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Global Stabilization of Exothermic Chemical Reactors under Input Constraints*

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A generic class of exothermic chemical reactors can be globally stabilized by state feedback with input saturations at an equilibrium that is unstable in open loop conditions. The control is robust against modelling uncertainties in the dependence of the kinetics with respect to temperature.

Key Words—Exothermic chemical reactors; nonlinear temperature control; state feedback controllers; global stabilization; robustness to uncertainties; input constraints, adaptive control.

Abstract—This paper is devoted to the temperature control and the stabilization under input constraints of exothermic chemical reactors. We first consider a reactor in which a single and exothermic reaction takes place and design state feedback controllers to achieve the global and robust stabilization under input constraints of the reactor. Then, we extend these results to a general class of exothermic reactors in which multiple coupled chemical reactions can take place. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

In this paper, we deal with the temperature control under input constraints of exothermic continuous chemical reactors. The problem of feedback stabilization under input constraints has been considered for a long time in the literature and can be traced back at least until Fuller (1969). In a paper by Sontag (1984), it was shown that linear systems $\dot{x} = Ax + Bu$ with A unstable cannot be stabilized in general with bounded feedback control. When the matrix A is critically stable, conditions for feedback stabilizability with input constraints are as given in Sussmann *et al.* (1994) (see also Teel (1995) and Lin *et al.* (1996)). Related conditions for nonlinear feedforward systems can be found in Teel (1992), while the problem of saturated feedback control for stable nonlinear systems is treated in Lin (1996). It is worth noting that, in the present paper, an important contribution is to show that a generic class of reaction systems can be globally stabilized by state feedback with input constraints at a hyperbolic equilibrium which is *unstable* in open loop conditions.

For the sake of illustration of the problem we are concerned with here, let us consider a continuous reactor in which a first order and exothermic reaction $A \rightarrow B$ takes place. Such a reactor can be described by the following equations:

$$\begin{cases} \dot{x}_A = -k(T)x_A + d(x_A^{in} - x_A), \\ \dot{x}_B = k(T)x_A - dx_B, \\ \dot{T} = bk(T)x_A + d(T^{in} - T) + e(T_w - T), \end{cases} \quad (1)$$

where x_A and x_B are the concentrations in the reactor of reactant A and the product B, respectively. T is the reactor temperature. x_A^{in} is the positive and constant concentration of reactant A in the feed flow. d and e are positive constants associated with the dilution rate and the heat transfer rate, respectively. b is a positive constant standing for the exothermicity of the reaction $A \rightarrow B$. T^{in} and T_w are the manipulated variables of the feed temperature and the coolant temperature, respectively. $k(T)$ is a non-negative and bounded function of the temperature.

According to the terminology of Viel *et al.* (1997) and Bastin and Levine (1993), system (1) is rewritten as:

$$\begin{cases} \dot{x}_A = -k(T)x_A + d(x_A^{in} - x_A), \\ \dot{x}_B = k(T)x_A - dx_B, \\ \dot{T} = bk(T)x_A - qT + u, \end{cases} \quad (2)$$

with $q = d + e$, while $u = dT^{in} + eT_w$ is the control input.

The open-loop reactor (with constant control input u) may exhibit three steady states, two of

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which are asymptotically stable and one of which is unstable (Aris and Admundson, 1958). The stable low temperature steady state, denoted by $(T^{S1}, \bar{x}_A^{S1}, \bar{x}_B^{S1})$, has such a low rate of conversion that it is not very desirable for economic reasons. The operation of the reactor at the stable high temperature steady state $(T^{S2}, \bar{x}_A^{S2}, \bar{x}_B^{S2})$ can lead to practical engineering difficulties, and for this reason is often not desirable in spite of its high conversion rate. Therefore, it is of great interest to operate the reactor at the intermediate unstable steady state $(T^U, \bar{x}_A^U, \bar{x}_B^U)$ (medium temperature and conversion rate). This explains the motivation for obtaining a control scheme such that closed loop dynamics are globally asymptotically stable at the intermediate open loop unstable steady state (see e.g. Abedekun and Schork, 1991; Cibrario, 1992; Alvarez-Ramirez, 1994).

Under the linearizing feedback controller

$$u(x_A, T) = \beta^*(T^* - T) + qT - bk(T)x_A, \quad (3)$$

where $T^* > 0$ is the temperature set point and $\beta^* > 0$ is a control design parameter, it is known (see e.g. Abedekun and Schork, 1991; Viel *et al.*, 1995) that the resulting closed-loop dynamics

$$\begin{cases} \dot{x}_A = -k(T)x_A + d(x_A^{in} - x_A), \\ \dot{x}_B = k(T)x_A - dx_B, \\ \dot{T} = \beta(T^* - T), \end{cases} \quad (4)$$

are globally asymptotically stable at $(T^*, \bar{x}_A, \bar{x}_B)$, where $(\bar{x}_A, \bar{x}_B)$ is the unique equilibrium point of the globally asymptotically stable zero dynamics:

$$\begin{cases} \dot{x}_A = -k(T^*)x_A + d(x_A^{in} - x_A), \\ \dot{x}_B = k(T^*)x_A - dx_B. \end{cases} \quad (5)$$

Therefore, by setting $T^* = T^U$, this feedback control strategy allows one to operate the reactor at the open loop unstable steady state $(T^U, \bar{x}_A^U, \bar{x}_B^U)$.

The main drawback of this result lies in the fact that no input constraints are imposed on the control action, although it is obvious that $u(t) = dT^{in}(t) + eT_w(t)$ is to be physically positive and bounded from above and from below:

$$0 < u^{min} \le u(t) \le u^{max},$$

where u^{min} and u^{max} are the positive constraints on the input u. The most direct way of handling the input constraints is to saturate the feedback controller:

$$u(x_A, T) = \underset{[u^{min}, u^{max}]}{\text{sat}} \{\beta^*(T^* - T) + qT - bk(T)x_A\}. \quad (6)$$

However, as will be illustrated later, input saturations can impair the nominal stabilizing property and lead to an unexpected and undesirable closed loop behaviour, as has been noted and discussed in the paper by Alvarez *et al.* (1991). For instance, we will show with a simulation example in Section 2.1 that, when saturated, the above feedback controller is no longer capable of cooling the exothermic reactor sufficiently to avoid stabilization at an undesired extraneous equilibrium point. From this fact, we are led to consider the concentration v of reactant A in the feed as an additional input. (Indeed, decreasing the quantity of reactant A in the feed is another way of cooling the exothermic reactor). We will show in Section 2.2 how to design and combine two feedback controllers $u(x_A, T)$ and $v(T)$ in order to solve the stabilization problem under input constraints (and, as a result, how to operate the reactor at the desirable open loop unstable steady state $(T^U, \bar{x}_A^U, \bar{x}_B^U)$). In Section 2.3, we present an extension of this feedback controller that achieves global stabilization of the reactor and is robust to large uncertainties on the dependence of the kinetics with respect to temperature. Finally, Section 3 is devoted to the extension of these results to a more general class of exothermic reactors.

2. CASE STUDY: THE SINGLE REACTION CASE

Let us first introduce two assumptions regarding the reactor system (2). These assumptions will be used throughout this section for control purposes.

Assumptions.

*h*1: the function $k(T)$ is non-negative, bounded and $k(0) = 0$.

*h*2: the input constraints u^{min} and u^{max} and the temperature set point T^* are such that the following inequality holds:

$$\forall x_A \in [0, x_A^{in}],$$
$$u^{max} > qT^* - bk(T^*)x_A > u^{min} > 0.$$

By Assumption h2, there exists a temperature interval $[T_1, T_2]$ such that $T_1 < T^* < T_2$ and the inequality $u^{max} > qT - bk(T)x_A > u^{min}$ is satisfied for all $(T, x_A) \in [T_1, T_2] \times [0, x_A^{in}]$. This assumption can be regarded as a kind of feasibility condition on the open loop system. Indeed, it implies that the static input corresponding to the equilibrium point $(T^*, \bar{x}_A, \bar{x}_B)$ belongs to the interval of input constraints. However, it is even much stronger than that: it might hold only for a very large range of the manipulated input. However, we shall see in Section 2.1 that the closed loop behaviour with the controller (6) can be unacceptable, even though the interval of constraints is large.

Consider the reactor system (2) under the saturated feedback control law (6). A first stability result is given in the following theorem:

Theorem 2.1. Under Assumptions h1 and h2, the dynamics of the controlled reactor (2)–(6) are such that:

(i) The domain $\Omega \times]0, T_2]$ with $\Omega = \{x_A \geq 0, x_B \geq 0, x_A + x_B \leq x_A^{in}\}$ is positively invariant.

(ii) The equilibrium point $(T^*, \bar{x}_A, \bar{x}_B)$ is asymptotically stable (relatively to the domain $\Omega \times]0, T_2]$) for sufficiently large β^*. □

Proof.

(i) We have $\dot{x}_A(x_A = 0) \geq 0$ and $\dot{x}_B(x_B = 0) \geq 0$. Hence, the concentrations remain non-negative provided that $x_A(t = 0) \geq 0$ and $x_B(t = 0) \geq 0$. Defining $Z = x_A + x_B$, we have $\dot{Z} = -d(Z - x_A^{in})$. Hence, $Z(t) \leq x_A^{in}$ for $Z(t = 0) \leq x_A^{in}$ and the compact set $\Omega = \{x_A \geq 0, x_B \geq 0, Z = x_A + x_B \leq x_A^{in}\}$. is positively invariant for the closed loop dynamics. Moreover, we have $\dot{T}(T = 0) \geq u^{min} > 0$. By Assumption h2, we obtain $u(x_A, T_2) = \max(u^{min}, \beta^*(T^* - T_2) + qT_2 - bk(T_2)x_A) \leq u^{max}$ and $\dot{T}(T_2) = \beta^*(T^* - T_2) < 0$ or $\dot{T}(T_2) = bk(T_2)x_A - qT_2 + u^{min} < 0$. Hence, the domain $\Omega \times]0, T_2]$ is positively invariant.

(ii) We will prove at first that the reactor temperature T is globally asymptotically stable at the set point T^* (relative to $]0, T_2]$). Assumption h2 and a continuity argument imply that there exist two temperatures $T'_1 = T'_1(\beta^*)$ and $T'_2 = T'_2(\beta^*)$ such that $T_1 < T'_1 < T^* < T'_2 < T_2$ and $u(x_A, T) = \beta^*(T^* - T) + qT - bk(T)x_A \in [u^{min}, u^{max}]$ on $[0, x_A^{in}] \times [T'_1, T'_2]$. The technique of proof that will be used is to show that the temperature trajectory is trapped in finite time within the positively invariant interval $[T'_1, T'_2]$.

Consider the case $0 < T(0) < T^*$. For any $T \in]0, T_1[$, we have

$$u(x_A, T) = \min(u^{max}, \beta^*(T^* - T) + qT - bk(T)x_A) \geq u^{min}$$

by choosing β^* large enough. So, for any $T \in]0, T_1[$ we obtain $T(T) = \beta^*(T^* - T) > 0$ or $\dot{T}(T) = bk(T)x_A - qT + u^{max} > -qT^* + u^{max} > 0$ by Assumption h2. For any $T \in [T_1, T'_1]$ we have

$$u(x_A, T) = \min(u^{max}, \beta^*(T^* - T) + qT - bk(T)x_A) \geq u^{min}$$

and $\dot{T}(T) = \beta^*(T^* - T) > 0$, or $\dot{T}(T) = bk(T)x_A - qT + u^{max} > -qT^* + u^{max} > 0$ by Assumption h2.

Consider the case $T^* < T(0) \leq T_2$. For any $T \in [T'_2, T_2]$, we obtain

$$u(x_A, T) = \max(u^{min}, \beta^*(T^* - T) + qT - bk(T)x_A) \leq u^{max}$$

and $\dot{T}(T) = \beta^*(T^* - T) < 0$ or $\dot{T}(T) = bk(T)x_A - qT + u^{min} < 0$.

Hence, the temperature trajectory is trapped in finite time within the positively invariant interval $[T'_1, T'_2]$. The temperature dynamics on $[T'_1, T'_2]$ are given by the asymptotically stable linear system $\dot{T} = \beta^*(T^* - T)$. So, we have proved that the reactor temperature T is globally asymptotically stable at the set point T^* (relative to $]0, T_2]$).

It remains now to show that the closed loop dynamics are globally asymptotically stable at $(T^*, \bar{x}_A, \bar{x}_B)$ where $(\bar{x}_A, \bar{x}_B)$ is the unique equilibrium point of the system (5). The isothermal dynamics:

$$\begin{cases} \dot{x}_A = -k(T^*)x_A + d(x_A^{in} - x_A), \\ \dot{x}_B = k(T^*)x_A - dx_B, \end{cases} \tag{7}$$

are globally asymptotically stable at $(\bar{x}_A, \bar{x}_B)$ (relative to Ω). By making use of Theorem A.1 in Appendix A in Viel *et al.* (1995), we conclude to the global asymptotic stability of the closed loop dynamics at $(T^*, \bar{x}_A, \bar{x}_B)$. ■

Theorem 2.1 states that, even if input saturations occur, the controlled reactor (2)–(6) can be stabilized on the temperature set point T^* provided that the initial conditions belong to the domain $\Omega \times]0, T_2]$. But this is not a global stability result, since the whole temperature range $]0, +\infty[$ is not covered.

2.1. *An example of undesired closed loop behaviour*

We shall now show with a simulation example that undesired closed loop behaviour may occur when the initial temperature condition belongs to $]T_2, +\infty[$. For this purpose, we consider the numerical values: $k(T) = k_0 \exp^{-k_1 T}$ (Arrhenius law), $k_0 = 7.2e + 10\ \mathrm{min}^{-1}$, $k_1 = 8700$ K, $d = 1.1\ \mathrm{min}^{-1}$, $x_A^{in} = 1$ mol/l, $b = 209.2$ K L/mol, $q = 1.25\ \mathrm{min}^{-1}$ and $u = 355$ K/min.

The three open loop steady states are shown in Fig. 1 (they correspond to the intersection between the rate of heat generation and the total rate of heat removal). We assume that the system is initially in open loop at the stable high temperature steady state $(T^{S2}, \bar{x}_A^{S2}, \bar{x}_B^{S2}) = (467.8, 0.002, 1.1)$ with a constant input $u = 355$ K/min.

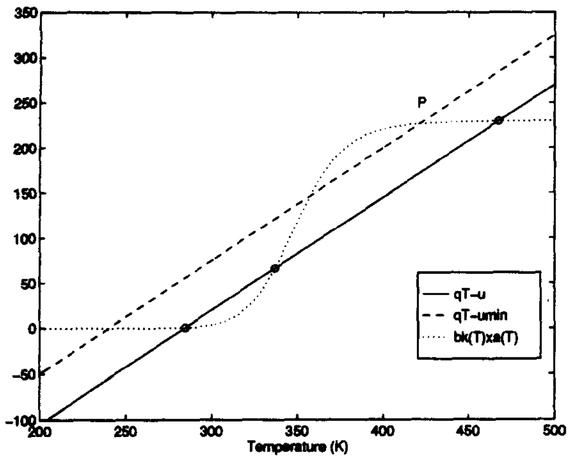

Fig. 1. The open loop steady states.

From time $t = 0$, the feedback control objective is to drive the system to the open loop unstable steady state $(T^{U}, \bar{x}_{A}^{U}, \bar{x}_{B}^{U}) = (337.1, 0.711, 0.29)$ and to stabilize the reactor at this equilibrium state.

For this purpose, we use the feedback control law (6) with $u^{\min} = 300$, $u^{\max} = 500$, $\beta^* = 5$ and the set point T^* set to the desired temperature $T^{U} = 337.1$. One can easily check that Assumptions h1 and h2 hold and that $T_2 = 341$.

As shown in Fig. 2, the control input u saturates at the lower bound $u^{\min}$ and the reactor is driven to the undesired extraneous equilibrium point P (see Fig. 1). This equilibrium point P corresponds to the high temperature open loop steady state when the input u is equal to $u^{\min}$. This behaviour can be easily explained by noting that, on the one hand, we have:

$$\begin{aligned} &\forall T \geq 370, \qquad \forall x_A \geq 0, \\ &\beta^*(T^* - T) + qT - bk(T)x_A \leq u^{\min}, \end{aligned} \tag{8}$$

and, on the other hand, the steady state $(T^{S2}, \bar{x}_{A}^{S2}, \bar{x}_{B}^{S2}) = (467.8, 0.002, 1.1)$ belongs to the basin of attraction of P. *The saturated feedback controller is not capable of cooling the exothermic reactor sufficiently.* One can easily check that the same scenario would occur for any β^* such that $\beta^* \geq 5$.

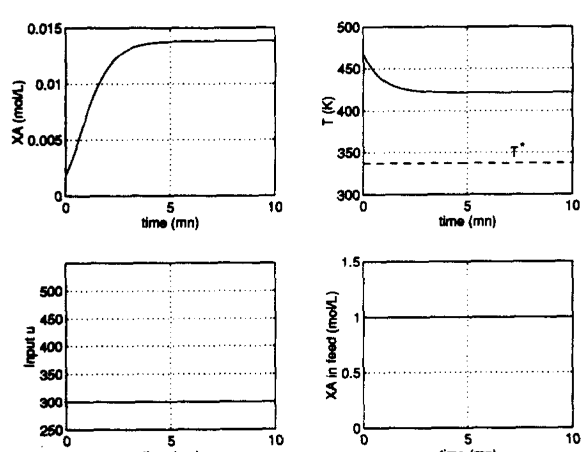

Fig. 2. An undesired closed loop behaviour.

2.2. *Global stabilization with input constraints*

From the simulation presented above, it is clear that we have to find another way of cooling the reactor in order to stabilize the system at the intermediate open loop unstable steady state. One obvious possibility is to decrease the concentration v of the reactant A in the feed, which should reduce the velocity of the exothermic reaction and hence have a cooling effect on the system. We are therefore led to consider the concentration v of reactant A in the feed as an additional input. The two-input reactor model we now consider is given by:

$$\begin{cases} \dot{x}_A = -k(T)x_A + d(v - x_A), \\ \dot{x}_B = k(T)x_A - dx_B, \\ \dot{T} = bk(T)x_A - qT + u, \end{cases} \tag{9}$$

where u and v are the manipulated heat and the manipulated concentration of reactant A in the feed, respectively.

Let β^* be such that $\beta^* > (qT_2 - u^{\min})/(T_2 - T^*)$. Consider the two feedback controllers $v(T)$ and $u(x_A, T)$, which are defined as follows:

$$\begin{aligned} v(T) &= x_A^{\text{in}}, \qquad \forall T \in]0; T_2[, \\ &= 0, \qquad \forall T \in [T_2, +\infty[, \end{aligned} \tag{10}$$

and

$$\begin{aligned} u(x_A, T) = \underset{[u^{\min}, u^{\max}]}{\text{sat}} \{&\beta(T)(T^* - T) \\ &+ qT - bk(T)x_A\}, \end{aligned} \tag{11}$$

with

$$\begin{aligned} \beta(T) &= \beta^*, && \forall T \in]0, T^*] \\ &= \min\left(\frac{qT - u^{\min}}{T - T^*}, \beta^*\right), && \forall T \in]T^*, T_2] \\ &= \frac{qT - u^{\min}}{T - T^*}, && \forall T \in [T_2, +\infty[. \end{aligned} \tag{12}$$

The role played by the feedback controller $v(T)$ is to set the concentration of reactant A in the feed to zero when the temperature of the reactor is high. The feedback controller $u(x_A, T)$ described by (11) and (12) is a modified version of the control law (6) with an adaptive gain $\beta(T)$. The next theorem shows that the feedback controllers (10)–(12) make the reactor system (9) *globally* asymptotically stable at the equilibrium point $(T^*, \bar{x}_A, \bar{x}_B)$.

Theorem 2.2. Under Assumptions h1 and h2, the dynamics of the controlled reactor (9)–(12) are such that:

(i) The domain $\Omega \times]0, +\infty[$ with $\Omega = \{x_A \geq 0, x_B \geq 0, x_A + x_B \leq x_A^{in}\}$ is positively invariant.

(ii) The equilibrium point $(T^*, \bar{x}_A, \bar{x}_B)$ is asymptotically stable (relatively to the domain $\Omega \times]0, +\infty[$) for β^* large enough.

Proof.

(i) The positive invariance of Ω can be proved as in item (i) of the proof of Theorem 2.1 by noting that $0 \geq v(T) \leq x_A^{in}$, while the positive invariance of the temperature interval $]0, +\infty[$ results from the fact that $\dot{T}(T = 0) \geq u^{min} > 0$.

(ii) Let us prove that the temperature trajectory is trapped in finite time within the positively invariant interval $[T_1', T_2']$ where T_1', T_2' are defined as in the proof of Theorem 2.1.

Consider the case $0 < T(0) < T^*$. Since $\beta(T) = \beta^*$, we are in the same situation as in the proof of Theorem 2.1.

Consider the case $T^* < T(0)$. For any $T \in [T_2', T_2]$, we obtain $u(x_A, T) = \max(u^{min}, \beta(T)(T^* - T) + qT - bk(T)x_A) \leq u^{max}$ and $\dot{T}(T) = \beta(T)(T^* - T) < 0$ or $\dot{T}(T) = bk(T)x_A - qT + u^{min} < 0$. For any $T \in]T_2, +\infty[$, we have $v(T) = 0$, $u(x_A, T) = u^{min}$ and $\dot{T}(T) < bk(T)x_A - qT_2 + u^{min} < bk(T)x_A - \varepsilon$ with $\varepsilon > 0$. Using the boundedness of $k(T)$ and the fact that x_A decreases towards 0 when $v(T) = 0$, we will have in finite time $\dot{T}(T) < 0$ for any $T > T_2$.

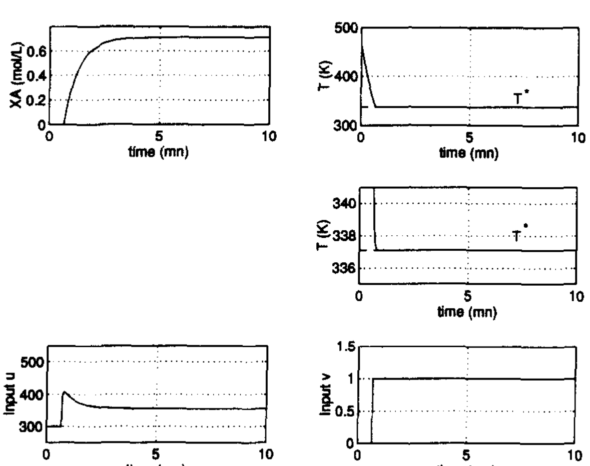

Fig. 3. Global stabilization with input constraints.

Hence, the temperature trajectory is trapped in finite time within the positively invariant interval $[T_1', T_2']$. The rest of the proof is similar to that of Theorem 2.1 (by noting that $v(T^*) = x_A^{in}$). □

Note that the control law (10) is discontinuous. In fact, Theorem 2.2 still holds when using any smooth feedback control $v(T)$ such that $v(T) \in [0, x_A^{in}]$, $v(T^*) = x_A^{in}$ and $v(T) = 0$ for $T \in [T_2, +\infty]$.

To illustrate Theorem 2.2, we show in simulation how our previous control objective (Section 2.1) can now be achieved: to drive the reactor from the open loop stable high temperature steady state $(T^{S2}, \bar{x}_A^{S2}, \bar{x}_B^{S2}) = (467.8, 0.002, 1.1)$ to the intermediate open loop unstable steady state $(T^U, \bar{x}_A^U, \bar{x}_B^U) = (337.1, 0.711, 0.29)$ with input constraints $u^{min} = 300$ and $u^{max} = 500$, set point $T^* = 337.1$ and parameter $\beta^* = 33 > (qT_2 - u^{min})/(T_2 - T^*)$. The simulation results are shown in Fig. 3. The reactor, as predicted by Theorem 2.2, is indeed driven to the intermediate open loop unstable steady state under input constraints.

2.3. *Robust global stabilization with input constraints*

Let us now consider the same control problem but assuming that the function $k(T)$ involved in the kinetics and the positive constant b are unknown. In other words, we consider a problem of robust global stabilization of the reactor under input constraints. This problem is mainly motivated by the fact that, in many instances, the function $k(T)$ may exhibit some uncertainty and deviates from the theoretical Arrhenius model.

This problem of robust global stabilization is solved in Viel *et al.* (1997) by using the heating rate u alone as a control action, without input constraints. The control design is based on input–output linearization, with an appropriate nonlinear dynamic extension.

We show hereafter that the same dynamic extension can be combined with the controller (10)–(12) to solve this problem of robust global stabilization under input constraints, provided the following additional assumption is satisfied:

*Assumption h*3: The function $k(T)$ is globally Lipschitz on $]0, +\infty[$. □

This is a very mild assumption which is satisfied by most plausible models of exothermicity, and in particular by the Arrhenius law.

A dynamic feedback controller $u(x_A, T)$ is defined as follows:

$$u(x_A, T) = \underset{[u^{\min}, u^{\max}]}{\mathrm{sat}} \left\{ \beta^*(T^* - T) + qT - f(T)\frac{\alpha\theta}{\alpha+\theta}x_A \right\}, \tag{13}$$
$$\dot{\theta} = (T - T^*)\theta x_A,$$

with

$$\begin{aligned} f(T) &= 1, && \forall T \in]0, T^*], \\ &= \frac{\beta^*(T^* - T) + qT - u^{\min}}{qT^* - u^{\min}}, && \forall T \in [T^*, T_l], \\ &= 0, && \forall T \in [T_l, +\infty[, \end{aligned} \tag{14}$$

and the temperature $T_l > T^*$ is such that

$$\beta^*(T^* - T_l) + qT_l - u^{\min} = 0.$$

The feedback controller $v(T)$ is defined as before:

$$\begin{aligned} v(T) &= x_A^{\mathrm{in}}, && \forall T \in]0; T_l[, \\ &= 0, && \forall T \in [T_l, +\infty[. \end{aligned} \tag{15}$$

Then, we have the following result.

Theorem 2.3. Under Assumptions h1, h2 and h3, the dynamics of the controlled reactor, (9), (13)–(15), are such that:

(i) The domain $\Omega \times]0, +\infty[\times]0, +\infty[$, with $\Omega = \{x_A \geq 0,\ x_B \geq 0,\ x_A + x_B \leq x_A^{\mathrm{in}}\}$, is positively invariant.
(ii) The equilibrium point $(T^*, \bar{x}_A, \bar{x}_B, \bar{\theta})$, where $\bar{\theta}$ is defined by $\alpha\bar{\theta}/(\alpha + \bar{\theta}) = bk(T^*)$, is asymptotically stable (relative to the domain $\Omega \times]0, +\infty[\times]0, +\infty[$) for $\beta^* > q$ large enough and for $\alpha > bk(T^*)$ such that $qT^* - \alpha x_A^{\mathrm{in}} > u^{\min}$ holds. □

Before proving Theorem 2.3, let us remark that, by means of Assumption h2, there really exists some $\alpha > bk(T^*)$ for which $qT^* - \alpha x_A^{\mathrm{in}} > u^{\min}$. Indeed, Assumption h2 implies that $qT^* - bk(T^*)x_A^{\mathrm{in}} > u^{\min}$. Hence, by continuity, there exists some $\alpha > bk(T^*)$ such that $qT^* - \alpha x_A^{\mathrm{in}} > u^{\min}$ holds.

Proof of Theorem 2.3.

(i) The positive invariance of $\Omega \times]0, +\infty[$ can be proved as in item (i) of the proof of Theorem 2.2, while the positive invariance of the interval $]0, +\infty[$ in θ results from the fact that $\dot{\theta}(\theta = 0) = 0$ (see Theorem 1.7, Chapter II in Bhatia and Szegö (1970)).
(ii) For the sake of clarity, the proof is organized in three successive steps.

Step 1: let β^* be large enough such that $T^* < T_l < T_2$. We show that the temperature trajectory is trapped in finite time within the positively invariant interval $[T_1, T_l]$, where $T_1 < T^*$ can be chosen arbitrarily close to T^*.

$\forall T \leq T_1$, we have

$$u(x_A, T) = \min\left(u^{\max}, \beta^*(T^* - T) + qT - \frac{\alpha\theta}{\alpha+\theta}x_A\right) \geq u^{\min}$$

by choosing β^* large enough. Hence, $\forall T \leq T_1$, we obtain

$$\dot{T}(T) = bk(T)x_A - qT + u^{\max} > -qT^* + u^{\max} > 0$$

by Assumption h2 or

$$\dot{T}(T) = \beta^*(T^* - T) + bk(T)x_A - \frac{\alpha\theta}{\alpha+\theta}x_A > 0$$

for β^* large enough.

$\forall T \geq T_l$, we have $u(x_A, T) = u^{\min}$ and $v(T) = 0$. $\forall T_l \leq T \leq T_2$, we obtain $\dot{T}(T) = bk(T)x_A - qT + u^{\min} < 0$ by means of Assumption h2. $\forall T > T_2$, we have in finite time $\dot{T}(T) < 0$, by using the same argument as in the proof of Theorem 2.2.

Step 2: let us prove now that $\forall T \in [T_1, T^*]$ we have

$$\beta^*(T^* - T) + qT - \frac{\alpha\theta}{\alpha+\theta}x_A \geq u^{\min},$$

$\forall T \in [T^*, T_l]$ we have

$$\beta^*(T^* - T) + qT - f(T)\frac{\alpha\theta}{\alpha+\theta}x_A \in [u^{\min}, u^{\max}],$$

$\forall T_1 \leq T \leq T^*$ we have

$$\beta^*(T^* - T) + qT - \frac{\alpha\theta}{\alpha+\theta}x_A \geq qT_1 - \alpha x_A^{\mathrm{in}} \geq u^{\min}$$

for T_1 close to T^* (since we have $qT^* - \alpha x_A^{\mathrm{in}} > u^{\min}$).

$\forall T \in [T^*, T_l]$, the reader can check that

$$\beta^*(T^* - T) + qT - f(T)\frac{\alpha\theta}{\alpha+\theta}x_A \geq u^{\min}$$

by using the definition of $f(T)$ and the fact that $qT^* - u^{\min} > \alpha x_A^{\text{in}}$. $\forall T \in [T^*, T_l]$, we also have

$$\beta^*(T^* - T) + qT - f(T)\frac{\alpha\theta}{\alpha + \theta}x_A \le u^{\max},$$

since $qT \le u^{\max}$ on $[T^*, T_l]$.

Step 3: we show that

$$V = \tfrac{1}{2}(T^* - T)^2 + \alpha \ln(\alpha + \theta) - bk(T^*)\ln\theta$$

is a Liapunov function on the domain $\Omega \times [T_1, T_l] \times]0, +\infty[$. (This function V was considered by Viel *et al.* (1997).)

Consider the domain $[T_1, T^*]$. We have

$$u(x_A, T) = \min\left(u^{\max}, \beta^*(T^* - T) + qT - \frac{\alpha\theta}{\alpha + \theta}x_A\right).$$

When $u(x_A, T) = u^{\max}$, using Assumption h3, we obtain for some $K_0 > 0$ (K_0 is independent of β^*):

$$\dot{V} \le K_0(T - T^*)^2 + (u^{\max} - qT)(T - T^*).$$

Hence, by Assumption h2 ($u^{\max} - qT > 0$), and for T_1 sufficiently close to T^* we have $\dot{V} \le 0$. When

$$u(x_A, T) = \beta^*(T^* - T) + qT - \frac{\alpha\theta}{\alpha + \theta}x_A,$$

using Assumption h3, we have for some $K_1 > 0$ (K_1 is independent of β^*):

$$\dot{V} \le (-\beta^* + K_1)(T^* - T)^2.$$

Hence, for β^* large enough, we have $\dot{V} \le 0$.

Consider the domain $[T^*, T_l]$. We obtain:

$$\dot{V} \le (-\beta^* + K_1)(T^* - T)^2 + \beta^* \frac{\alpha x_A^{\text{in}}}{qT^* - u^{\min}}(T^* - T)^2.$$

Let $K_2 > 0$ be defined by $K_2 = \alpha x_A^{\text{in}}/(qT^* - u^{\min})$. Hence, we have

$$\dot{V} \le (\beta^*(K_2 - 1) + K_1)(T^* - T)^2.$$

Since $qT^* \times \alpha x_A^{\text{in}} > u^{\min}$, we have $K_2 < 1$ and for β^* large enough we obtain $\dot{V} \le 0$.

(iii) The rest of the proof is similar to that of Theorem 4.1 (see also Corollary 4.1) in Viel *et al.* (1997) (use of LaSalle's invariance principle and some ω-limit arguments.) □

To illustrate Theorem 2.3, let us consider the same scenario as before, i.e. the stabilization of the open loop unstable steady state $(T^U, \bar{x}_A^U, \bar{x}_B^U) = (337.1, 0.711, 0.29)$ from the open loop stable steady state $(T^{S2}, \bar{x}_A^{S2}, \bar{x}_B^{S2}) = (467.8, 0.002, 1.1)$ with respect to the constraints $u^{\min} = 300$ and $u^{\max} = 500$. We set $T^* = 337.1$ and we choose $\alpha = 120$ (the inequality $qT^* - \alpha x_A^{\text{in}} > u^{\min}$ is then satisfied) and $\beta^* = 33$. (The reader can check a posteriori that $\alpha > bk(T^*)$.) We have $T_l = 340.9$. The simulation results are shown in Fig. 4. The reactor stabilizes the open loop unstable steady state with input constraints, in spite of the unknown kinetics.

3. GENERALIZATION TO THE CASE OF MULTIPLE REACTIONS

Our purpose in this section is to extend our previous results to a general class of exothermic continuous stirred tank reactors (CSTRs).

3.1. *System description*

We consider CSTRs in which m exothermic reactions take place involving n ($n > m$) chemical species and described by:

$$\begin{cases} \dot{x} = Cr(x, T) + d(x^{\text{in}} - x), \\ \dot{T} = B(x, T) - qT + u. \end{cases} \tag{16}$$

In this system:

- x is the vector of the concentrations of the involved chemical species, $\dim x = n$.
- x^{in} is the vector of non-negative and constant feed concentrations, $\dim x^{\text{in}} = n$.
- T is the reactor temperature.
- $r(x, T)$ is the vector of reaction kinetics with $\dim r = m$ and $r^T(x, T) = (r_1(x, T), r_2(x, T), \ldots, r_m(x, T))$. (Here and throughout, y^T stands for the transpose vector of y.) Moreover, we have

$$r_i(x, T) = k_i(T)\varphi_i(x), \tag{17}$$

where $k_i(T)$ is a positive and bounded functions of the temperature (for instance, the Arrhenius law) and $\varphi_i(x)$ is a non-negative

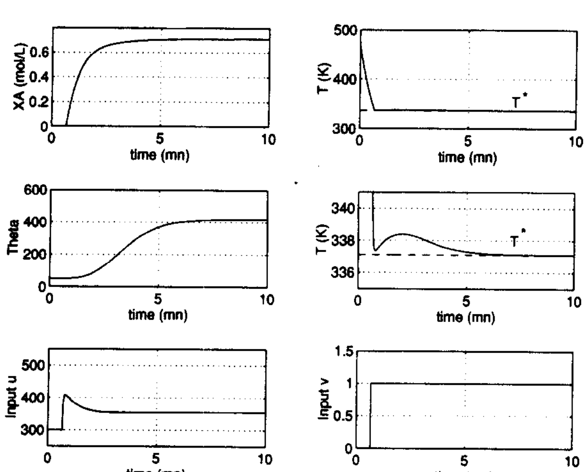

Fig. 4. Robust global stabilization with input constraints.

function of the concentrations that vanishes if $x_j = 0$ for some reactant j involved in the ith reaction.

- C is the stoichiometric matrix, $\dim C = n \times m$.
- $B(x, T)$ is the non-negative reaction heat. We have
$$B(x, T) = \sum_i b_i r_i(x, T), \tag{18}$$
where the coefficients b_i are positive constants.
- d is the positive and constant dilution rate.
- q is a positive and constant heat transfer coefficient.
- u is the input, i.e. the manipulated heat.

Such a formalism for the description of (bio)-chemical reactors has been used previously in the literature (see e.g. Bastin and Dochain, 1990; Dochain *et al.*, 1992; Bastin and Levine, 1993).

Example. Consider a CSTR in which the exothermic reactions $A \rightarrow B$ and $B + 2C \rightarrow D$ take place. The reactor is fed with the reactants A and C. The dynamics of the process are described by the model (16), with the following definitions;

$$x = (x_A, x_B, x_C, x_D)^T,$$
$$x^{in} = x_A^{in}, 0, x_C^{in}, 0)^T,$$
$$C = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -2 \\ 0 & 1 \end{pmatrix},$$
$$r(x, T) = (r_1(x, T), r_2(x, T))^T,$$
$$B(x, T) = b_1 r_1(x, T) + b_2 r_2(x, T).$$

Moreover, if we assume that the first reaction $A \rightarrow B$ is of first order with respect to A and the second reaction $B + 2C \rightarrow D$ is of first order with respect to B and of second order with respect to C, we have:

$$r_1(x, T) = k_1(T)\varphi_1(x) = k_1(T)x_A,$$
$$r_2(x, T) = k_2(T)\varphi_2(x) = k_2(T)x_B x_C^2. \quad \square$$

Let us introduce the following assumption.
Assumption H0 (principle of mass conservation). There exists a positive vector, $w = (w_1, w_2, \ldots, w_n)^T$, $w_j > 0$, $j = 1, \ldots, n$ such that $w^T C = 0$. □

This assumption implies that the reaction system is mass-conserving; in other words, what is produced in the reaction system cannot be larger than what is consumed. It also enables us to state a useful result on the boundedness of the concentrations in a chemical reactor described by the model (16).

Lemma 3.1. Uniform boundedness. Under Assumption H0, the concentrations $x_j(t)$ remain non-negative for all t if $x_j(0) \geq 0$ and, furthermore, admit as a positively invariant domain the compact set $\Omega = \{x \in \mathscr{R}^n : \forall j, \ x_j \geq 0, \ w^T x \leq w^T x^{in}\}$. Therefore, the concentrations are uniformly bounded with respect to the temperature trajectory. □

The proof of this lemma is given in Viel *et al.* (1997). As a consequence of Lemma 3.1, throughout the rest of the paper the vector of concentrations x will be restricted to the bounded set Ω.

Example (*continued*). Assumption H0 is satisfied with $w = (1, 1, 1, 3)^T$. Therefore, by Lemma 3.1, the set $\Omega = \{x_A \geq 0, x_B \geq 0, x_C \geq 0, x_D \geq 0, w^T x \leq w^T x^{in}\}$ is a positively invariant domain for the chemical reactor under consideration. □

Let us now consider the saturated feedback controller

$$u(x, T) = \underset{[u^{min}, u^{max}]}{\text{sat}} \{\beta^*(T^* - T) + qT - B(x, T)\}, \tag{19}$$

where u^{min} and u^{max} are the positive constraints on the input u. The problem we are going to focus on is the state feedback stabilization with input constraints of the controlled reactor (16)–(19) at the temperature set point T^*. This problem will be studied under the following assumptions.

Assumptions.
*H*1. The functions $k_i(T)$ are non-negative, bounded and $k_i(0) = 0$.
*H*2. The input constraints u^{min} and u^{max} and the temperature set point T^* are such that the following inequality holds:
$$\forall x \in \Omega,$$
$$u^{max} > qT^* - B(x, T^*) > u^{min} > 0.$$
*H*3. The isothermal dynamics $\dot{x} = Cr(x, T^*) + d(x^{in} - x)$ are asymptotically stable (relative to the set Ω) at the single equilibrium point $\bar{x} \in \Omega$.

Assumption H2 implies that there exists a set of temperatures $[T_1, T_2]$ such that $T_1 < T^* < T_2$ and the inequality $u^{max} > qT - B(x, T) > u^{min}$ is satisfied for all $(T, x) \in [T_1, T_2] \times \Omega$. Let us also point out that, in spite of the global asymptotic stability of the isothermal dynamics which coincide with the zero-dynamics (Assumption H3), the overall dynamics of the chemical reactor can be open loop unstable. The reader can refer to Feinberg (1987) and Rouchon (1992), who give some sufficient conditions on the kinetic scheme, so that the minimum-phase Assumption H3 holds.

Theorem 3.1. Under Assumptions H0, H1 and H2, the dynamics of the controlled reactor (16)–(19) are such that:

(i) The domain $\Omega \times]0, T_2]$ is positively invariant.

(ii) The reactor temperature T converges asymptotically to the temperature set point T^* ($\forall (x(0), T(0)) \in \Omega \times]0, T_2]$) for β^* large enough.

If, in addition, Assumption H3 holds, then we have:

(iii) The equilibrium point $(\bar{x}, T^*)$ is asymptotically stable (relative to the domain $\Omega \times]0, T_2]$) for β^* large enough. □

Proof.

(i) The positive invariance of Ω results from Lemma 3.1, while the positive invariance of the temperature interval $]0, T_2]$ can be proved as in item (i) of the proof of Theorem 2.1 by replacing $bk(T)x_A$ by $B(x, T)$.

(ii) The convergence of T can be proved as in item (ii) of the proof of Theorem 2.1 by replacing $bk(T)x_A$ by $B(x, T)$.

(iii) The result follows from Theorem A.1 of Appendix A given in Viel *et al.* (1997), using Lemma 3.1. □

Example (*continued*). It can be shown that Assumption H3 holds. Then, under Assumptions H1 and H2, the feedback controller

$$u(x, T) = \operatorname*{sat}_{[u^{\min}, u^{\max}]} \{\beta^*(T^* - T) + qT - b_1 k_1(T) x_A^2 - b_1 k_2(T) x_B x_C^2\}$$

is such that the controlled reactor is asymptotically stable at the equilibrium point $(\bar{x}, T^*)$ relative to the positively invariant domain $\Omega \times]0, T_2]$ for β^* large enough. Moreover, $\bar{x} = (\bar{x}_A, \bar{x}_B, \bar{x}_C, \bar{x}_D) \in \Omega$ is the unique solution of

$$\begin{cases} 0 = -k_1(T^*)\bar{x}_A + d(x_A^{in} - \bar{x}_A), \\ 0 = k_1(T^*)\bar{x}_A - k_2(T^*)\bar{x}_B \bar{x}_C^2 - d\bar{x}_B, \\ 0 = -2k_2(T^*)\bar{x}_B \bar{x}_C^2 + d(x_C^{in} - \bar{x}_C), \\ 0 = k_2(T^*)\bar{x}_B \bar{x}_C^2 - d\bar{x}_D. \end{cases}$$ □

Theorem 3.1 is an extension of Theorem 2.1 to a general class of exothermic CSTRs. We have shown in our case study (Section 2) that an undesired extraneous equilibrium point occurs when the initial temperature does not belong to the stability domain $]0, T_2]$. For more complex reactors, it is likely that other phenomena, such as unstable limit cycles, may occur. It is therefore of interest to devise similar generalizations of Theorems 2.2 and 2.3 to the class of chemical reactors under consideration.

3.2. *Global stabilization with input constraints*

From now on, we consider the multi-input reactor system

$$\begin{cases} \dot{x}_1 = C_1 r(x, T) + f(v - x_1), \\ \dot{x}_2 = C_2 r(x, T) + d(x_2^{in} - x_2), \\ \dot{T} = B(x, T) - qT + u, \end{cases} \tag{20}$$

where the state x has been split into two substates x_1 and x_2, and the matrices C_1 and C_2 are defined accordingly. The substate x_1 is a set of chemical reactants such that $r(x_1 = 0, x_2 = x_2^{in}, T) = 0$. The input vector v stands for the concentrations of the reactants x_1 fed into the reactor.

Example (*continued*). Here, we have $x_1 = x_A$ and $x_2 = (x_B, x_C, x_D)$ and

$$C_1 = (-1 \quad 0),$$

$$C_2 = \begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$x_2^{in} = (0, x_C^{in}, 0)^T.$$

v is the manipulated concentration of reactant A fed into the reactor. □

Let β^* be such that $\beta^* > (qT_2 - u^{\min})/(T_2 - T^*)$. Consider the feedback controllers $v(T)$ and $u(x, T)$, which are defined as follows:

$$\begin{aligned} v(T) &= x_1^{in}, && \forall T \in]0; T_2[\\ &= 0, && \forall T \in [T_2, +\infty[, \end{aligned} \tag{21}$$

and

$$u(x, T) = \operatorname*{sat}_{[u^{\min}, u^{\max}]} \{\beta(T) \cdot (T^* - T) + qT - B(x, T)\}, \tag{22}$$

with

$$\begin{aligned} \beta(T) &= \beta^*, && \forall T \in]0, T^*], \\ &= \min\left(\frac{qT - u^{\min}}{T - T^*}, \beta^*\right), && \forall T \in]T^*, T_2], \\ &= \frac{qT - u^{\min}}{T - T^*}, && \forall T \in [T_2, +\infty[. \end{aligned} \tag{23}$$

Then, we have the following result.

Theorem 3.2. Under Assumptions H0, H1 and H2, the dynamics of the controlled reactor (20)–(23) are such that:

(i) The domain $\Omega \times]0,+\infty[$ is positively invariant.

(ii) The reactor temperature T converges asymptotically to the temperature set point $T^*(\forall(x(0), T(0)) \in \Omega \times]0,+\infty[)$ for β^* large enough.

If, in addition, Assumption H3 holds, then we have:

(iii) The equilibrium point $(\bar{x}, T^*)$ is asymptotically stable (relative to the domain $\Omega \times [0,+\infty[$) for β^* large enough. □

Proof.

(i) The proof of the positive invariance of $\Omega \times]0,+\infty[$ is as in item (i) of that of Theorem 2.2.

(ii) The convergence of T can be proved as in item (ii) of the proof of Theorem 2.2, by replacing $bk(T)x_A$ by $B(x, T)$.

(iii) The proof is as in item (iii) of that of Theorem 3.1. □

Example (*continued*). By application of Theorem 3.2, we deduce that, under Assumptions H1 and H2, the feedback controllers:

$$v(T) = x_A^{in}, \qquad \forall T \in]0; T_2[$$
$$\quad = 0, \qquad \forall T \in [T_2,+\infty[$$

and

$$u(x, T) = \underset{[u^{min},u^{max}]}{\text{sat}} \{\beta(T)(T^* - T) + qT - b_1k_1(T)x_A - b_2k_2(T)x_Bx_C^2\}$$

are such that the controlled reactor is asymptotically stable at the equilibrium point $(\bar{x}, T^*)$ relative to the positively invariant domain $\Omega \times]0,+\infty[$ for β^* large enough. Remember that $\bar{x} = (\bar{x}_A, \bar{x}_B, \bar{x}_C, \bar{x}_D) \in \Omega$ is the unique solution of

$$\begin{cases} 0 = -k_1(T^*)\bar{x}_A + d(x_A^{in} - \bar{x}_A), \\ 0 = k_1(T^*)\bar{x}_A - k_2(T^*)\bar{x}_B\bar{x}_C^2 - d\bar{x}_B, \\ 0 = -2k_2(T^*)\bar{x}_B\bar{x}_C^2 + d(x_C^{in} - \bar{x}_C), \\ 0 = k_2(T^*)\bar{x}_B\bar{x}_C^2 - d\bar{x}_D. \end{cases}$$ □

3.3. *Robust global stabilization with input constraints*

We address here the same control problem as in Section 3.2, but we consider that the non-negative functions $k_i(T)$ involved in the reaction kinetics $r(x, T)$ and the positive constants b_i are unknown. As said earlier, this situation is motivated by the fact that, in many practical cases, the functions $k_i(T)$ that are given according to empirical Arrhenius laws present large uncertainties. Hence, the question now is about the robust global stabilization under input constraints of a general class of exothermic CSTRs. For this purpose, we set the following assumption.

*Assumption H*4. The functions $k_i(T)$ are globally Lipschitz on $]0,+\infty[$. □

Let the control design parameter β^* and the m positive constants α_i $(i = 1, \dots, m)$ be such that $\beta^* > q$ and $\forall i, \alpha_i > b_ik_i(T^*)$. Let $\bar{\theta} = (\dots, \bar{\theta}_i, \dots)^T \in \mathcal{R}_+^m$ and the temperature $T_l > T^*$ be given by:

$$\beta^*(T^* - T_l) + qT_l - u^{min} = 0,$$
$$\frac{\alpha_i\bar{\theta}_i}{\alpha_i + \bar{\theta}_i} = b_ik_i(T^*).$$

Then, consider the feedback controllers $v(T)$ and $u(x, T)$ defined below;

$$v(T) = x_1^{in}, \qquad \forall T \in]0; T_l[$$
$$\quad = 0, \qquad \forall T \in [T_l, +\infty[\qquad (24)$$

and

$$u(x, T) = \underset{[u^{min},u^{max}]}{\text{sat}} \beta^*(T^* - T) + qT - f(T) \sum \frac{\alpha_i\theta_i}{\alpha_i + \theta_i} \varphi_i(x), \quad (25)$$

$$\dot{\theta}_i = (T - T^*)\theta_i\varphi_i(x), \qquad i = 1, \dots, m,$$

with

$$f(T) = 1, \qquad \forall T \in]0, T^*]$$
$$= \frac{\beta^*(T^* - T) + qT - u^{min}}{qT^* - u^{min}}, \qquad \forall T \in [T^*, T_l]$$
$$= 0, \qquad \forall T \in [T_l,+\infty[. \qquad (26)$$

Theorem 3.3. Under Assumptions H0, H1, H2 and H4 the dynamics of the controlled reactor ((20), (24)–(26)) are such that:

(i) The domain $\Omega \times]0,+\infty[\times \mathcal{R}_+^m$ is positively invariant.

(ii) There exists $\beta^* > q$ large enough and $\alpha_i > b_ik_i(T^*)$, $i = (1, \dots, m)$, such that the reactor temperature T converges asymptotically to the temperature set point T^* and the variables θ_i are bounded $(\forall(x(0), T(0), \theta(0)) \in \Omega \times]0,+\infty[\times\mathcal{R}_+^m)$.

If, in addition, Assumption H3 holds, then we have:

(iii) There exists $\beta^* > q$ large enough and $\alpha_i > b_i k_i(T^*)$, $i = (1, \dots, m)$, such that (x, T) globally converges to $(\bar{x}, T^*)$ $(\forall (x(0), T(0), \theta(0)) \in \Omega \times]0, +\infty[\times \mathscr{R}_+^m)$. □

Proof. The proof is similar to that of Theorem 2.3, by replacing $bk(T)x_A$ by $B(x, T)$. □

Example (*continued*). By application of Theorem 3.3, we know that under Assumptions H1, H2 and H4 the feedback controllers:

$$v(T) = x_A^{in}, \qquad \forall T \in]0; T_l[$$
$$= 0, \qquad \forall T \in [T_l, +\infty[,$$
$$u(x, T) = \underset{[u^{min}, u^{max}]}{\text{sat}} \beta^*(T^* - T) + qT - f(T)\left(\frac{\alpha_1 \theta_1}{\alpha_1 + \theta_1} x_A - \frac{\alpha_2 \theta_2}{\alpha_2 + \theta_2} x_B x_C^2\right),$$
$$\dot{\theta}_1 = (T - T^*)\theta_1 x_A,$$
$$\dot{\theta}_2 = (T - T^*)\theta_2 x_B x_C^2,$$

are such that the controlled reactor is asymptotically stable at the eqilibrium point $(\bar{x}, T^*, \bar{\theta})$ relative to the positively invariant domain $\Omega \times]0, +\infty[\times \mathscr{R}_+^2$ for some $\beta^* > q$ and for some $\alpha_1 > b_1 k_1(T^*)$ and $\alpha_2 > b_2 k_2(T^*)$. The equilibrium point $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2)$ is given by

$$\frac{\alpha_1 \bar{\theta}_1}{\alpha_1 + \bar{\theta}_1} = b_1 k_1(T^*), \qquad \frac{\alpha_2 \bar{\theta}_2}{\alpha_2 + \bar{\theta}_2} = b_2 k_2(T^*). \quad □$$

4. CONCLUSION

From a control point of view, exothermic chemical reactors are nonlinear challenging processes due to their instability features (multiple open loop steady states that are either locally asymptotically stable or unstable) and their capability of leading to thermal runaways. In this paper, we have designed various state feedback control structures for exothermic CSTRs that achieve global stabilization of the process, that are robust to large uncertainties on the dependence of the kinetics with respect to the temperatures and that handle input constraints along the closed loop trajectories. Theorems 2.3 and 3.3 can be considered as the main results of the paper. These results have been motivated and explained in detail by considering the case of a CSTR in which the exothermic reaction A → B takes place.

From a practical point of view, it is well known that reactor operation at an open loop unstable steady state often corresponds to an optimal process performance (see e.g. Bruns and Bailey, 1975). Hence, we provide control tools that can achieve in a realistic manner this objective. The robustness and input constraints issues have been considered in conjunction with the stabilization aspect. On the other hand, our controllers involve high gain feedback. In practice, as the simulation of Section 2.3 illustrates, the values of the gain are however not excessive.

To be implemented, our state feedback controllers require the on-line knowledge of the full state of the system: the set of concentrations and the temperature. In the case of partial state measurement, we can use the robust state observer proposed by Bastin and Dochain (1990) and Dochain *et al.* (1992). It has been shown in Viel *et al.* (1997) that the incorporation of such an observer in the loop does not impair the nominal global stabilization and robustness properties of a state feedback controller.

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