectively). Odd train numbers refer to trains of line A, even numbers to trains of line B.

Several control policies are implemented.

Case 1. Free system: no control is applied. It can be seen in Figure 8 that the delays at the last station become very important (500 s for the first train and 180 s for the 9th train). The delays are propagated because a minimum distance between successive trains has to be ensured.

Case 2. Optimal control law (equations (20), (21)), with constraints, with $p = 5$ and $q = 0$. it can be seen in Figure 9 that the delays are smaller than for the free system and that, from the 4th train, the delays are reduced from one station to the other.

Case 3. Optimal control law with constraints, with $p = 0$ and $q = 5$. The delays are larger than in case 2, but the interval regularity is good. In fact all the trains show the same delay, increasing slowly from one station to the next (Figure 10).

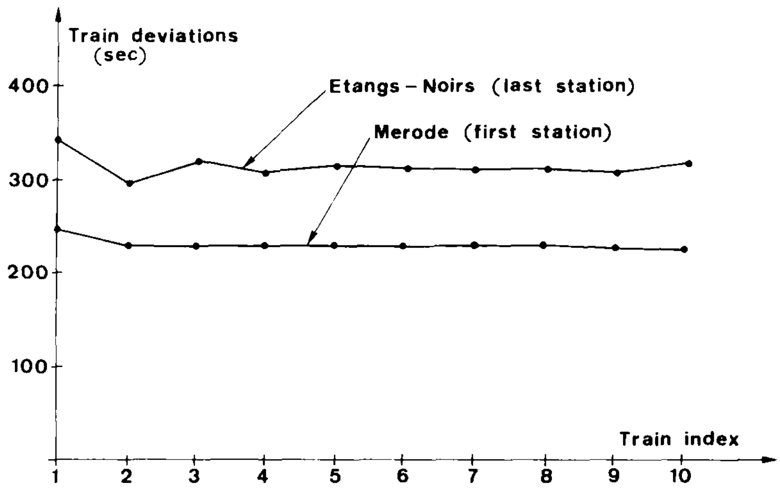


Figure 10. Controlled system with $p = 0$, $q = 5$

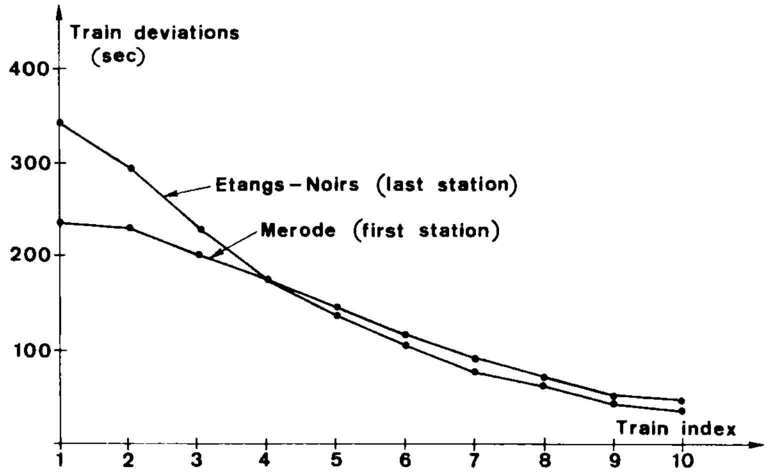


Figure 11. Controlled system with $p = 1$, $q = 1$

Case 4. Optimal control law with constraints, with $p = 1$ and $q = 1$ (Figure 11). This case constitutes an acceptable trade-off between the two objectives.

Clearly the regulation significantly improves the behaviour of the system. A better result can of course be obtained if the severe constraints on the control variables are somewhat relaxed.

7. CONCLUSION

1. The real time state-space representation has been shown to present significant advantages: the simple form of the corresponding dynamical matrices, the possibility of implementation for real control and the simple form of the one-step optimal control. It is therefore particularly suited to on-line traffic regulation.
2. It is easy to extend the basic presentation to take into account several problems: constraints on the control, physical and security constraints, complex network with merging and branching. These extensions have been investigated in simulations of the Brussels line.
3. It is possible to consider a more sophisticated model for the system. The staying time model (equation (3)) can be modified in order to involve not only the passengers entering the vehicle, but also the number of passengers coming out of the train. Considering the load of the train as an additional variable the real time model structure is the same as for the present model. An example of model extension is given in the Appendix.

APPENDIX: EXAMPLE OF MODEL EXTENSION

Instead of considering as in eqution (3) that the modification of the staying time s_k^i is only proportional to the waiting time between two successive trains and therefore to the number of entering passengers, it is possible to take into account the effect of exiting passengers. Equation (3) then becomes

$$s_{k+1}^i = D + w_{k+1}^i + \alpha(o_{k+1}^i + m_{k+1}^i) \tag{24}$$

where o^i_{k+1} and m^i_{k+1} are, respectively, the numbers of passengers exiting and entering train i at station $(k+1)$, α is the delay rate representing the time necessary for one passenger to get on or off a train, and where D and w^i_{k+1} have the same meanings as in equation (3).

The number of exiting passengers is assumed to be proportional to the number of passengers in the train, i.e. the load of the train:

$$o^i_{k+1} = \beta^i_{k+1}\, l^i_k \tag{25}$$

where l^i_k is the load of train i, between stations k and $(k+1)$, and β^i_{k+1} is a proportionality factor depending on the station $(k+1)$ and on the hour (i).

The evolution of the load is given immediately by

$$l^i_{k+1} = l^i_k + m^i_{k+1} - o^i_{k+1} \tag{26}$$

The number of entering passengers is assumed to be proportional to the waiting time between successive trains:

$$m^i_{k+1} = \gamma^i_{k+1}\,(t^i_{k+1} - s^i_{k+1} - t^{i-1}_{k+1}) \tag{27}$$

where γ^i_{k+1} represents the effect of the waiting time on the number of passengers.

By elimination of o^i_{k+1} and m^i_{k+1} between equations (24) to (27), the system is completely described by two linear equations relating the staying times and the loads of the trains.

$$s^i_{k+1} = a^i_{k+1} l^i_k + b^i_{k+1}(t^i_{k+1} - t^{i-1}_{k+1}) + c^i_{k+1}(D + w^i_{k+1}) \tag{28}$$

and

$$l^i_{k+1} = d^i_{k+1} l^i_k + l^i_{k+1}(t^i_{k+1} - t^{i-1}_{k+1}) - \gamma^i_{k+1} c^i_{k+1}(D + w^i_{k+1}) \tag{29}$$

where a^i_{k+1}, b^i_{k+1}, c^i_{k+1}, d^i_{k+1} and e^i_{k+1} are defined in terms of α, β_{k+1}, γ_{k+1}:

$$a^i_{k+1} = \frac{\alpha\beta^i_{k+1}}{1+\alpha\gamma^i_{k+1}},\quad b^i_{k+1} = \frac{\alpha\gamma^i_{k+1}}{1+\alpha\gamma^i_{k+1}},\quad c^i_{k+1} = \frac{1}{1+\alpha\gamma^i_{k+1}},$$
$$d^i_{k+1} = 1 - \beta^i_{k+1} - \gamma^i_{k+1} a^i_{k+1},\quad e^i_{k+1} = \gamma^i_{k+1}\,(1 - b^i_{k+1})$$

Defining a nominal evolution of t^i_k and l^i_k, corresponding to $w^i_k = u^i_k = 0$, i.e.

$$T^i_{k+1} = T^i_k + R_k + a^i_{k+1} L^i_k + b^i_{k+1}(T^i_{k+1} - T^{i-1}_{k+1}) + c^i_{k+1} D \tag{30}$$

and

$$L^i_{k+1} = d^i_{k+1} L^i_k + e^i_{k+1}(T^i_{k+1} - T^{i-1}_{k+1}) - \gamma^i_{k+1} c^i_{k+1} D \tag{31}$$

and the two-dimensional vector of the deviations with respect to the nominal values T^i_k and L^i_k, i.e.

$$\mathbf{X}^i_k = \begin{pmatrix} x^i_k \\ y^i_k \end{pmatrix} = \begin{pmatrix} t^i_k - T^i_k \\ l^i_k - L^i_k \end{pmatrix} \tag{32}$$

we obtain the following matrix equation for the evolution of $\mathbf{X}^i_k$

$$\mathbf{X}^i_{k+1} = \mathbf{A}^i_{k+1}\mathbf{X}^i_k + \mathbf{B}^i_{k+1}\mathbf{X}^{i-1}_{k+1} + \mathbf{C}^i_{k+1} u^i_k + \mathbf{D}^i_{k+1} w^i_{k+1} \tag{33}$$

with

$$\mathbf{A}^i_{k+1} = \begin{bmatrix} \dfrac{1}{1-b^i_{k+1}} & \dfrac{a^i_{k+1}}{1-b^i_{k+1}} \\ \dfrac{e^i_{k+1}}{1-b^i_{k+1}} & d^i_{k+1} + \dfrac{a^i_{k+1}e^i_{k+1}}{1-b^i_{k+1}} \end{bmatrix}, \mathbf{B}^i_{k+1} = \begin{bmatrix} \dfrac{-b^i_{k+1}}{1-b^i_{k+1}} & 0 \\ \dfrac{-e^i_{k+1}}{1-b^i_{k+1}} & 0 \end{bmatrix}$$

$$\mathbf{C}^i_{k+1} = \begin{bmatrix} \dfrac{1}{1-b^i_{k+1}} \\ \dfrac{e^i_{k+1}}{1-b^i_{k+1}} \end{bmatrix}, \ \mathbf{D}^i_{k+1} = \begin{bmatrix} \dfrac{c^i_{k+1}}{1-b^i_{k+1}} \\ \dfrac{c^i_{k+1}e^i_{k+1}}{1-b^i_{k+1}} - \gamma^i_{k+1}c^i_{k+1} \end{bmatrix}$$

Equation (33) is a linear relationship between $\mathbf{X}^i_{k+1}$, $\mathbf{X}^i_k$, $\mathbf{X}^{i-1}_{k+1}$, u^i_k and w^i_{k+1}, and is completely similar to equation (6), with the difference that $\mathbf{X}^i_k$ is now a two-component vector. It is therefore straightforward to extend the RTM to equation (33). We can define a quadratic performance index taking account not only of the deviations from the nominal time schedule and the deviations of the waiting time as in equation (17), but also of the deviations of the actual load from its nominal value L^i_k. The resulting optimal control policy is a linear state feedback control. The implementation of the control policy needs not only the knowledge of the departure times but also of the actual load of the trains. This last information is obtained by counting the entering and exiting passengers. This model needs the estimation of the parameters of the line: α, β^i_k and γ^i_k.

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