

# Robust Global Stabilisation of Stirred Tank Reactors by Saturated Output Feedback\*

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*This paper is concerned with the robust stabilisation of a wide class of unstable processes operated in continuous stirred tank reactors. For non-isothermal reactors, the regulated output is the temperature. For isothermal reactors, the regulated output is the single concentration of either an initial reactant or a final product. In both cases the proposed control law is a dynamic output feedback which can be interpreted as a straightforward modification of a standard PI controller. In accordance with the engineering constraints, the control action is positive and saturated. With this control law, it is shown that the output variable may be regulated at a prescribed set point despite a wide kinetic uncertainty. Furthermore, the controller achieves a global stabilisation of the process in its domain of physical existence. When the reactor is minimum phase, the closed loop has a single equilibrium which is globally asymptotically stable. When the reactor is non-minimum phase, a global practical output regulation with state boundedness is obtained with the same controller.*

**Keywords:** Chemical reactors; Non-linear control; PID control; Robust control

\*This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.

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## 1. Introduction

In this paper, we are concerned with the control of continuous stirred tank reactors represented in Fig. 1, where endothermic and/or exothermic reactions take place between chemical and/or biological species.

These processes can be described by a dynamic model of the form

$$\dot{x} = f(x, T) + d(x^{in} - x) \quad (1)$$

$$\dot{T} = h(x, T) + d(T^{in} - T) + e(T_c - T) \quad (2)$$

$$y = T \quad (3)$$

where  $x = (x_1, \dots, x_n)' \in \mathbb{R}_+^n$  is the reactor compo-

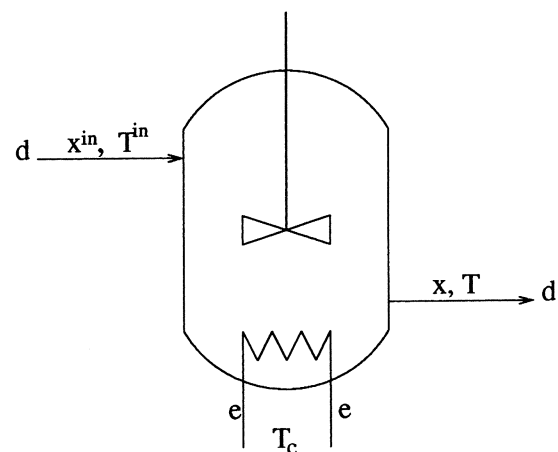


Fig. 1. A continuous stirred tank reactor (CSTR).

sition (i.e., the vector of the species concentrations in the reactor),  $f(x, T) = (f_1(x, T), \dots, f_n(x, T))'$  is a vector of uncertain non-linear kinetic expressions,  $y = T \in \mathbb{R}_+$  is the absolute temperature,  $h(x, T)$  is the reaction heat,  $d \in \mathbb{R}_+$  is the dilution rate,  $x^{in} = (x_1^{in}, \dots, x_n^{in})' \in \mathbb{R}_+^n \setminus \{0\}$  is the feed composition,  $T^{in} \in \mathbb{R}_+ \setminus \{0\}$  is the feed temperature,  $e \in \mathbb{R}_+ \setminus \{0\}$  is the heat transfer rate between the reactor and the heat exchanger and  $T_c \in \mathbb{R}_+$  is the coolant temperature.

The system (1)–(2)–(3) can be rewritten as

$$\dot{x} = f(x, y) + d(x^{in} - x) \quad (4)$$

$$\dot{y} = h(x, y) - (d + e)y + u \quad (5)$$

where  $u = dT^{in} + eT_c \in \mathbb{R}_+$  is the control input which corresponds to the heat supply rate. We assume that

- H1: The map  $(f, h): \mathbb{R}_+^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n \times \mathbb{R}$  is locally Lipschitz  
 H2: For all  $i \in \{1, \dots, n\}$ ,  $f_i(x, y) \geq 0$  if  $x_i = 0$  and  $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+$   
 H3:  $f(x, 0)$  and  $h(x, 0) = 0$  for all  $x \in \mathbb{R}_+^n$   
 H4:  $\sum_{i=1}^n f_i(x, y) = 0$  for all  $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+$   
 H5: There exists a vector  $\tilde{h} \in \mathbb{R}_+^n$  such that  $h(x, y) + \tilde{h}'f(x, y) = 0$  for all  $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+$

Assumption H1 is quite standard and satisfied by most kinetic models proposed in the literature. Assumption H2 expresses the basic physical fact that any species which is missing in the reactor ( $x_i = 0$ ) may be produced by the reactions in which it is involved as a product but cannot be consumed by the reactions in which it is involved as a reactant. Assumption H3 expresses the thermodynamic postulate that no reaction can occur at absolute zero. Assumption H4 expresses that the involved chemical reactions are mass conservative. Similarly, Assumption H5 expresses the energy conservation.

Under these assumptions, it is worth noting that the system (4)–(5) is positive (see, for example, Luenberger [29]):

$$\text{if } \forall i : x_i(0) \geq 0, y(0) \geq 0 \text{ and } u(t) \geq 0 \text{ for all } t \geq 0$$

then  $\forall i : x_i(t) \geq 0$  and  $y(t) \geq 0$  for all  $t \geq 0$

This property will be proved in Lemma 1. It is of paramount importance since the concentrations  $x_i(t)$ , the absolute temperature  $y(t)$  and the heat supply rate  $u(t)$  are physically non-negative variables.

We are first interested in the robust stabilisation of the reactor by means of the regulation of the temperature  $y$  at a desired set point  $y^*$  under the constraint that the control input is positive and saturated. The robustness is to be understood as the robustness of the

control law against uncertainties in the kinetic expressions which satisfy assumptions H1–H2–H3–H4–H5. More precisely, our main contribution in Section 2 will be to show that this control objective can be achieved with a simple saturated temperature feedback controller. When the system is minimum phase, it will be shown in Theorem 5 that the closed loop has a single equilibrium satisfying  $y = y^*$  which is globally asymptotically stable for all initial compositions in a domain  $\Omega$  of physical interest; i.e.,  $\Omega = \{x \in \mathbb{R}_+^n \mid x_1 + \dots + x_n \leq x_1^{in} + \dots + x_n^{in}\}$ . When the system is non-minimum phase, it will be shown in Theorem 7 that a global practical temperature regulation with state boundedness can be achieved with the same controller. The results will be illustrated with a simple simulation experiment.

In the second part of the paper, we are interested in the control of the composition when the reactor operates under *isothermal* conditions. The objective is to regulate the concentration of an initial reactant or a final product at a desired set point. This control objective may also be achieved with a simple positive saturated output feedback controller having similar stabilisation properties.

## 2. Temperature Control

The engineering motivation for the temperature control of continuous stirred tank non-isothermal reactors relies on the fact that the operation of these processes at unstable equilibrium points may correspond to an optimal performance. As a matter of illustration, let us consider a chemical reactor in which the following single irreversible exothermic reaction takes place:



This process can be described by

$$\dot{x}_1 = -r(x_1, y) + d(x_1^{in} - x_1) \quad (7)$$

$$\dot{x}_2 = r(x_1, y) - dx_2 \quad (8)$$

$$\dot{y} = \nu r(x_1, y) - (d + e)y + u \quad (9)$$

where  $x_1, x_2$  are the concentrations in [mol/l] of the reactant  $X_1$  and the product  $X_2$  respectively,  $y$  is the absolute temperature in [K],  $u$  is the control input (i.e., the heat supply rate in [K/min]),  $d = 1$  [min<sup>-1</sup>],  $x_1^{in} = 1$  [mol/l],  $\nu = 250$  [K(mole/l)] and  $e = 0.5$  [min<sup>-1</sup>]. The kinetic expression (first order, Arrhenius) is defined by

$$r(x, y) = \gamma_1 \exp(-\gamma_2/y)x_1$$

where  $\gamma_1 = \exp(25)$  [min<sup>-1</sup>] and  $\gamma_2 = 9000$  [K]. It

can be observed that this model satisfies assumptions H1–H2–H3–H4–H5. All the equilibrium points  $(\bar{x}_1, \bar{x}_2, \bar{y})$  can be parametrised by the constant input  $\bar{u}$  as follows:

$$250k(\bar{y})/(1 + k(\bar{y})) = 1.5\bar{y} - \bar{u} \quad (10)$$

$$\bar{x}_1 = 1/(1 + k(\bar{y})) \quad (11)$$

$$\bar{x}_2 = k(\bar{y})\bar{x}_1 \quad (12)$$

with  $k(y) = \exp(25) \exp(-9000/y)$ . Since equations (11)–(12) are linear in  $\bar{x}_1$  and  $\bar{x}_2$  for fixed  $\bar{y}$ , there are as many equilibrium points as intersection points between the left-hand side of Eq. (10) which represents the heat creation and its right-hand side which represents the heat exchange. For instance, if  $\bar{u} = 450$ , there are three intersection points (as shown in Fig. 2) which correspond to three possible equilibria of the reactor. The low- and high-temperature equilibrium points are asymptotically stable, while the intermediate one is unstable. This can be shown simply by looking at the Jacobian linearisation of the model around the equilibria (see, for example, Cibrario [15]). Often, it is not desirable to operate the process either at the low-temperature equilibrium point for economic reasons (extinction) or at the high-temperature equilibrium point for safety reasons (ignition). It is thus optimal to operate the process at the intermediate unstable equilibrium point. This explains the motivation for a controller which can globally stabilise the closed-loop system at this intermediate open-loop unstable steady state.

## 2.1. A Brief Review of the Literature

The temperature control of continuous stirred tank reactors in non-isothermal conditions can be traced

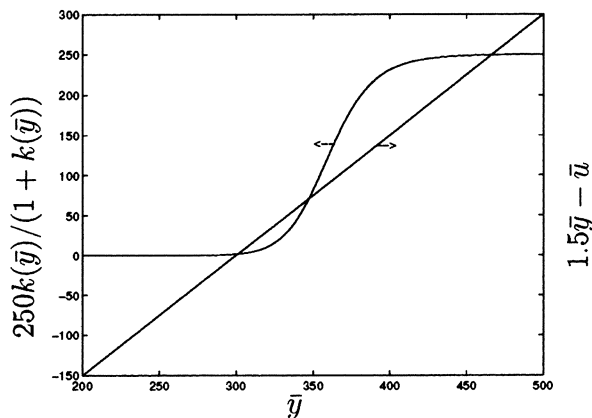


Fig. 2. The three equilibrium points of the reactor.

back to the pioneering papers [7] and has given rise, since then, to an abundant literature [4,17,18,21,28]. However, in this brief review, we limit ourselves to papers which give rigorous proofs of global stabilisation, especially when some robustness to kinetic uncertainties is guaranteed. It is nevertheless worth mentioning the paper [5] which highlights the difficulties that result from the saturation of the control input.

The global output feedback stabilisation of an exothermic reactor with one single reaction has been proved in Cibrario [15] under PI control. A similar result has been obtained in Adebekun and Schork [2] by state feedback using linearisation techniques. However, in these papers, an exact knowledge of the process model is required. An adaptive version based on a Lyapunov design has been given in Adebekun and Schork [3]. An extension to an exothermic reactor with two consecutive reactions has been presented in Adebekun [1]. In these two papers, the uncertainty is nevertheless restricted to lie in some constants that enter linearly in the model of the process. A result of robust global practical stabilisation by time-varying state feedback of an exothermic reactor with one single reaction has been given in Alvarez-Ramirez [6] assuming only the knowledge of bounding functions on the kinetic uncertainties. All these results concern specific examples.

As stated above, we are interested in non-isothermal continuous stirred tank reactors for which the cooling dynamics can be neglected. The intuitive motivation for this simplification is that the cooling dynamics are fast, while the reactor dynamics are slow. A rigorous treatment of this problem combined with a robust control approach has been developed in Christofides et al. [14], where a simulation experiment to an exothermic reactor with two parallel reactions is also given.

Our results in the present paper (see also [24,33,34]) extend all these previous results on the following points:

1. The control design is made robust against kinetic uncertainties which do not depend explicitly on time but are not restricted otherwise (even to be within some known bounding functions).
2. We use output (temperature) feedback only (not state feedback). Furthermore, the proposed controller is guaranteed to achieve global stabilisation with a positive *saturated* control action in accordance with the physical reality.

## 2.2. Open-Loop Analysis

A key property of the open-loop system (4)–(5) which is important for our subsequent analysis is given in the following lemma.

**Lemma 1.** If the input  $u(t)$  is continuous, bounded and positive for all  $t \geq 0$ , then for every bounded initial condition  $(x(0), y(0)) \in \Omega \times \mathbb{R}_+$ , there is one and only one solution  $(x(t), y(t))$  of (4)–(5) which is bounded, positive and contained in  $\Omega \times \mathbb{R}_+$  for all  $t \geq 0$ .

*Proof.* By H2,  $\dot{x}_i \geq dx_i^n \geq 0$  for all  $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+$  with  $x_i = 0$ . By H3,  $\dot{y} = u(t) > 0$  for all  $(x, t)$  with  $x \in \mathbb{R}_+^n$ . Hence,  $(x(t), y(t))$  is in  $\mathbb{R}_+^n \times ]0, \infty[$  for all  $t \in [0, w[$ , the time domain for which the solution is defined. By H4, the time derivative of the function  $M(x) = x_1 + \dots + x_n$  along the solutions of (4)–(5) is  $\dot{M}(x) = -d(M(x) - M^m)$  with  $M^m = x_1^m + \dots + x_n^m$ , which implies that  $\Omega$  is positively invariant. Hence  $x(t)$  is positive and bounded for all  $t \in [0, w[$ . By H5, the time derivative of the function  $W(x, y) = \tilde{h}'x + y$  along the solutions of (4)–(5) is  $\dot{W}(x, y) = -(d + e)W(x, y) + d\tilde{h}'x^n + e\tilde{h}'x + u(t)$ , which implies that  $W(x(t), y(t))$  is bounded for all  $t \in [0, w[$ . Furthermore, in  $\mathbb{R}_+^n \times ]0, \infty[$ , if  $y \rightarrow \infty$  then  $W(x, y) \rightarrow \infty$ . Hence  $y(t)$  is bounded for all  $t \in [0, w[$ . Finally, the existence of the solutions for all  $t \geq 0$  follows from H1 and Theorem 2.4 in Khalil [26].  $\square$

## 2.3. Statement of the Control Law

The control objective is to regulate the output  $y$  of the system (4)–(5) at the set point  $y^*$ .

We begin with a classical linear PI controller:

$$\begin{aligned} u &= [k_0 + k_p (y^* - y)] + \theta \\ \dot{\theta} &= k_i (y^* - y) \end{aligned}$$

The first term  $k_0 + k_p (y^* - y)$  is the proportional action (with a constant offset  $k_0$ ). The second term  $\theta$  is the integral action. An important drawback of this controller is that no constraints are imposed to the control action. In practice, it is obvious that the control  $u = dT^n + eT_c$  must be physically bounded from below and from above:

$$0 < u_0 \leq u \leq u_m$$

We introduce therefore saturations on the proportional and integral actions as follows (see Figs 3 and 4):

$$\sigma(y) = \underset{[u_0, \sigma^m]}{\text{sat}} (k_0 + k_p (y^* - y))$$

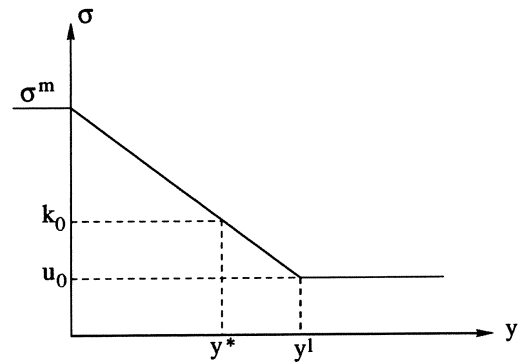


Fig. 3. Saturated proportional action.

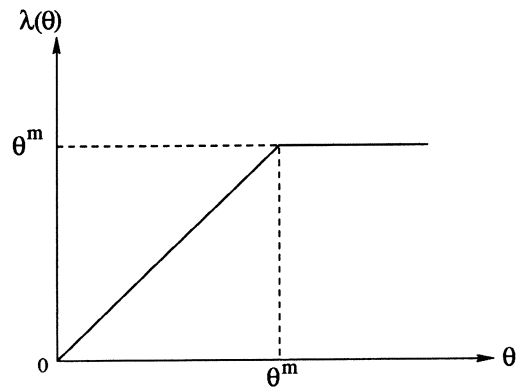


Fig. 4. Saturated integral action.

$$\lambda(\theta) = \underset{[0, \theta^m]}{\text{sat}} (\theta)$$

The saturation function is defined as

$$\underset{[a, b]}{\text{sat}}(s) = \min \{b, \max\{a, s\}\}$$

The control law is then modified as

$$\begin{aligned} u &= \sigma(y) + \lambda(\theta) \\ \dot{\theta} &= k_i (y^* - y) + \lambda(\theta) - \theta \end{aligned}$$

An anti-windup term  $\lambda(\theta) - \theta$  is added in the second equation. This term is zero when the integral action does not saturate, while it is positive when  $\theta < 0$  and negative when  $\theta > \theta^m$  in order to mitigate the excursions of  $\theta$  outside the interval  $[0, \theta^m]$ .

Furthermore, when the temperature is too high, we want to guarantee that the control input is set at its minimal value  $u_0$  whatever the value of  $\theta$ . The integral term  $\lambda(\theta)$  is therefore pre-multiplied by the function (see Fig. 5)

$$\rho(y) = \underset{[0, 1]}{\text{sat}} \left( \frac{y^l - y}{y^l - y^*} \right)$$

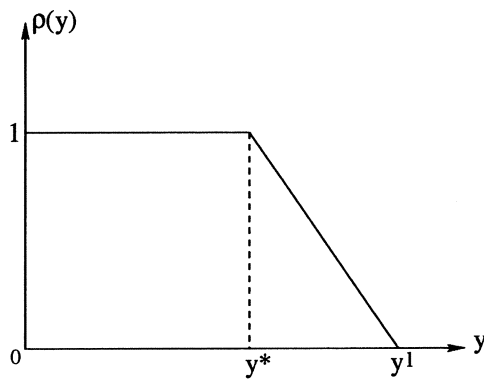


Fig. 5. Graph of the function  $\rho(y)$ .

in order to progressively attenuate the effect of the integral action when  $y \gg y^*$ .

The control law is then finally written as follows:

$$u = \sigma(y) + \rho(y)\lambda(\theta) \quad (13)$$

$$\dot{\theta} = k_i(y^* - y) + \lambda(\theta) - \theta \quad (14)$$

while the constants  $\sigma^m$  and  $y^l$  are defined as follows:

$$\sigma^m = k_0 + k_p y^* \quad y^l = y^* + (k_0 - u_0)/k_p$$

with  $k_0 > u_0 > 0$ .

It is easy to check that, with these definitions, the control action is saturated as follows:

$$0 < u_0 \leq u(t) \leq \sigma^m + \theta^m$$

We assume furthermore that the lower bound  $u_0$  is chosen to satisfy the following assumption:

H6:  $h(x, y^*) - (d + e)y^* + u_0 < 0$  for all  $x \in \Omega$

This assumption implies by continuity, that there exists an interval  $[y^-, y^+]$  with  $y^- < y^* < y^+$  such that whatever  $y \in [y^-, y^+]$ ,  $h(x, y) - (d + e)y + u_0 < 0$  for all  $x \in \Omega$ .

It is worth noting that the proportional and integral actions are saturated separately in contrast to standard saturated PI controllers. Furthermore, the integral action is here forced to zero whenever the proportional action is set to zero.

As we shall see later on, the proposed control law is sufficient to achieve our control objective when all the reactions are endothermic (i.e.,  $h(x, y) \leq 0$  for all  $x \in \Omega$  and all  $y > 0$ ). However, when some of the reactions are exothermic, it may happen that the controller is not capable of cooling the reactor enough owing to the positivity constraint  $u(t) \geq u_0$  for all  $t \geq 0$  or, in more technical terms, that Assumption H6 may not be sufficient to guarantee that the region where the control input is saturated at  $u_0$  will not be attracting. In that case, it is clear that we have to find another way to cool

the reactor. An obvious possibility is to decrease the reactant concentrations in order to decrease the reaction rates and so the heat creation. We therefore make use of the feed composition as an additional control action. The system (4)–(5) is then rewritten as

$$\dot{x} = f(x, y) - dx + v \quad (15)$$

$$\dot{y} = h(x, y) - (d + e)y + u \quad (16)$$

where the extra feedback control action  $v$  is defined as follows:

$$v = \delta(y) \quad (17)$$

with

$$\delta(y) = d \operatorname{sat}_{[0,1]} \left( \frac{y^l - y}{y^l - y^*} \right) x^{in}$$

## 2.4. Closed-Loop Analysis

### 2.4.1. Preliminaries

Let us write down the equations of the system (15)–(16) under dynamic output feedback (13)–(14)–(17) as follows:

$$\begin{aligned} \dot{x} &= f(x, y) - dx + \delta(y) \\ \dot{y} &= h(x, y) - (d + e)y + \sigma(y) + \rho(y)\lambda(\theta) \end{aligned} \quad (18)$$

$$\dot{\theta} = k_i(y^* - y) + \lambda(\theta) - \theta$$

Notice that the right-hand side of (18) is locally Lipschitz from  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$  to  $\mathbb{R}^{n+2}$ . We first prove that the solutions of (18) are bounded.

**Lemma 2.** For every bounded initial condition  $(x(0), y(0), \theta(0))$  in  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ , there is one and only one solution  $(x(t), y(t), \theta(t))$  of (18) which is in  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$  and bounded for all  $t \geq 0$ .

*Proof.* By definition of the control law, the following control input bounds hold:

$$\forall t \geq 0 : 0 < u_0 \leq u(t) \leq \sigma^m + \theta^m$$

Using the fact that  $0 \leq v_1(t) + \dots + v_n(t) \leq d(x_1^{in} + \dots + x_n^{in})$  for all  $t \geq 0$ , it follows from the proof of Lemma 1 that  $(x(t), y(t))$  is in  $\Omega \times \mathbb{R}_+$  and bounded independently of  $\theta(t)$  for all  $t \in [0, w]$ , the time domain for which the solution is defined. Now, the following differential inequalities hold:

$$\begin{aligned} \dot{\theta} &\leq \bar{\tau} + \theta^m - \theta \\ \dot{\theta} &\geq \underline{\tau} - \theta \end{aligned}$$

where  $\bar{\tau} = \sup_{t \in [0, w]} \{-k_i(y^* - y(t))\}$  and  $\underline{\tau} = \inf_{t \in [0, w]} \{-k_i(y^* - y(t))\}$ . Hence,  $\theta(t)$  is bounded for all  $t \in [0,$

w]. The existence of the solutions for all  $t \geq 0$  follows from Theorem 2.4 in Khalil [26].  $\square$

By definition of  $y'$ , we can always choose  $k_0$  such that  $y' \in [y^-, y^+]$ . This allows us to establish in the following lemma that the closed-loop trajectories leave the region where  $y \geq y'$  (i.e., the region where the control input is saturated at  $u_0$ ) in finite time.

**Lemma 3.** For the closed-loop system (18), there exists a finite time  $t_0 \geq 0$  such that  $y(t) < y'$  for all  $t \geq t_0$ .

*Proof.* By H5, the time derivative of the function  $W(x, y) = \tilde{h}'x + y$  along the solutions of (18) is  $\dot{W}(x, y) = -d\tilde{h}'x - (d + e)y + u_0 < 0$  at  $y \geq y'$  due to H6, which also implies that at  $y = y'$ ,  $\dot{y} < 0$  for all  $x \in \Omega$ . Using Lemma 2, the result follows immediately.  $\square$

Without loss of generality, it is then supposed henceforth that  $y(0) - y' < 0$ . Hence the system dynamics are restricted to the set  $\Omega' = \{x \in \Omega, 0 < y < y'\}$ , which is a positively invariant set for the closed-loop system (18) which reduces to

$$\begin{aligned} \dot{x} &= f(x, y) - dx + \delta(y) \\ \dot{y} &= h(x, y) - (d + e)y + k_0 + k_p(y^* - y) + \rho(y)\lambda(\theta) \\ \dot{\theta} &= k_i(y^* - y) + \lambda(\theta) - \theta \end{aligned} \quad (19)$$

#### 2.4.2. Minimum-Phase Systems

From Eqs (4)–(5), it is clear that the zero dynamics of the system (1)–(2)–(3) are

$$\dot{x} = f(x, y^*) + d(x^{in} - x) \quad (20)$$

These dynamics can be interpreted also as the isothermal dynamics of the reactor. We now assume that the system (19) is minimum phase in the sense that the zero dynamics (20) have a single equilibrium point  $\bar{x}$  which is exponentially stable relative to  $\Omega$ . More precisely, this assumption is formulated as follows:

H7: There exists a continuously differentiable function  $W_0 : \Omega$  to  $\mathbb{R}$  such that, for all  $x \in \Omega$ :

1.  $W_0(x) \geq \alpha \|x - \bar{x}\|^2$
2.  $\|\nabla W_0(x)\| \geq \beta \|x - \bar{x}\|$
3.  $\nabla W_0(x) [f(x, y^*) + d(x^{in} - x)] \leq -\gamma \|x - \bar{x}\|^2$  where  $\alpha, \beta, \gamma$  are positive constants, while  $\nabla W_0(x)$  is the gradient of  $W_0(x)$ .

Under this minimum-phase assumption, we can

prove the following result concerning the system (19).

**Theorem 4.** There exists  $k_0 > 0, k_p > 0, k_i > 0$  and  $\theta^m > 0$  such that the equilibrium point  $(\bar{x}, y^*, \bar{\theta})$  of (19) with  $\bar{\theta} = (d + e)y^* - h(\bar{x}, y^*) - k_0$  is asymptotically stable relative to  $\Omega' \times \mathbb{R}$ .

*Proof.* Let  $\theta^m > 0$  be such that  $\theta^m + (d + e)y^* - h(\bar{x}, y^*) - k_0 > 0$ . Then, by H6, the following equation has a unique solution  $\bar{\theta}$  which is positive, provided  $k_0 > 0$  is suitably chosen

$$h(\bar{x}, y^*) - (d + e)y^* + k_0 + \lambda(\bar{\theta}) = 0$$

Let  $W(x, y, \theta) = W_0(x) + \frac{1}{2}(y - y^*)^2 + k_i^{-1} \int_{\bar{\theta}}^{\theta} (\lambda(\vartheta) - \bar{\theta}) d\vartheta$  be a continuously differentiable function defined from  $\Omega' \times \mathbb{R}$  to  $\mathbb{R}$ . Then it is easily verified that

$$W(x, y, \theta) \geq \chi \left( \begin{array}{c} \|x - \bar{x}\| \\ \|y - y^*\| \\ \|\theta - \bar{\theta}\| \end{array} \right)$$

where  $\chi(\cdot)$  is an unbounded strictly increasing function that is zero at zero. Now, the time derivative of  $W(x, y, \theta)$  along the solutions of (19) is as follows:

$$\begin{aligned} \dot{W}(x, y, \theta) &= \nabla W_0(x) (f(x, y) - dx + \delta(y)) \\ &\quad + (y - y^*) (h(\bar{x}, y) - (d + e)y \\ &\quad + \bar{\theta} + k_0 + k_p(y^* - y)) \\ &\quad + (y - y^*) (h(x, y) - h(\bar{x}, y)) \\ &\quad + (y - y^*) \lambda(\theta) (\rho(y) - 1) \\ &\quad + k_i^{-1} (\lambda(\theta) - \bar{\theta}) (\lambda(\theta) - \bar{\theta}) \end{aligned}$$

Using the fact that the right-hand side of (19) is globally Lipschitz on the closure of  $\Omega' \times \mathbb{R}$ , it follows from H7 that for all  $(x, y, \theta) \in \Omega' \times \mathbb{R}$

$$\begin{aligned} \dot{W}(x, y, \theta) &\leq -\gamma \|x - \bar{x}\|^2 \\ &\quad - (k_p + d + e - \eta_1) \|y - y^*\|^2 \\ &\quad + (\beta \eta_2 + \eta_3) \|x - \bar{x}\| \|y - y^*\| \end{aligned}$$

where  $\eta_1 = \eta_3$  is the Lipschitz constant relative to  $h(x, y)$ , while  $\eta_2$  is the Lipschitz constant relative to  $f(x, y) - dx + \delta(y)$ . So,  $\dot{W}(x, y, \theta) \leq 0$  provided  $(k_p + d + e - \eta_1)\gamma - (\beta \eta_2 + \eta_3)^2/4 > 0$ . The asymptotic stability of the equilibrium point  $(\bar{x}, y^*, \bar{\theta})$  is then derived from La Salles' invariance theorem. The result follows immediately.  $\square$

Theorem 4 shows that the system (19) has a single equilibrium point which is globally asymptotically

stable in  $\Omega' \times \mathbb{R}$ . Now, it must be observed that this system represents in fact the behaviour of the closed-loop dynamics (18) after the time instant  $t_0$ . This is, however, sufficient to induce the global stabilisation of the full system (15)–(16) in  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$  as stated in the following theorem.

**Theorem 5.** The dynamic output feedback (13)–(14)–(17) globally stabilises the system (15)–(16) at the equilibrium point which satisfies  $y = y^*$  with respect to  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ .

*Proof.* Using Lemma 3, the result follows from Theorem 4.  $\square$

A very important consequence of the proof of Theorem 4 is that the tuning of the positive design parameters  $k_0$ ,  $k_p$ ,  $k_i$ , and  $\theta^m$  is robust against a *full kinetic uncertainty* satisfying assumptions H1–H2–H3–H4–H5. Indeed, it suffices to choose the offset  $k_0$  close enough to  $u_0$  in order to satisfy the conditions  $k_0 - u_0 > 0$ ,  $y^* + (k_0 - u_0)/k_p \in [y^-, y^+]$  and  $h(\bar{x}, y^*) - (d + e)y^* + k_0 < 0$ , the proportional gain  $k_p$  and the upper bound  $\theta^m$  large enough to satisfy the conditions  $(k_p + d + e - \eta_1)\gamma - (\beta\eta_2 + \eta_3)^2/4 > 0$  and  $\theta^m + (d + e)y^* - h(\bar{x}, y^*) - k_0 > 0$  respectively, while the integral gain  $k_i$  can be arbitrarily assigned. Some knowledge of the kinetics is, however, requested in order to be able to verify the minimum-phase assumption, but this knowledge is not required for the controller tuning.

#### 2.4.3. Non-Minimum-Phase Systems

If the minimum-phase assumption H7 is not satisfied, practical output regulation can be achieved with the same controller. By practical output regulation, we mean that the regulation error  $y(t) - y^*$  can be made arbitrarily small in finite time:

**Theorem 6.** For any  $\epsilon > 0$ , there exists  $k_p > 0$  such that any solution of (19) initialised at  $t_0$  in  $\Omega' \times \mathbb{R}$  satisfies the following condition:

$$\begin{aligned} \forall t \geq t_0 : |y(t) - y^*| \\ \leq |y(t_0) - y^*| \exp(-t) + \epsilon(1 - \exp(-t)) \end{aligned} \quad (21)$$

*Proof.* The time derivative of  $W(y) = \frac{1}{2}(y - y^*)^2$  along the solutions of (19) is

$$\begin{aligned} \dot{W}(y) = (y - y^*) (h(x, y) - \\ (d + e)y + k_0 + k_p(y^* - y) + \rho(y)\lambda(\theta)) \end{aligned}$$

Using the fact that  $(y - y^*) (h(x, y) - (d + e)y + k_0 + \rho(y)\lambda(\theta))$  is bounded on the closure of  $\Omega' \times ]-\infty, \infty[$ , it follows that for all  $(x, y, \theta) \in \Omega' \times ]-\infty, \infty[$

$$\dot{W}(y) \leq -2k_p W(y) + \eta$$

where  $\eta$  is the bound of  $|(y - y^*) (h(x, y) - (d + e)y + k_0 + \rho(y)\lambda(\theta))|$ . Defining  $\epsilon$  as  $\sqrt{\eta/[2k_p]}$ , the result follows from Theorem 3.2, Chapter VIII in Rouche and Mawhin [31].  $\square$

Theorem 6 shows that the output of the system (19) is practically regulated at  $y = y^*$  in  $\Omega' \times \mathbb{R}$ . This is sufficient to induce the practical output regulation of the full system (15)–(16) in  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$  as stated in the following theorem.

**Theorem 7.** The dynamic output feedback (13)–(14)–(17) practically regulates the output of the system (15)–(16) at  $y = y^*$  with respect to  $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ .

*Proof.* Using Lemma 3, the result follows immediately from Theorem 6.  $\square$

An important consequence of the proof of Theorem 6 is that, whatever  $k_0$ ,  $k_i$  and  $\theta^m$  are, the higher the proportional gain  $k_p$  is the smaller the asymptotic output regulation error is, *independently of the kinetics*.

## 2.5. Example

We go back to the example (7)–(8)–(9) given in Section 2, which is rewritten as

$$\dot{x}_1 = -k(y)x_1 - x_1 + v \quad (22)$$

$$\dot{x}_2 = k(y)x_1 - x_2 \quad (23)$$

$$\dot{y} = 250k(y)x_1 - 1.5y + u \quad (24)$$

with  $k(y) = \exp(25 - 900/y)$ .

Let  $y^* = 350$  and  $u_0 = 375$ . Assumption H6 is then clearly satisfied since the following inequality holds:

$$250k(350) - 525 + 375 < 0$$

Assumption H7 is also satisfied. Indeed, the zero dynamics (see (20)) which are written as

$$\dot{x}_1 = -k(350)x_1 - x_1 + 1 \quad (25)$$

$$\dot{x}_2 = k(350)x_1 - x_2 \quad (26)$$

are linear asymptotically stable.

The following values of the design parameters have been selected:

$$k_0 = 380 \quad k_p = 3 \quad k_i = 1 \quad \theta^m = 100$$

The initial concentrations  $x_1$  and  $x_2$  are set to zero, while the initial temperature is  $y(0) = 445$  K.

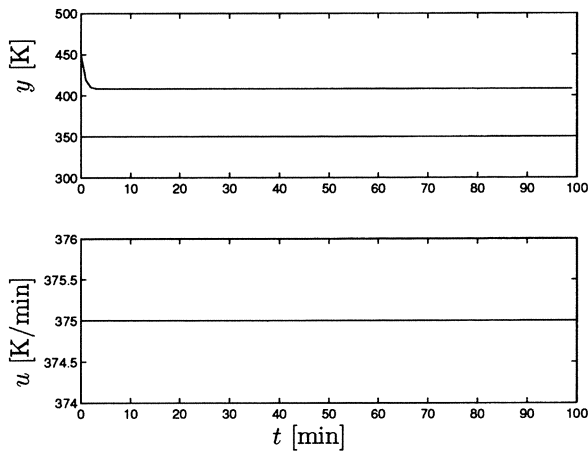


Fig. 6. Undesirable closed-loop behaviour when the reactant feed rate is kept constant.

In Fig. 6, we first observe what happens if the process is under control of the non-linear controller (13)–(14) but with a constant inlet reactant feed rate  $v = \text{constant}$ . In that case, the control input  $u$  is continuously saturated at its minimal value  $u_0$  and the temperature may not converge to the desired set point. The reactor is driven to the high-temperature equilibrium point corresponding to  $u_0$ . There is an undesirable additional equilibrium point (shown in Fig. 7) which is asymptotically stable and belongs to the region of the state space where the control input saturates.

Suppose now that the extra control action (17) is used. The closed-loop behaviour is then shown in Fig. 8, where we observe that the control objective is now effectively achieved. The temperature converges to the desired set point and the closed-loop system now has a single equilibrium which is globally asymptotically stable (though it was open-loop unstable!).

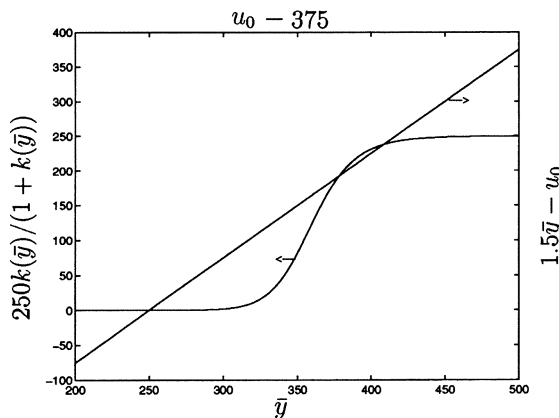


Fig. 7. The equilibria corresponding to a constant input  $u_0 = 375$ .

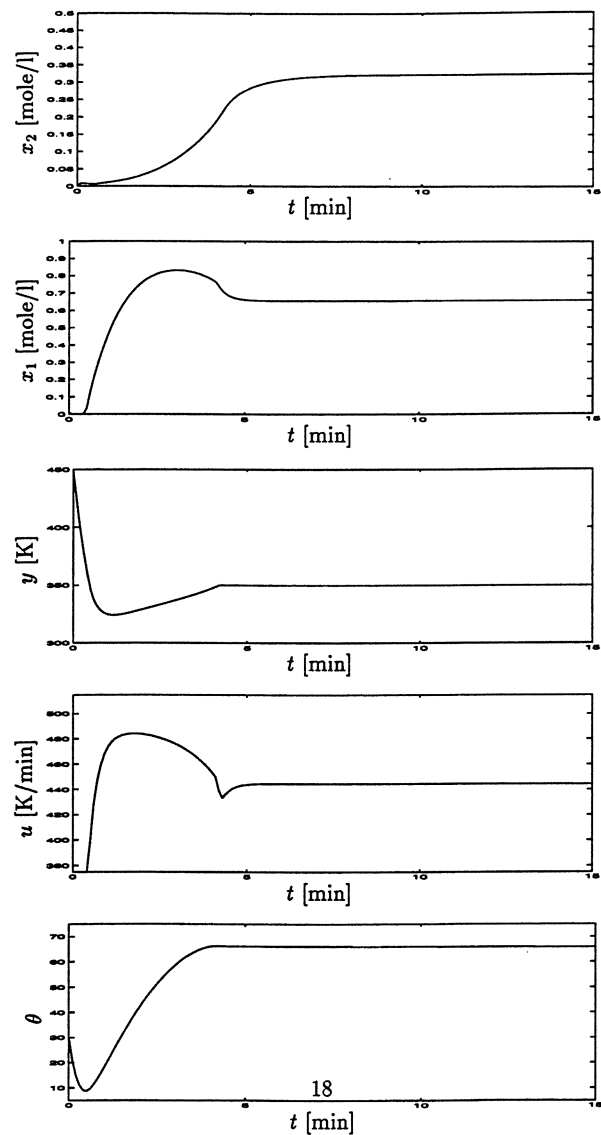


Fig. 8. Effective closed-loop behaviour.

### 3. Composition Control in Isothermal Reactors

We now consider a reactor operating under isothermal conditions where the temperature is supposed to be maintained constant, possibly with the regulator which has been described in the previous section. The process is described by the following single-input single-output model:

$$\dot{x} = f(x) + u(x^{in} - x) \quad (27)$$

$$y = x_1 \quad (28)$$

where  $x = (x_1, \dots, x_n)' \in \mathbb{R}_+^n$  is the reactor composition (i.e., the vector of the species concentrations in the reactor),  $f(x) = (f_1(x), \dots, f_n(x))' \in \mathbb{R}^n$  is a



vector of uncertain non-linear kinetic expressions and  $x^{in} = (x_1^{in}, \dots, x_n^{in})' \in \mathbb{R}_+^n \setminus \{0\}$  is the known constant feed composition, while the control input  $u \in \mathbb{R}_+$  is the dilution rate (proportional to the volumetric feed rate) and the output  $y$  is one measured species concentration of the process.

The map  $f(x)$  satisfies the following conditions:

- H1b: The map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}^n$  is locally Lipschitz  
 H2b: For all  $i \in \{1, \dots, n\}$ ,  $f_i(x) \geq 0$  if  $x \in \mathbb{R}_+^n$  and  $x_i = 0$   
 H3b: For all  $x \in \mathbb{R}_+^n$ ,  $\sum_{i=1}^n f_i(x) = 0$

The underlying physical motivation of these assumptions has been given in Section 1.

The control objective is to regulate the concentration  $y = x_1$  at a desired set point  $y^*$  under the constraint that the control is positive and saturated. However, the choice of  $y$  and  $y^*$  is not free and we have the following additional assumption:

- H4b: The regulated output variable  $y = x_1$  is the concentration of either an initial reactant or a final product  $X_1$ .  
 If  $X_1$  is an initial reactant, then  $0 < y^* < y^{in} = x_1^{in}$   
 If  $X_1$  is a final product, then  $0 = y^{in} < y^*$

An initial reactant is a species which is a reactant of at least one reaction but is neither a product nor a catalyst of any reaction. Necessarily an initial reactant can only be consumed and must be provided to the reactor from the outside. A final product is a species which is produced by at least one reaction but is neither a reactant nor a catalyst of any reaction.

### 3.1. Brief Review of the Literature

Compared with the literature on temperature control in non-isothermal reactors (which was reviewed above), there are relatively few papers dealing with composition control in isothermal reactors and most of the available publications concern biological reactors.

Early references (e.g., [13,16,19]) were devoted to output feedback regulation by PID control. More recently, the issue of composition control by state feedback linearisation was considered, e.g., in [20,22,23,30]. In these papers, exact feedback linearisation for stirred tank reactor models was discussed under the assumption of a perfect knowledge of the kinetic functions.

In this paper, we are concerned with the situation

where the kinetic functions are not known. The objective is to design composition controllers that have robustness against kinetic uncertainties.

In cases where the kinetic uncertainty is represented by unknown constant parameters, adaptive techniques can be used. In the case of linear parametrisation, several successful applications have been given in Bastin and Dochain [11]. The case of a specific non-linearly parametrised reactor model has been recently treated in Boskovic [12].

A more interesting situation occurs when the kinetic function is expressed as  $f(x) = f_0(x) + \partial f(x, t)$ , where  $f_0(x)$  is a known nominal model and  $\partial f(x, t)$  represents the unknown modelling uncertainty. This problem has been recently considered in two papers, by Kravaris and Palanki [27] and by Arkun and Calvet [8].

In Kravaris and Palanki [27], a Lyapunov redesign approach [10] is followed to guarantee the robustness of a linearising state feedback controller under a fairly restrictive matching condition. A proof of robust global practical stabilisation is given, provided the zero dynamics of the nominal process are input-to-state stable in the sense of Sontag and Wang [32]. The method is illustrated with an application of composition control of a chemical reactor with robustness against an unmodelled first-order side reaction.

It is shown in Arkun and Calvet [8] that the matching condition can be relaxed by using a Lyapunov redesign based on an algebraic Riccati equation. The method is also illustrated with a chemical reactor with an unmodelled side reaction. However, only a semi-global practical stabilisation can be guaranteed under a minimum-phase assumption.

Our contribution (see also Jadot [24]) differs from these previous works on the following points:

1. In Kravaris and Palanki [27] and Arkun and Calvet [8], the kinetic uncertainty is restricted to have an additive form satisfying a specific matching condition which is often unrealistic in practice. In the next section, the control design is made robust against a much wider kinetic uncertainty which is only required to satisfy the mild and quite realistic assumptions H1b to H3b.
2. In Kravaris and Palanki [27] and Arkun and Calvet [8], the composition control is achieved through state feedback linearisation, which requires in particular full state on-line measurements. In the next section, we propose an output feedback control law, which requires only the on-line measurement of the single concentration of either an initial reactant or a final product.

3. In the present paper, the proposed controller is guaranteed to achieve global stabilisation with a non-negative control action, in accordance with the physical reality (indeed, the control input is a feeding flow rate, usually provided by a pump, which cannot be negative in practice). This is an important point because it is well known that the blind saturation of a controller which has not been cautiously designed may impair its nominal stabilisation properties.

### 3.2. Statement of the Control Law

Let us recall that the control objective is to regulate the concentration  $y$  at the set point  $y^*$ . To achieve this objective, we propose a simple output feedback controller which is quite similar to the temperature controller that we have used above. The control law is written as follows:

$$u = \frac{1}{|y^{in} - y|} (\sigma(y) + \rho(y) \lambda(\theta)) \quad (29)$$

$$\dot{\theta} = k_i \pi (y^* - y) + \lambda(\theta) - \theta \quad (30)$$

with

$$\sigma(y) = \text{sat}_{[u_0, \sigma^m]} (k_0 + k_p \pi (y^* - y))$$

$$\lambda(\theta) = \text{sat}_{[0, \theta^m]} (\theta)$$

$$\rho(y) = \text{sat}_{[0, 1]} \left( \frac{y^l - y}{y^l - y^*} \right)$$

where

$$\pi = \text{sgn}(y^{in} - y^*)$$

$$\sigma^m = \sup_{0 < y < z^{in}} \{k_0 + k_p \pi (y^* - y)\}$$

$$y^l = y^* + \pi k_0 / k_p$$

The parameter  $\pi$  is a constant equal to +1 or -1 depending on the sign of  $y^{in} - y^*$ . In other words,  $\pi = +1$  if the controlled species is an initial reactant and -1 if it is a final product (see Assumption H4b).

The underlying motivation for the structure of this control law is totally parallel to that of the temperature controller. It is not repeated here.

### 3.3. Stability Properties of the Closed Loop

When an isothermal reactor of the form (27)–(28) is under the control of the proposed output feedback controller (29)–(30), the stability properties of the

closed loop can be analysed by following an argumentation similar to the analysis of temperature control that we have presented in detail in Section 2. A comprehensive treatment of this analysis can be found in the thesis [24] and in the report [25]. The results of this analysis can be summarised as follows. When the reactor is minimum phase, the closed loop has a single equilibrium globally asymptotically stable in the positive orthant. This equilibrium is located inside the subset  $\Sigma_{inv}$  defined as follows:

$$\Sigma_{inv} = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = \sum_{i=1}^n x_i^{in} \right\}$$

This set  $\Sigma_{inv}$  is usually called a reaction invariant in the chemical engineering literature (see, e.g., Asbjornsen [9]). When a reactor is non-minimum phase, the controller achieves a global practical output regulation with state boundedness of the system.

## 4. Conclusion

The robust stabilisation by output feedback of a wide class of processes operated in continuous stirred tank reactors has been achieved relative to the domain of physical interest. For non-isothermal reactors, the regulated output is the temperature. For isothermal reactors, the regulated output is the single concentration of either an initial reactant or a final product. In both cases, the proposed control law is a dynamic output feedback which can be interpreted as a non-linear straightforward modification of a standard PI controller. The output variable may be regulated at a prescribed set point despite a wide kinetic uncertainty. The controller achieves a global stabilisation with a positive and saturated control action.

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