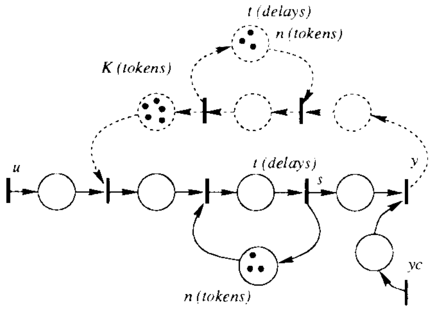

Fig. 4. Kanban cell of the feedback has been modified.

V. Conclusion

First, we have presented a method to synthesize the greatest linear and causal feedback in order to keep the transfer relation of the open-loop system. Second, we have proposed a method to modify a pull control system in order to delay as far as possible the input of unprocessed parts without changing the output in regard to the customer's demand. Both methods allow reducing the work-in-process without changing the system performance. They are based on residuation theory and dioid properties. The solutions proposed to solve the two previous problems are relatively reminiscent with the pole placement method well known in the conventional linear system theory.

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Output Deadbeat Control of Nonlinear Discrete-Time Systems with One-Dimensional Zero Dynamics: Global Stability Conditions

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***Abstract*— Output deadbeat control in one step is considered for a class of discrete-time systems described by a nonlinear single-input/single-output recursive representation. Global stability conditions are established for the particular subclass of systems with *one-dimensional zero dynamics*. The results are illustrated with applications to polynomial and neural dynamical systems.**

***Index Terms*—Deadbeat control, discrete time, nonlinear systems, stability.**

I. Introduction

In the black box approach to modeling nonlinear processes, classes of nonlinear functions, such as polynomial Hammerstein models or neural networks, are being used to represent dynamic relationships between input and output variables [8], [9]. In some sense these are the natural nonlinear versions of the well-known linear ARX system representations. This motivates us to focus attention on how such input–output models may be used for control purposes.

A classical objective in control system design is to require that the system output should reach its desired value as quickly as possible. A typical and widely used example of such a minimum time design in digital control systems is the well-known output *deadbeat* controller.

In a linear framework, output deadbeat controllers (also called one-step-ahead controllers [5]) and their stochastic counterparts minimum variance controllers [1] have been studied for a long time and numerous practical applications have been reported in the literature. "The study of linear deadbeat controllers has given much insight into the properties of linear systems and it seems worthwhile to investigate output deadbeat controllers for nonlinear systems" (see Glad [4]). The purpose of the present paper is to contribute to this investigation with an input–output approach.

We limit ourselves to a particular class of nonlinear single-input/single-output systems of the form $y_{t+1} = \varphi(y_t, y_{t-1}, \cdots, y_{t-\mu}, u_t, u_{t-1})$. Here y_t is the system output and u_t is the input. We consider only the specific subclass of systems with *one-dimensional zero dynamics*, that is the right-hand side of the model equation depends only on u_t and u_{t-1} but not on earlier values of the input.

Under the assumption that output deadbeat control is feasible we identify conditions under which set point regulation with global stability of the closed-loop system is guaranteed. An important point in our result is that the feedback law need not be continuous, which indeed is the norm in output deadbeat control. This makes the stability analysis/result nontrivial as classical tools such as Lyapunov theory are not (directly) applicable to discontinuous maps. The stability question, in the case of set point regulation, can be seen to be

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equivalent to global stability of the zero dynamics. The main results of the paper are contained in Theorems 3.1 and 3.6 where global stability conditions are given. Although dynamic systems defined by a scalar continuous map on the real line are well studied, see, e.g., [3], we believe that our stability results are novel.

The paper is organized as follows. The control problem is formulated in Section II. The global stability conditions are given in Section III and illustrated with simple polynomial examples. In Section IV we discuss an application to a class of neural network dynamical systems. Section V concludes the paper.

II. Problem Formulation

A. System Description

We consider nonlinear discrete-time SISO systems represented as

$$y_{t+1} = \varphi\big(Y_t^\mu, u_t, u_{t-1}\big) \quad t \in N \tag{1}$$

where t is the time index, u_t is a scalar control input, and y_t is the scalar output. Y_t^μ is a vector of past outputs

$$Y_t^\mu = (y_t, y_{t-1}, \cdots, y_{t-\mu}) \quad \mu \in \mathsf{N}. \tag{2}$$

We assume that the map $\varphi : \mathsf{R}^{\mu+1} \times \mathsf{R} \times \mathsf{R} \to \mathsf{R}$ is continuous in its arguments and satisfies the following.

Assumption 2.1 (Feasibility): For all $(Y, w) \in \mathsf{R}^{\mu+1} \times \mathsf{R}$ there exists at least one $v \in \mathsf{R}$ such that

$$\varphi(Y, v, w) = 0. \tag{3}$$

The above assumption states that it is in principle feasible to regulate the system output to zero and to maintain it at zero. We like to stress that this is not a trivial assumption (see [2] for a discussion of this issue).

B. Output deadbeat Control

An output deadbeat control law for a system of the form (1) is a feedback compensator capable of regulating the system output to zero in a single time step. More precisely we define the following.

Definition 2.2 (Output deadbeat Control): An output deadbeat control law for a system of the form (1) is a feedback law

$$u_t = \alpha\big(Y_t^\mu, u_{t-1}\big) \tag{4}$$

where the map $\alpha : \mathsf{R}^{\mu+1} \times \mathsf{R} \to \mathsf{R}$ is such that for all $(Y, w) \in \mathsf{R}^{\mu+1} \times \mathsf{R}$

$$0 = \varphi(Y, \alpha(Y, w), w) \tag{5}$$

Assumption 2.1 guarantees the existence of a (control) value v for any variable (Y, w) such that $\varphi(Y, v, w) = 0$. Because the equation $\varphi(Y, v, w) = 0$ may for any one (Y, w) have multiple solutions v, the definition of the control law $\alpha(Y, w)$ involves two steps. At any instant t find (using a root solving algorithm) all possible values for v such that $\varphi(Y_t^\mu, v, u_{t-1}) = 0$, then implement u_t as a particular value v according to some *choice criterion*.

C. Closed-Loop Dynamics

Introduce the map $g : \mathsf{R} \to \mathsf{R} : w \to g(w) = \alpha(0^\mu, w)$ (here 0^μ is a $\mu + 1$ dimensional vector, all of whose entries are zero).

It is clear from the definition of output deadbeat control law that the closed-loop dynamics [described by (1) and (4)] are essentially governed by the equation

$$\begin{aligned} y_{t+1} &= 0 \quad \forall t \geq 0 \\ u_{t+1} &= g(u_t) \quad \forall t \geq \mu + 1. \end{aligned} \tag{6}$$

The properties of the closed-loop are hence entirely characterized by the scalar map g. The equation $u_{t+1} = g(u_t)$ can be interpreted as representing the *zero dynamics* of the closed-loop system.

In the special case of the deadbeat control of a linear ARX system, it is well known that the closed loop is stable iff the system is *minimum phase*, that is if the zeros of the system are strictly inside the unit circle. Indeed, in such a case, the closed-loop system (1)–(4) has a unique globally attracting and stable fixed point.

A natural nonlinear extension of this linear stability condition is as follows.

Definition 2.3 (Stability): A deat beat control law $u_t = \alpha(Y_t^\mu, u_{t-1})$ is stabilizing if there exists a bounded closed interval $\mathcal{A} \subset \mathsf{R}$ such that:

- the interval $\mathcal{A}$ is invariant under $g : g(\mathcal{A}) \subset \mathcal{A}$.
- the interval $\mathcal{A}$ is stable under g : for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$g^k(\mathcal{B}_\delta) \subset \mathcal{B}_\epsilon \quad \forall k > 0 \tag{7}$$

  where $\mathcal{B}_\epsilon$ and $\mathcal{B}_\delta$ denote an ϵ-neiborhood and a δ-neiborhood of $\mathcal{A}$. (An ϵ-neiborhood of $\mathcal{A}$ is an interval defined by $\mathcal{B}_\epsilon = \{u \in \mathsf{R} : \inf_{x\in\mathcal{A}} |x - u| < \epsilon\}$, see, e.g., [6, p. 301]).
- the interval $\mathcal{A}$ is attracting under g : for any initial condition $u_0 \in \mathsf{R}$, the solution u_t of the zero dynamics $u_{t+1} = g(u_t)$ converges to the interval $\mathcal{A}$, that is

$$\lim_{t\to\infty} \left[\inf_{x\in\mathcal{A}} |x - u_t|\right] = 0. \tag{8}$$

Remark that this definition requires the existence of an attracting *interval*, which is weaker than requiring the existence of an attracting *point*. In the linear case, this distinction is not relevant because all attractors are fixed points. In contrast, in the nonlinear case, this definition enables us to consider zero dynamics having other (periodic or chaotic) attractors than fixed points.

It is the purpose of the next section to establish global stability conditions for these zero dynamics and hence for the output controlled system (1) and (4).

III. Global Stability Conditions

Typically the map g is discontinuous, even if φ is continuous. This discontinuity is closely related to the nonuniqueness of the solutions v of $\varphi(Y, v, w) = 0$ for given (Y, w).

In this section, we present two main results. First, we consider the situation where the map g has a bounded closed invariant interval in the sense of Definition 2.3 and is continuous on the real line outside this interval (all the dicontinuities are inside the interval). In this case we will provide necessary and sufficient conditions for this invariant interval to be globally stable and attracting (Theorem 3.1). Then we consider the case where the map g is piecewise continuous and has a unique *fixed point* $u^* = g(u^*)$. In this case we will provide sufficient conditions under which this fixed point is globally asymptotically stable (Theorem 3.5).

Assume that the map $g : \mathsf{R} \to \mathsf{R}$ has a bounded closed invariant interval $\mathcal{A} \triangleq [a, b] \subset \mathsf{R}$. Denote by G the graph of g outside $\mathcal{A} \times \mathcal{A}$, i.e., $G = \{(x, g(x)) : x \in \mathsf{R} \setminus \mathcal{A}\}$. Denote by G^{-1} the reciprocal graph $G^{-1} = \{(g(x), x) : x \in \mathsf{R} \setminus \mathcal{A}\}$. The graph G is partitioned in a left graph G_L and a right graph G_R defined as

$$G_L = \{(x, g(x)) \in G : x < a\} \quad G_R = \{(x, g(x)) \in G : x > b\}. \tag{9}$$

The reciprocal graphs G_L^{-1} and G_R^{-1} are defined similarly.

Theorem 3.1: Let $\mathcal{A} \triangleq [a, b] \subset \mathsf{R}$ and $g : \mathsf{R} \to \mathsf{R}$ such that:

- $\mathcal{A}$ is invariant under $g : g(\mathcal{A}) \subset \mathcal{A}$;
- g is continuous on $\mathsf{R}\backslash]a, b[$.

Then $\mathcal{A}$ is a globally attracting interval of the iterative map $u_{t+1} = g(u_t)$ iff the following conditions hold:

$$\forall x < a \quad g(x) > x \tag{10}$$

$$\forall x > b \quad g(x) < x \tag{11}$$

$$\forall x < a : \exists (x, z) \in G_R^{-1} \quad g(x) < z \tag{12}$$

$$\forall x > b : \exists (x, z) \in G_L^{-1} \quad g(x) > z. \tag{13}$$

Proof: See the Appendix. □

Example 3.2: Consider the following system:

$$y_{t+1} = \left(y_t + u_t^2 - 2u_t - u_{t-1}^2 + 2\right)\left(y_t^2 + u_t - u_{t-1} - 1\right). \tag{14}$$

For this example, the *first step* of any deadbeat control algorithm consists of calculating the roots $u(t)$ of

$$\left(y_t + u_t^2 - 2u_t - u_t^2 + 2\right)\left(y_t^2 + u_t - u_{t-1} - 1\right) = 0. \tag{15}$$

There are obviously three roots which can be written as

$$u_t^{(1)} = 1 - \sqrt{u_{t-1}^2 - 1 - y_t} \tag{16}$$

$$u_t^{(2)} = 1 + \sqrt{u_{t-1}^2 - 1 - y_t} \tag{17}$$

$$u_t^{(3)} = u_{t-1} + 1 - y_t^2. \tag{18}$$

To design a deadbeat control algorithm, we then have to specify a criterion of choice between these three roots at each time step.

The following choice criterion is proposed:

$$\begin{aligned} \text{if } u_{t-1} \geq \sqrt{5}, &\quad \text{then } u_t = u_t^{(1)} \\ \text{if } 1 < u_{t-1} < \sqrt{5}, &\quad \text{then } u_t = u_t^{(2)} \\ \text{if } -1 \leq u_{t-1} \leq 1, &\quad \text{then } u_t = u_t^{(3)} \\ \text{if } u(t-1) < -1, &\quad \text{then } u_t = u_t^{(2)}. \end{aligned}$$

In this case, the map g corresponding to the zero-dynamics is defined as

$$\begin{aligned} &u_t = 1 - \sqrt{u_{t-1}^2 - 1} \quad \text{if } u_{t-1} \geq \sqrt{5} \\ &u_t = 1 + \sqrt{u_{t-1}^2 - 1} \quad \text{if } 1 < u_{t-1} < \sqrt{5} \\ &u_t = u_{t-1} + 1 \quad \text{if } -1 \leq u_{t-1} \leq 1 \\ &u_t = 1 + \sqrt{u_{t-1}^2 - 1} \quad \text{if } u_{t-1} < -1. \end{aligned}$$

It is easy to check (see [2]) that the interval $\mathcal{A} = [1 - \sqrt{8}, 3]$ satisfies the conditions of Theorem 3.1. $\mathcal{A}$ is bounded, closed, and invariant. The map g is continuous outside $\mathcal{A}$ while the graphs G and G^{-1} do not intersect and satisfy conditions (10)–(13). We conclude that the output deadbeat control law is stabilizing in the sense of Definition 2.3. The signal u_t is a bounded deterministic chaos with an hyperbolic strange attractor. The chaotic nature of u_t is proved in [2].

This example illustrates two key points.

1) In contrast to the linear case, the design of an output deadbeat control law for a nonlinear system is not a trivial task. The issue of finding explicit design rules is an interesting open question.
2) In the linear case, the output deadbeat controller is unique and it is stabilizing if and only if the system is minimum phase. From the foregoing example, we see that for nonlinear systems several output deadbeat control laws can coexist, depending on the choice criterion which is formulated. Some of these control laws can be stabilizing while others are not.

An interesting special situation occurs when the invariant interval $\mathcal{A}$ reduces to a single fixed point $u^* = g(u^*)$. In that case, we have the following corollary of Theorem 3.1.

Corollary 3.3: Assume that g is continuous on R with a single fixed point u^*. Then u^* is a globally asymptotically stable fixed point iff it is locally asymptotically stable and $G \cap G^{-1} = \emptyset$.

Proof: This corollary is a straightforward consequence of Theorem 3.1. Indeed it suffices to check that the statement "u^* is locally asymptotically stable and $G \cap G^{-1} = \emptyset$" is equivalent to conditions (10)–(13) when $a = b = u^*$. □

Comment 3.4: If g was not only continuous but differentiable as well (C^2 to be precise), Corollary 3.3 would follow from Sarkovskii's theorem and a property of the ω limit set of the orbits of one dimensional maps given in [3, Ch. 4, Th. A]. Our result here is stronger since it applies to maps that are continuous but not necessarily differentiable.

The stability conditions of Corollary 3.3 can be extended to the case where g is only piecewise continuous, with a unique fixed point $u^* = g(u^*)$.

Theorem 3.5: Let $g : \mathsf{R} \to \mathsf{R}$ be a piecewise continuous function. Assume that there exists a unique fixed point $u^* \in \mathsf{R}$ such that $u^* = g(u^*)$. Assume that g is continuous at u^*. Let u^* be locally asymptotically stable. u^* is a global attractor if there exist continuous bounding functions g_m and g_M satisfying $g_m(x) \leq g(x) \leq g_M(x)$ for all $x \in \mathsf{R}$ with $g_m(u^*) = g_M(u^*) = u^*$ and such that $G_m \cap G_M^{-1} = \emptyset$ and $G_M \cap G_m^{-1} = \emptyset$.

Proof: This theorem is a direct consequence of Theorem 3.1 and Corollary 3.3. □

Example 3.6: Consider the following polynomial input–output system:

$$y_{t+1} = 2y_t + u_t\left(u_t^2 - 6\right) + 0.7u_{t-1}. \tag{19}$$

In order to find an output deadbeat control law we need to solve the equation $0 = 2y + v^3 - 6v + 0.7w$ for v given (y, w). Clearly, there exists a real v that satisfies this equation, and depending on the value (y, w) there may be three possible choices for v.

Regardless of the precise choice that defines uniquely the deadbeat control law $u_t = \alpha(y_t, u_t)$ and the corresponding zero dynamics $u_t = g(u_{t-1})$ the behavior of u_t must satisfy the identity

$$u_t^3 - 6u_t = -0.7u_{t-1}. \tag{20}$$

It is clear that the function g must necessarily be discontinuous. The design of the output deadbeat control law could be viewed as attempting to select the choice criterion in such a manner as to satisfy the conditions of global stability in Theorem 3.5. In this particular example it can be verified that the choice criterion *select least magnitude solution* achieves the desired result. The output deadbeat control law is then defined as

$$\begin{aligned} u_t &= \alpha(y_t, u_{t-1}) \\ &= \arg\min\left\{|u_t| : 2y_t + u_t^3 - 6u_t + 0.7u_{t-1} = 0\right\}. \end{aligned} \tag{21}$$

It can be checked that the conditions of Theorem 3.5 are satisfied since the two graphs G and G^{-1} of the zero-dynamics do not intersect. The zero-dynamics have a single globally attracting fixed point at $u^* = 0$.

It is also interesting to observe that, for *any* choice criterion (for example: *select the greatest magnitude solution*), the zero dynamics will be stable in the sense of Theorem 3.1 with a globally attracting invariant stable interval $\mathcal{A} = [-4, +4]$.

Application to a Class of Neural Dynamical Systems

In this section, we consider a class of dynamical systems where the function φ is implemented under the form of a two-layered recurrent neural network with one hidden layer and one linear output layer

$$y_t = \sum_{i=1}^{N} \omega_i \sigma(a_i y_t + b_i u_t + c_i u_{t-1} + d_i) \quad (22)$$

where N is the number of hidden nodes and $\sigma(\cdot) : \mathsf{R} \to [-1, 1]$ is a smooth monotonic (sigmoid) function such that $\lim_{x\to\infty} \sigma(x) = 1$ and $\lim_{x\to-\infty} \sigma(x) = -1$. We assume that the weights ω_i satisfy the following conditions:

$$\omega_i b_i > 0 \quad \text{and} \quad \left|\frac{c_i}{b_i}\right| < 1 \quad i = 1, 2, \cdots N. \quad (23)$$

Under this condition, it can be shown that there exists a globally stabilizing output deadbeat controller for that system.

Lemma 4.1 (Existence and Uniquenes of Deadbeat Control): For any neural dynamical system of the form (22) the feasibility Assumption 2.1 is satisfied and there exists a unique output deadbeat control law.

Proof: Consider

$$\varphi(y, v, w) = \sum_{i=1}^{N} \omega_i \sigma(a_i y + b_i v + c_i w + d_i). \quad (24)$$

This function is monotic in v for all (y, w), indeed

$$D_v\varphi(y, v, w) = \sum_{i=1}^{i=N} \omega_i b_i D\sigma(a_i y + b_i v + c_i w + d_i) > 0 \quad (25)$$

and such that

$$\lim_{v\to\pm\infty} \varphi(y, v, w) = \pm \sum_{i=1}^{N} |\omega_i|. \quad (26)$$

It follows that there exists a unique $\alpha(y, w)$ such that $\varphi(y, \alpha(y, w), w) = 0$. □

Lemma 4.2 (Global Stability with Deadbeat Control): The neural dynamical system (22) under condition (23) has globally asymptotically stable behavior under deadbeat control in that the output is regulated to zero in one step and the zero dynamics have a single globally attracting fixed point.

Proof: For the system (22) the zero dynamics are governed by the map $u_{t+1} = g(u_t)$ where g is implicitly defined as

$$\sum_{i=1}^{i=N} \omega_i \sigma(b_i g(w) + c_i w + d_i) = 0. \quad (27)$$

We have for its derivative evaluated at w

$$Dg(w) = -\frac{\sum_{i=1}^{i=N} \omega_i c_i D_i}{\sum_{i=1}^{i=N} \omega_i b_i D_i}. \quad (28)$$

Here $D_i = D\sigma(b_i g(w) + c_i w + d_i) > 0$. Hence, it follows that g is well defined and C^1 everywhere. Moreover, from the expression

$$Dg(w) = \sum_{i=1}^{i=N} \left(\frac{\omega_i b_i D_i}{\sum_{i=1}^{i=N} \omega_i b_i D_i}\right)\left(\frac{c_i}{b_i}\right) \quad (29)$$

we conclude that $|Dg(w)| < 1$ because of condition (23). The result then follows from Corollary 3.3. □

V. Conclusion

This paper has dealt with output deadbeat control for a class of SISO discrete-time nonlinear systems with one-dimensional zero dynamics. Under the assumption that set point regulation is feasible, we have presented sufficient conditions for the global stability of the closed-loop system. The issue has been illustrated with examples of systems with polynomial nonlinearities and an application to a class of neural dynamical systems. A more comprehensive study of the special case of planar polynomial systems can be found in [7].

The analysis of the present paper was restricted to the specific class of systems with one-dimensional zero dynamics. One difficult open question is to extend the stability results of Section III to higher order zero-dynamics. Conservative sufficient stability conditions can obviously be easily stated. The issue of finding necessary and sufficient conditions is much more intricate. The obstruction originates from the fact that the nonintersection condition (i.e., $G \cap G^{-1} = \emptyset$) does not prevent the existence of periodic orbits for higher dimension (or otherwise stated, the Sarkovskii's theorem is valid only for one-dimensional systems).

Appendix
Proof of Theorem 3.1

Without loss of generality, we may assume that the invariant interval $\mathcal{A}$ is centered at the origin: $\mathcal{A} \triangleq [-a, a]$. Conditions (10)–(13) can then be rewritten as

$$\forall x < -a \quad g(x) > x \quad (30)$$

$$\forall x > a \quad g(x) < x \quad (31)$$

$$\forall x < -a : \exists(x, z) \in G_R^{-1} \quad g(x) < z \quad (32)$$

$$\forall x > a : \exists(x, z) \in G_L^{-1} \quad g(x) > z. \quad (33)$$

We thus have to prove that these conditions (30)–(33) are sufficient and necessary for $\mathcal{A}$ to be a globally attracting interval of the iterative map g. The notations G_R and G_L have been defined in (9).

First, we observe that the invariance of $\mathcal{A}$ under g and the continuity of g on $\mathsf{R}\backslash\text{int}(\mathcal{A})$ implies that

$$\lim_{\substack{x\to-a\\x\le-a}} g(x) \in [-a, a] \qquad \lim_{\substack{x\to a\\x\ge a}} g(x) \in [-a, a]. \quad (34)$$

Proof of Sufficiency: There exists a function $h : \mathsf{R}\backslash\mathcal{A} \to \mathsf{R}$ that has the following properties.

1) h is continuous and monotonically decreasing.
2) $h(-a) = a$ and $h(a) = -a$.
3) The graph of the function h is located "between" the graphs G and G^{-1}, which is made technically precise as follows:

$$h(x) > g(x) \quad \forall x < -a \quad (35)$$

$$h(x) < g(x) \quad \forall x > a \quad (36)$$

$$h(x) < z \quad \forall x < -a \text{ such that } \exists(x, z) \in G_R^{-1} \quad (37)$$

$$h(x) > z \quad \forall x > a \text{ such that } \exists(x, z) \in G_L^{-1}. \quad (38)$$

Under the assumptions of the theorem, it is easy to check that such a function is guaranteed to exist. In particular, inequalities (35) to (38) follow from the fact that G and G^{-1} do not intersect, while the decreasing monotonicity is made possible because g is a function.

Now we define the following function $V(u_t)$:

$$V(u_t) = \begin{cases} |u_t| + |h(u_t)|, & \text{if } u_t \in R\backslash\mathcal{A} \\ 2a, & \text{if } u_t \in \mathcal{A} \end{cases} \quad (39)$$

We observe that this function V is positive and continuous for all $u_t \in \mathsf{R}$. We are now going to show that it is a Lyapunov function for the map g outside the interval $\mathcal{A}$.

Suppose that $u_t > a$. Then, under conditions (35) to (38), we have three possibilities for $g(u_t)$: 1) $a < g(u_t) < u_t$; 2) $-a \le g(u_t) \le a$; and 3) $h(u_t) < g(u_t) < -a$. Let us examine these three possibilities and show that, in each case, necessarily $V(u_{t+1}) < V(u_t)$.

1) $a < g(u_t) < u_t$, then $a < u_{t+1} < u_t$ and $|h(u_{t+1})| < |h(u_t)|$ because h is monotonically decreasing. Hence $V(u_{t+1}) < V(u_t)$.

2) $-a \leq g(u_t) \leq a$, then $u_{t+1} \in \mathcal{A}$ and $V(u_{t+1}) = 2a < V(u_t)$.
3) $h(u_t) < g(u_t) < -a$, then $u_{t+1} < -a$ and $\exists z$ such that $(u_{t+1}, z) \in G_R^{-1}$. Then

$$\begin{aligned} V(u_{t+1}) = |u_{t+1}| + |h(u_{t+1})| &< |u_{t+1}| + |z| \\ &= |u_t| + |g(u_t)| \\ &< |u_t| + |h(u_t| = V(u_t). \end{aligned}$$

The first inequality follows from condition (37), the equality follows from the definition of G_R^{-1}, and the second inequality follows from condition (36).

A parallel argument can be used if we suppose that $u_t < -a$. Hence, we have shown that $V(u_t)$ is a Lyapunov function along the trajectories of the map g outside $\mathcal{A}$ and $\mathcal{A}$ is globally attracting.

Proof of Necessity: First, we observe that the global attractivity of $\mathcal{A}$ necessarily implies that the graphs G and G^{-1} do not intersect; $G \cap G^{-1} = \emptyset$. Otherwise, g would have either fixed points or periodic orbits outside $\mathcal{A}$ and $\mathcal{A}$ could not be globally attracting. This implies in particular that G does not intersect the first bisector $\mathcal{B} = \{(x, x)\}$.

Now, assume that condition (30) is not satisfied. Since $G \cap \mathcal{B} = \emptyset$ we have $g(x) < x \; \forall x < -a$. Then, necessarily $g(-a) = -a$ in view of (34). It is then easy to check (e.g., with a simple graphical analysis) that the map g is expansive for any $u_t < -a$, with $u_{t+1} < u_t$. Obviously, if condition (31) is not satisfied, the same holds for any $u_t > a$.

Now assume that condition (32) is not satified. This implies that the symmetrical condition (33) is not satisfied as well. Since $G \cap G^{-1} = \emptyset$ we have $g(-a) = a$, $g(x) > a \; \forall x < a$, $g(a) = -a$ and $g(x) < -a \; \forall x > a$. It is easy to check that these properties imply that the map g is globally expansive for any $u_t \notin \mathcal{A}$.

We have shown that $\mathcal{A}$ cannot be globally attracting if the conditions (30)–(33) are not all satisfied. This completes the theorem.

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Measurement and Identification of Nonlinear Systems Consisting of Linear Dynamic Blocks and One Static Nonlinearity

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***Abstract*—This paper considers the measurement and the identification of nonlinear time-invariant single-input/single-output (SISO) systems, consisting of a multivariable linear dynamic system and one static nonlinear SISO system. This includes Wiener–Hammerstein systems in a linear feedback loop. The nonparametric identification of the frequency response functions of the linear parts are obtained without measuring the signals over the static nonlinearity. Measurements on an electronic circuit demonstrate the usability of this identification scheme.**

***Index Terms*—Nonlinear estimation, nonlinear systems.**

I. INTRODUCTION

The model description for lumped linear systems is relatively straightforward. Modeling nonlinear systems is less evident due to the huge amount of possible nonlinear structures (nonlinear state-space models, Volterra series [1], NARMAX models [2], the Wiener theory [1], neural networks, ...).

This paper considers nonlinear dynamic time-invariant single-input/single-output (SISO) systems which contain only one static nonlinear time-invariant SISO building block, represented in Fig. 1 nonlinear systems. It is assumed that:

- $G_{11}(\omega)$, $G_{12}(\omega)$, $G_{21}(\omega)$, and $G_{22}(\omega)$ represent linear, time-invariant SISO systems;
- $g(x)$ represents a real valued polynomial $\sum_{k=0} b_k x^k$.

It is assumed that the Volterra series expansion of this system converges. Hence, the output of the system $y_1(t)$ can be written as

$$\begin{aligned} y_1(t) = \sum_{k=1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} & g_{[k]}(t - t_1, \cdots, t - t_k) \\ & \cdot u_1(t_1) \cdots, u_1(t_k)\, dt_1 \cdots dt_k \end{aligned}$$

in the time domain where $g_{[k]}(t - t_1, \cdots, t - t_k)$ represents the kth-order Volterra kernel. The frequency domain representation is given by

$$\begin{aligned} Y_1(\omega) = \sum_{k=1} \frac{1}{(2\pi)^{k-1}} & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} U_1(\omega - \omega_1) \\ & \cdot U_1(\omega_1 - \omega_2) \cdots U_1(\omega_{k-1}) G_{[k]}(\omega - \omega_1, \\ & \omega_1 - \omega_2, \cdots, \omega_{k-1})\, d\omega_1 \cdots d\omega_{k-1} \end{aligned}$$

where $G_{[k]}(\omega_1, \cdots, \omega_k)$ represents the kth-order Volterra kernel in the frequency domain which corresponds with the kth-dimensional Fourier transform of $g_{[k]}(t_1, \cdots, t_k)$ [1]. $U_1(\omega)$ represents the Fourier transform of $u_1(t)$.

This model set includes, e.g., Wiener–Hammerstein models put in a linear feedback loop (Fig. 2). In all generality, these nonlinear

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