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Adaptive control of nonlinear systems with nonlinear parameterization

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Abstract

An adaptive feedback regulation scheme is proposed for a class of single input nonlinear systems, with nonlinear parameterizations. A proof of local regulation is given. The results are validated through a simulation study.

Keywords: Nonlinearly parameterized nonlinear systems; Adaptive stabilization; Lyapunov functions; Backstepping design

1. Introduction

In recent years, the regulation of uncertain nonlinear systems by adaptive feedback linearization has stimulated many research studies. A basic motivation is the important drawback of the exact feedback linearization, which relies on assumed perfect cancellation of the plant nonlinearities. Indeed, the perfect knowledge of the nonlinearities required for cancellation is not appropriate to uncertain systems and has therefore led to the use of adaptive control in order to bring robustness to the feedback linearization in case of parametric uncertainties.

To solve the regulation problem of uncertain linearly parameterized nonlinear systems, two trends have appeared. The first trend consists in introducing a growth condition on the plant nonlinearities (like the Lipschitz condition by Sastry and Isidori [14]), or a specific growth condition on a Lyapunov function (as in Praly et al. [13]). The second trend, with which we are concerned, is to impose a certain canonical form to the system (i.e. to restrict the location of the parameters entering the model).

First, Taylor et al. have introduced in [15] the *strict matching condition*, which means that the parametric uncertainty can only appear in equations including a control term.

Later, Kanellakopoulos et al. [3] have enlarged the considered class of systems by introducing a less restrictive *extended matching condition*.

Finally, still weaker geometric conditions have been introduced by Kanellakopoulos et al. in [4] leading to the concept of systems in *pure parametric feedback form*, which allow a step-by-step design of an adaptive

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regulation algorithm by using the so-called *backstepping* technique. For this class of systems, an *overparameterized backstepping* algorithm is proposed in [4], while an improved *nonoverparameterized* version, which therefore brings stronger stability properties and reduces the dynamic order of the controller, is described in [10].

In this paper, we consider a class of systems which is a nonlinearly parameterized extension of the class of systems in *pure parametric feedback form*. The systems we consider are therefore said to be in *nonlinearly parameterized pure feedback form*. The control algorithm we propose is itself an extension of the *nonoverparameterized backstepping* algorithm of [10]. An underlying motivation of our approach is that it often arises that some nonlinearly parameterized physical models cannot be "reparameterized" into the *pure parametric feedback form*. There is thus a clear incentive to try to relax the often nonphysical assumption of a linear parameterization. In practice, nonlinear parameterizations naturally arise on physical systems such as cascaded reactions in bioreactors (see [5] and [6]) or DC-to-DC boost converters (see [8]). Other attempts to deal with nonlinear parameterization in adaptive output feedback control of nonlinear systems can be found in Marino and Tomei [11, 12] and Byrnes et al. [1]. Moreover, a sliding-mode extension of the adaptive backstepping technique for nonlinearly parameterized systems is introduced in [7] and is applied to biotechnological systems in [5] and [6].

Our method is based on a first-order Taylor approximation which transforms the nonlinearly parameterized system into a form which is analogous (but not identical) with the *pure feedback form*. It is then shown that the "natural" Lyapunov function for the approximate linearized system remains, at least locally, a valid Lyapunov function for the "true" system (i.e. the system with the nonlinear parameterization).

The paper is organized as follows: Section 2 describes the class of studied nonlinear systems. Section 3 deals with the design of the adaptive *backstepping* regulation technique applied to the linearly parameterized approximate closed-loop system. Section 4 deals with the stability analysis of the "true" nonlinearly parameterized closed-loop system. Finally, in Section 5, we have shown simulations before concluding.

2. Problem statement

In this paper, we consider the class of systems, which we call *nonlinearly parameterized pure feedback form*, described by the following equations:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \gamma_i(x_1,\dots,x_{i+1},\theta), \quad 1 \leqslant i \leqslant n-1, \\ \dot{x}_n &= \gamma_0(x) + \gamma_n(x,\theta) + (\beta_0(x) + \beta_n(x,\theta))u, \end{aligned} \tag{1}$$

where $x = (x_1,x_2,\dots,x_n)^{\mathrm{T}} \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}$ is the control input and $\theta \in \mathbb{R}^p$ is an unknown constant parameter vector.

Furthermore, we assume that $\gamma_0,\gamma_1,\dots,\gamma_n$, and β_0,β_n are smooth functions of their arguments such that the following structural property is satisfied:

Assumption 1.

$$\gamma_0(0,0,\dots,0)=\gamma_1(0,0,\theta)=\cdots=\gamma_n(0,\dots,0,\theta)=0 \quad \text{and} \quad \beta_0(0,0,\dots,0)\neq 0, \ \forall\theta\in\mathbb{R}^p.$$

This class of systems is clearly a nonlinearly parameterized version of the well-known *pure parametric feedback form* introduced in [4] by Kanellakopoulos et al. This quasi-triangular form allows a step-by-step design of an adaptive regulation algorithm using the so-called *backstepping* technique without any growth condition.

The objective, in this paper, is to derive an adaptive state feedback controller to regulate system (1) at a zero equilibrium point ($x = 0$).

3. Stability analysis for an approximate closed-loop system

3.1. Backstepping algorithm in case of a linear parameterization

Let us first give a brief review of the *nonoverparameterized backstepping* control regulation scheme, derived from a Lyapunov step-by-step design, and developed by Krstić et al. [10] for a system in *pure parametric feedback form* given by

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \phi_i(x_1,\ldots,x_{i+1})\theta, \quad 1 \leqslant i \leqslant n-1, \\ \dot{x}_n &= \phi_0(x) + \phi_n(x)\theta + (\psi_0(x) + \psi_n(x)\theta)u, \end{aligned} \tag{2}$$

where

$$\phi_0(0,0,\ldots,0) = \phi_1(0,0) = \cdots = \phi_n(0,\ldots,0) = 0 \quad \text{and} \quad \psi_0(0,0,\ldots,0) \neq 0.$$

Instead of the true values $\theta = [\theta_1,\ldots,\theta_p]^{\mathrm{T}}$, which are unknown, the controller is designed using parameter estimates $\hat{\theta} = [\hat{\theta}_1,\ldots,\hat{\theta}_p]^{\mathrm{T}}$.

The following change of coordinates is introduced:

$$z_i = x_i - \alpha_{i-1}, \quad 1 \leqslant i \leqslant n$$

given the following so-called stabilizing functions: $\alpha_0 = 0$,

$$\alpha_i(x_1,\ldots,x_{i+1},\hat{\theta}) = -z_{i-1} - c_i z_i - \omega_i\hat{\theta} + \sum_{k=1}^{i}\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\tau_i + \sum_{k=2}^{i-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma\omega_i^{\mathrm{T}}z_k,$$

with $z_0 = 0$, given the following so-called tuning functions:

$$\tau_0 = 0, \qquad \tau_i(x_1,\ldots,x_{i+1},\hat{\theta}) = \tau_{i-1} + \Gamma\omega_i^{\mathrm{T}}z_i,$$

with

$$\omega_i(x_1,\ldots,x_{i+1},\hat{\theta}) = \phi_i - \sum_{k=1}^{i}\frac{\partial\alpha_{i-1}}{\partial x_k}\phi_k.$$

The following Lyapunov function V is considered:

$$V = \frac{1}{2}\sum_{k=1}^{n} z_k^2 + \frac{1}{2}(\theta - \hat{\theta})^{\mathrm{T}}\Gamma^{-1}(\theta - \hat{\theta}), \tag{3}$$

with Γ a diagonal positive-definite matrix.

The parameter update law is derived in order to make the derivative of V independent of the unknown term $(\theta - \hat{\theta})$, in the following way:

$$\dot{\hat{\theta}} = \left[\tau_n + \Gamma\left(1 - \frac{\partial\alpha_{n-1}}{\partial x_n}\right)\psi_n^{\mathrm{T}}(x)uz_n\right]. \tag{4}$$

Finally, the control law u stabilizing the uncertain linearly parameterized system (2) at the equilibrium point $x = 0$, is obtained by making $\dot{V}$ negative semi-definite (i.e. $\dot{V} = -\sum_{k=1}^{n}c_k z_k^2$ with positive c_k), as follows:

$$u = \frac{\alpha_n - (1 - \partial\alpha_{n-1}/\partial x_n)\phi_0}{(1 - \partial\alpha_{n-1}/\partial x_n)[\psi_0 + \psi_n(\hat{\theta} - \sum_{k=2}^{n}\Gamma(\partial\alpha_{k-1}/\partial\hat{\theta})^{\mathrm{T}}z_k)]}. \tag{5}$$

Obviously, this control law is feasible only if the denominator of u is nonzero. The *feasibility region* is defined as the bounded set $\tilde{\mathscr{F}} = \tilde{B}_x \times \tilde{B}_\theta$ containing $(0,\theta)$ such that $\forall(x,\hat{\theta}) \in \tilde{\mathscr{F}}$:

$$\left|1 - \frac{\partial\alpha_{n-1}}{\partial x_n}\right| > 0, \quad 1 \leqslant i \leqslant n-1$$

$$\left|\psi_0(x) + \psi_n(x)\left(\hat{\theta} - \sum_{k=2}^{n}\Gamma\left(\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right)^{\mathrm{T}} z_k\right)\right| > 0$$

3.2. *Backstepping algorithm in case of a nonlinear parameterization*

In order to extend to system (1) the *nonoverparameterized backstepping* algorithm described in the previous subsection, we introduce the following *adaptive parameter linearization approximation* (A.P.L. Approximation) which transforms the *nonlinearly parameterized pure feedback* system into a form which is analogous with the *pure parametric feedback form.*

A.P.L. Approximation. The functions $\gamma_i(x,\theta)$, for $1 \leqslant i \leqslant n$ and $\beta_n(x,\theta)$ are replaced by their first-order Taylor approximation as follows:

$$\gamma_i(x,\theta) \simeq \gamma_i(x,\hat{\theta}) + \left.\frac{\partial\gamma_i}{\partial\theta}\right|_{\hat{\theta}}(\theta - \hat{\theta}), \quad i = 1,\dots n$$

$$\beta_n(x,\theta) \simeq \beta_n(x,\hat{\theta}) + \left.\frac{\partial\beta_n}{\partial\theta}\right|_{\hat{\theta}}(\theta - \hat{\theta}).$$

With this approximation, system (1) is transformed into the following approximate system:

$$\begin{aligned}
\dot{x}_i &= x_{i+1} + \gamma_i(x_1,\dots,x_{i+1},\hat{\theta}) + \left.\frac{\partial\gamma_i}{\partial\theta}\right|_{\hat{\theta}}(\theta - \hat{\theta}), \quad 1 \leqslant i \leqslant n-1, \\
\dot{x}_n &= \gamma_0(x) + \gamma_n(x,\hat{\theta}) + \left.\frac{\partial\gamma_n}{\partial\theta}\right|_{\hat{\theta}}(\theta - \hat{\theta}) + \left(\beta_0(x) + \beta_n(x,\hat{\theta}) + \left.\frac{\partial\beta_n}{\partial\theta}\right|_{\hat{\theta}}(\theta - \hat{\theta})\right)u.
\end{aligned} \tag{6}$$

This system can be compared with the *pure parametric feedback form* system (2), which can also be written in the following way:

$$\begin{aligned}
\dot{x}_i &= x_{i+1} + \phi_i(x_1,\dots,x_{i+1})\hat{\theta} + \phi_i(x_1,\dots,x_{i+1})(\theta - \hat{\theta}), \quad 1 \leqslant i \leqslant n-1, \\
\dot{x}_n &= \phi_0(x) + \phi_n(x)\hat{\theta} + \phi_n(x)(\theta - \hat{\theta}) + (\psi_0(x) + \psi_n(x)\hat{\theta} + \psi_n(x)(\theta - \hat{\theta}))u.
\end{aligned} \tag{7}$$

The similarity of forms between systems (6) and (7) is used to derive the new parameter update law and the new control law which stabilize the approximate system (6).

The multiplicative coefficients of the parameter error $(\theta - \hat{\theta})$ are different in the two cases: $(\partial\gamma_i/\partial\theta)|_{\hat{\theta}}$ instead of $\phi_i(x_1,\dots,x_{i+1})$ in the first $(n-1)$ equations and $(\partial\gamma_n/\partial\theta)|_{\hat{\theta}} + (\partial\beta_n/\partial\theta)|_{\hat{\theta}}u$ instead of $\phi_n(x) + \psi_n(x)u$ in equation n.

Furthermore, the terms $\phi_i(x_1,\dots,x_{i+1})\hat{\theta}$ are changed into $\gamma_i(x_1,\dots,x_{i+1},\hat{\theta})$, for $1 \leqslant i \leqslant n$ and $\psi_n(x)\hat{\theta}$ into $\beta_n(x,\hat{\theta})$.

Therefore, the adaptive *backstepping* algorithm, consisting of Eqs. (4) and (5), is slightly modified.

We have the new following expressions of the α_i and the τ_i :

$$\alpha_i(x_1,\dots,x_{i+1},\hat{\theta}) = -z_{i-1} - c_i z_i - \omega_i + \sum_{k=1}^{i}\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\tau_i + \sum_{k=1}^{i-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma\bar{\omega}_i^{\mathrm{T}} z_k,$$

$$\tau_i(x_1,\dots,x_{i+1},\hat{\theta}) = \tau_{i-1} + \Gamma\bar{\omega}_i^{\mathrm{T}} z_i,$$

with

$$\omega_i(x_1,\ldots,x_{i+1},\hat{\theta}) = \gamma_i - \sum_{k=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_k}\gamma_k,$$

$$\bar{\omega}_i(x_1,\ldots,x_{i+1},\hat{\theta}) = \frac{\partial \gamma_i}{\partial \hat{\theta}} - \sum_{k=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_k}\frac{\partial \gamma_k}{\partial \hat{\theta}}.$$

Then the new parameter update law for the approximate system (6) is given by

$$\dot{\hat{\theta}} = \left[\tau_n + \left(1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right) \Gamma \left(\frac{\partial \beta_n}{\partial \hat{\theta}}\right)^{\mathrm{T}} u z_n\right], \tag{8}$$

and the control u is expressed in the following way:

$$u = \frac{\alpha_n - (1 - \partial\alpha_{n-1}/\partial x_n)\gamma_0}{(1 - \partial\alpha_{n-1}/\partial x_n)\left[\beta_0 + \beta_n(x,\hat{\theta}) - \sum_{k=2}^{n}(\partial\beta_n/\partial\hat{\theta})\Gamma(\partial\alpha_{k-1}/\partial\hat{\theta})^{\mathrm{T}} z_k\right]}. \tag{9}$$

Here, it is worth noting that by construction, the parameter update law can be factorized in the following way:

$$\dot{\hat{\theta}} = \left[\mathscr{T} + \left(1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right) \Gamma \frac{\partial \beta_n}{\partial \hat{\theta}}^{\mathrm{T}} e_n^{\mathrm{T}} u\right] z, \tag{10}$$

with $\mathscr{T}$ a $p \times n$ matrix such that $\tau_n = \mathscr{T} z$ and e_n the n-unit vector $(0,0\cdots 0,1)^{\mathrm{T}}$.

The *feasibility region* is easily determined as above and is defined as the bounded set $\mathscr{F} = B_z \times B_\theta \subset \mathbb{R}^n \times \mathbb{R}^p$, containing $(0,\theta)$ such that $\forall (z,\hat{\theta}) \in \mathscr{F}$:

$$\left|1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right| > 0, \quad 1 \leqslant i \leqslant n-1$$

$$\left|\beta_0(x) + \beta_n(x)\left(\hat{\theta} - \sum_{k=2}^{n} \Gamma \left(\frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\right)^{\mathrm{T}} z_k\right)\right| > 0.$$

Finally, an *approximate closed-loop system* consisting of the linearized system (6) under the adaptive control law (9)–(10) is given by

$$\begin{aligned} \dot{z} &= A_z z + \left[\mathscr{T} + \left(1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right) \Gamma \frac{\partial \beta_n}{\partial \hat{\theta}}^{\mathrm{T}} e_n^{\mathrm{T}} u\right]^{\mathrm{T}} \Gamma^{-1}(\theta - \hat{\theta}), \\ \dot{\hat{\theta}} &= \left[\mathscr{T} + \left(1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right) \Gamma \frac{\partial \beta_n}{\partial \hat{\theta}}^{\mathrm{T}} e_n^{\mathrm{T}} u\right] z, \end{aligned} \tag{11}$$

with

$$A_z(z,\hat{\theta}) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & -c_2 & 1+\sigma_{2,3} & \cdots & \cdots & \sigma_{2,n} \\ 0 & -1-\sigma_{2,3} & -c_3 & & \cdots & \sigma_{3,n} \\ \vdots & & & & & \vdots \\ 0 & & & -\alpha 1 & -c_{n-1} & 1+\sigma_{n-1,n} \\ 0 & -\sigma_{2,n} & \cdots & & -1-\sigma_{n-1,n} & -c_n \end{bmatrix}$$

and

$$\sigma_{i,k} = -\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \bar{\omega}_k^{\mathrm{T}}, \quad 2 \leqslant i \leqslant n-1, \ k > i, \ k \neq n,$$

$$\sigma_{i,n} = -\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \left(\bar{\omega}_n^{\mathrm{T}} + \left(1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right) \left(\frac{\partial \beta_n}{\partial \hat{\theta}}\right)^{\mathrm{T}} u \right), \quad 2 \leqslant i \leqslant n-1.$$

Stability analysis for the approximate closed-loop system (11). This analysis is quite similar to the one done by Kanellakopoulos et al. in [4]. Since $\dot{V} = -\sum_{k=1}^{n} c_k z_k^2$, with positive c_k, it is straightforward that

$$\dot{V} \leqslant -c_{\min} \|z\|^2,$$

with $c_{\min}$ the minimum of the c_i, $1 \leqslant i \leqslant n$.

This proves the uniform stability of the equilibrium: $z = 0, \hat{\theta} = \theta$ of the adaptive system (11), according to Lyapunov arguments.

An estimate $\Omega \subset \mathscr{F}$ of the region of attraction of this equilibrium is obtained as follows. Using Assumption 1, it is straightforward that the point $z = 0, \hat{\theta} = \theta$ coincides with the point $x = 0, \hat{\theta} = \theta$. Let $\Omega(c)$ be the invariant set of (11) defined by $V < c$, and let c^* be the largest constant c such that $\Omega(c) \subset \mathscr{F}$. Then, as in [9], an estimate of the region of attraction is given by

$$\Omega = \Omega(c^*) = \{(z, \hat{\theta}) \mid V(z, \hat{\theta}) < c^*\} \quad \text{with } c^* = \arg \sup_{\Omega(c) \subset \mathscr{F}} \{c\}.$$

Finally, using the LaSalle invariance principle, it is easily shown that the closed-loop system is such that $\forall (z(0), \hat{\theta}(0)) \in \Omega$, we have $\lim_{t \to \infty} z(t) = 0$.

Inductively, and as in [4], it can be concluded that system (6) is locally regulated around the equilibrium point $x = 0$.

Remark 1. Note that different adaptation gains (i.e. matrix Γ) can be found such that the estimate of the region of attraction Ω is maximized by a better fit of $\mathscr{F}$.

4. Stability analysis for the exact closed-loop system

The previous stability analysis has been carried out for the approximate closed-loop system. From now on, x, z and $\hat{\theta}$ denote the state variables of the exact system. Let us consider the situation when the control law $u(x, \hat{\theta})$ and the parameter update law $\dot{\hat{\theta}}(x, \hat{\theta})$, derived in the previous section, are applied to the exact system (1). The objective of this section is to prove that the Lyapunov function for the approximate system remains a Lyapunov function for the exact system, at least locally. The Lyapunov function expression for the exact system, denoted V, is the same as above and is given by (3).

From the previous section and more precisely from (11), we know a truncated expression of the derivative of z:

$$\dot{z}_{\mathrm{tr}} = f(z, 0) + Df(z, \theta - \hat{\theta})|_{(z,0)} \cdot (\theta - \hat{\theta})$$

with the expression of function f given by (11).

Using Assumption 1, it is easy to show that the Taylor series of f, parameterized in $\hat{\theta}$, around $z = 0$, necessarily begins with a first-order term in z.

Therefore, we can rewrite the dynamics of the system state variable z in the following way:

$$f(z, 0) = C(z, \hat{\theta}) z \tag{12}$$

with C an $n \times n$ matrix.

Moreover, the closed-loop dynamics for the exact system are written as follows: [1]

$$\dot{z} = \dot{z}_{tr} + \tfrac{1}{2}D^2 f(z, \theta - \hat{\theta})|_{(z,0)} \cdot (\theta - \hat{\theta})^2 + \mathcal{O}^3(\theta - \hat{\theta}), \tag{13}$$

with

$$D^2 f(z, \theta - \hat{\theta})|_{(z,0)} \cdot (\theta - \hat{\theta})^2 = \left(\left(\sum_{j,k=1}^{p} \left. \frac{\partial^2 f_i}{\partial \theta_j \partial \theta_k} \right|_{(z,0)} (\theta_j - \hat{\theta}_j)(\theta_k - \hat{\theta}_k) \right) \right)_{i=1\ldots n}$$

and $\mathcal{O}^3(\theta - \hat{\theta})$ represents the higher-order terms of the Taylor series, i.e. a function of $(\theta - \hat{\theta})^3$ and all higher powers of $(\theta - \hat{\theta})$.

Then, the derivative of the Lyapunov function for the exact system is given by

$$\begin{aligned} \dot{V} &= z^{\mathrm{T}}\dot{z} - (\theta - \hat{\theta})^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}} \\ &= \dot{V}_{tr} + (z)^{\mathrm{T}}[\tfrac{1}{2}D^2 f(z, \theta - \hat{\theta})|_{(z,0)} \cdot (\theta - \hat{\theta})^2 + \mathcal{O}^3(\theta - \hat{\theta})], \end{aligned} \tag{14}$$

with $\dot{V}_{tr} = (z)^{\mathrm{T}}\dot{z}_{tr} - (\theta - \hat{\theta})^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}}$.

In the previous section, the following relation has been obtained:

$$\dot{V}_{tr} \leqslant -c_{\min}\|z\|^2,$$

with $c_{\min}$ the minimum of the c_i, $1 \leqslant i \leqslant n$.

Now, let us examine the second part of the Lyapunov function derivative denoted $\Delta\dot{V}$ (with $\Delta\dot{V} = \dot{V} - \dot{V}_{tr}$). First, the expression $\mathcal{O}^3(\theta - \hat{\theta})$ is written in the following way:

$$\begin{aligned} \mathcal{O}^3(\theta - \hat{\theta}) &= \frac{1}{3!}\left(\left(\sum_{j,k,l=1}^{p} \left. \frac{\partial^3 (Cz)_i}{\partial \theta_j \partial \theta_k \partial \theta_l} \right|_{\hat{\theta}} (\theta_j - \hat{\theta}_j)(\theta_k - \hat{\theta}_k)(\theta_l - \hat{\theta}_l) \right) \right)_{i=1\ldots n} + \text{h.o.t.} \\ &= \bar{C}(z, \hat{\theta}) \cdot (\theta - \hat{\theta})^3 z + \text{h.o.t.} \end{aligned}$$

Therefore, using (12), a part of expression (15) is given in the following way:

$$z^{\mathrm{T}}\mathcal{O}^3(\theta - \hat{\theta}) = z^{\mathrm{T}}[\bar{C}(z, \hat{\theta}) + \text{h.o.t.}]z.$$

Then, $\Delta\dot{V}$ can be split into two terms $\Delta\dot{V}_1$ and $\Delta\dot{V}_2$, given by

$$\Delta\dot{V}_1 = \tfrac{1}{2}z^{\mathrm{T}}D^2 f(z, \theta - \hat{\theta})|_{(z,0)} \cdot (\theta - \hat{\theta})^2 = \tfrac{1}{2}z^{\mathrm{T}}D^2(Cz) \cdot (\theta - \hat{\theta})^2$$

and

$$\Delta\dot{V}_2 = z^{\mathrm{T}}[\bar{C}(z, \hat{\theta}) + \text{h.o.t.}]z.$$

On the one hand, we have

$$\Delta\dot{V}_1 = \tfrac{1}{2}(\theta - \hat{\theta})^{\mathrm{T}} z^{\mathrm{T}} \left(\left(\sum_{j,k=1}^{p} \left. \frac{\partial^2 C_i}{\partial \theta_j \partial \theta_k} \right|_{\hat{\theta}} \right) \right)_{i=1\ldots n} z(\theta - \hat{\theta}).$$

Hence, for any positive constant μ_1, there exist two bounded sets B_{z1} (with $B_{z1} \subset B_z$) and $B_{\theta 1}$ (with $B_{\theta 1} \subset B_\theta$), respectively, in the neighbourhood of $z = 0$ and $\hat{\theta} = \theta$, such that $\forall z \in B_{z1}, \forall \hat{\theta} \in B_{\theta 1}$:

$$\Delta\dot{V}_1 \leqslant \mu_1\|z\|^2\|\theta - \hat{\theta}\|^2.$$

[1] To avoid the useless notational complexity of the tensor product, we use the shortened notation: $(\theta - \hat{\theta})^n \triangleq \underbrace{(\theta - \hat{\theta}) \otimes \cdots \otimes (\theta - \hat{\theta})}_{n}$

On the other hand, by applying Theorem 8.14.3 of Dieudonné [2], with the two bounded sets $B_{z2} \subset B_z$ and $B_{\theta 2} \subset B_\theta$, we have $\forall \mu_2 > 0, \exists r > 0$, such that $\forall \hat{\theta} \in B_{\theta 2}$ satisfying $\|\theta - \hat{\theta}\| < r$ (i.e. $B(\theta, r) = B_{\theta 2}$) and $\forall z \in B_{z2}$:

$$\|\bar{C}(z, \hat{\theta}) + \text{h.o.t.}\| \leqslant \mu_2 \|\theta - \hat{\theta}\|^2.$$

Therefore, we know that the following second inequality is locally satisfied $\forall z \in B_{z2}, \forall \hat{\theta} \in B_{\theta 2}$:

$$\Delta \dot{V}_2 \leqslant \mu_2 \|z\|^2 \|\theta - \hat{\theta}\|^2.$$

Finally, defining the bounded sets $\bar{B}_z = B_{z1} \cap B_{z2}$ and $\bar{B}_\theta = B_{\theta 1} \cap B_{\theta 2}$, the following inequality is obtained, $\forall z \in \bar{B}_z, \forall \hat{\theta} \in \bar{B}_\theta$:

$$\dot{V} \leqslant -c_{\min} \|z\|^2 + (\mu_1 + \mu_2) \|\theta - \hat{\theta}\|^2 \|z\|^2.$$

Since $\bar{B}_\theta$ is a bounded set, let $\Delta\theta^2_{\max} = \sup_{\hat{\theta} \in \bar{B}_\theta} \|\theta - \hat{\theta}\|^2$. Now, if we choose $c_{\min}$ such that, for some positive ζ:

$$c_{\min} \geqslant \zeta + (\mu_1 + \mu_2) \Delta\theta^2_{\max},$$

then we have a negative semi-definite Lyapunov function derivative given by

$$\dot{V} \leqslant -\zeta \|z\|^2$$

and hence the point $z = 0$, $\hat{\theta} = \theta$ is a uniformly stable equilibrium point for the exact system (13).

Note that the region of attraction $\bar{\Omega}$ is included in the feasibility region $\bar{\mathscr{F}} = \bar{B}_x \times \bar{B}_\theta$, and that the estimate of this region of attraction may be more conservative than the previous estimate of the region of attraction Ω.

Moreover, as in the previous section, it can be shown that system (1) is locally regulated at the equilibrium point $x = 0$.

5. Simulations

Consider the following two-dimensional nonlinear system in *nonlinearly parameterized pure feedback form*,

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_2 e^{\theta x_1}, \\ \dot{x}_2 &= x_1 x_2^2 + e^{\theta x_2} u. \end{aligned} \tag{15}$$

Let us now describe the *backstepping* procedure developed in Section 3.

Step 1: Consider the following change of coordinates:

$$z_1 = x_1 \quad \text{and} \quad z_2 = x_2 - \alpha_1.$$

The one-dimensional subsystem (first equation of (15)) is to be stabilized with respect to $V_1 = \frac{1}{2} z_1^2 + \frac{1}{2}(\hat{\theta} - \theta)^{\mathrm{T}} \Gamma^{-1} (\hat{\theta} - \theta)$.

After having rewritten the first equation of (15) in the new coordinates (z_1, z_2) and after having used the A.P.L. Approximation (i.e. first-order Taylor approximation), we derive a first temporary update law τ_1 (the so-called tuning function) in order to make the derivative of the Lyapunov function V_1 independent of the unknown term $(\hat{\theta} - \theta)$:

$$\dot{\hat{\theta}} = \tau_1(x_1, x_2, \hat{\theta}) = \Gamma x_1 x_2 (1 + \hat{\theta} x_1) e^{\hat{\theta} x_1}.$$

Then, we impose the following stabilizing function α_1 so that the derivative of V_1 is negative semi-definite (i.e. $\dot{V}_1 = -c_1 z_1^2$ with no pseudo-control error ($z_2 = 0$)):

$$\alpha_1(x_1, x_2, \hat{\theta}) = -c_1 z_1 - \hat{\theta} x_2 e^{\hat{\theta} x_1}.$$

Since $z_2 \neq 0$ and $\dot{\hat{\theta}} \neq \tau_1$, the resulting equations are given by

$$\begin{aligned} \dot{z}_1 &= z_2 - c_1 z_1 + x_2(1 + \hat{\theta} x_1) e^{\hat{\theta} x_1}(\theta - \hat{\theta}), \\ \dot{V}_1 &= -c_1 z_1^2 + z_1 z_2 + (\hat{\theta} - \theta)^{\mathrm{T}} \Gamma^{-1}(\dot{\hat{\theta}} - \tau_1). \end{aligned} \tag{16}$$

Step 2: If we rewrite in the new coordinates (z_1, z_2) the whole system (15), approximated with the A.P.L. Approximation, then the strategy is once again to make negative semi-definite the derivative of the new extended Lyapunov function $V_2 = V_1 + \frac{1}{2} z_2^2$. In order to achieve that aim, we design the following parameter update law:

$$\dot{\hat{\theta}} = \tau_1 + \Gamma z_2 \left[\left(1 - \frac{\partial \alpha_1}{\partial x_2} \right) x_2 e^{\hat{\theta} x_2} u - \frac{\partial \alpha_1}{\partial x_1} x_2 (1 + \hat{\theta} x_1) e^{\hat{\theta} x_1} \right],$$

and the following feedback control law u given by

$$u = \frac{-z_1 - c_2 z_2 - (1 - \partial\alpha_1/\partial x_2) x_1 x_2^2 + (\partial\alpha_1/\partial x_1) x_2 (1 + \hat{\theta} e^{\hat{\theta} x_1}) + (\partial\alpha_1/\partial\hat{\theta})(\tau_1 - \Gamma z_2 (\partial\alpha_1/\partial x_1) x_2 (1 + \hat{\theta} x_1) e^{\hat{\theta} x_1})}{(1 - \partial\alpha_1/\partial x_2)(1 - \Gamma z_2 x_2 (\partial\alpha_1/\partial\hat{\theta})) e^{\hat{\theta} x_2}}.$$

Note that for this example, we know from the algorithm developed in Section 3, that we can find an update law and a feedback control law, only if the following conditions:

$$1 + \hat{\theta} e^{\hat{\theta} x_1} \neq 0, \quad 1 + \Gamma(x_2 + c_1 x_1 + \hat{\theta} x_2 e^{\hat{\theta} x_1})(1 + \hat{\theta}) x_2^2 e^{\hat{\theta} x_1} \neq 0 \quad \text{and} \quad e^{\hat{\theta} x_2} \neq 0,$$

are satisfied (note that the third one is always true).

These conditions give the feasibility region of the system.

Simulation results: Fig. 1 shows the simulation results for:
- two different sets of initial conditions: $\hat{\theta}_0 = 1.7, x_0 = (0.3, 0.3)^{\mathrm{T}}$, and $\hat{\theta}_0 = 4, x_0 = (0.3, 0.3)^{\mathrm{T}}$,
- the nominal parameter $\theta_n = 0.5$,
- the adaptation gain $\Gamma = 5$ – and the constants $c_1 = 2.2, c_2 = 4$, $T = 0.005$ (simulation period).

We observe that the adaptive control algorithm is really local. Indeed, $\hat{\theta}_0 = 1.7$ is within the basin of attraction: (x_1, x_2) converges to zero and $\hat{\theta}$ is bounded and converges to a constant value. In contrast, $\hat{\theta}_0 = 4$ is outside of the basin of attraction since the state variables x_1, x_2 and the parameter $\hat{\theta}$ diverge to infinity. In Fig. 1, it has only been possible to point out this phenomenon on the parameter curve, the divergence rate being too fast for the other variables.

Fig. 2 illustrates the point emphasized in Remark 1: By changing the adaptation gain in the previous simulation (namely $\Gamma = 0.003$), the region of attraction Ω is maximized by a better fit of $\mathscr{F}$. Indeed, with this new value of Γ, for the initial condition $\hat{\theta}_0 = 4$, the closed-loop is stable and the regulation objective is achieved, albeit with a deterioration of the performances in terms of time response.

6. Conclusion

In this paper, we have extended the adaptive backstepping technique to a class of nonlinearly parameterized nonlinear systems. We have introduced the A.P.L. Approximation consisting of a first-order Taylor approximation, and we have shown that this approximation is good enough for a local adaptive *backstepping* regulation of nonlinearly parameterized systems. A proof of local asymptotic stability based on Lyapunov arguments has

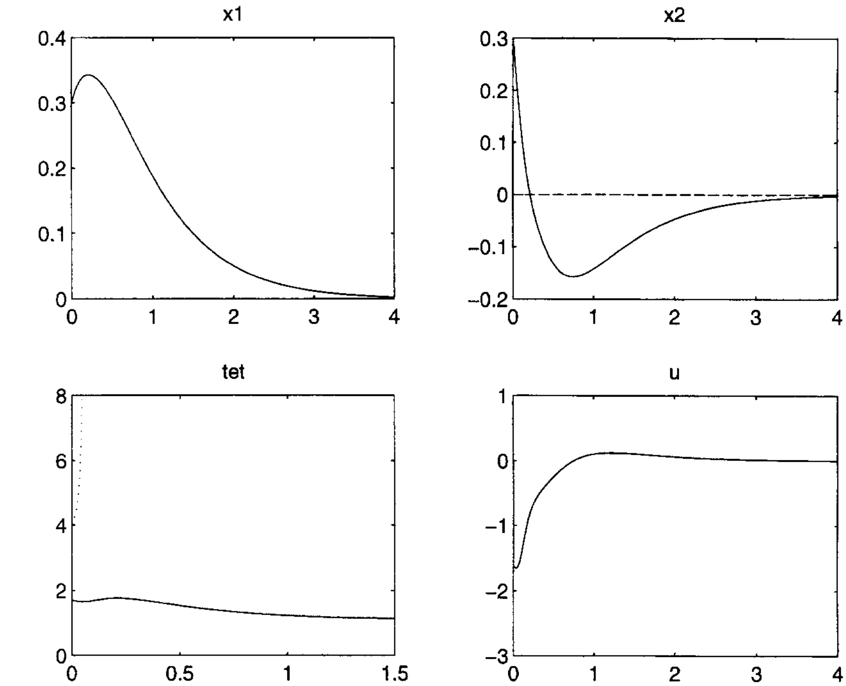

Fig. 1. (–) $\hat{\theta}_0 = 1.7$; (···) $\hat{\theta}_0 = 4$.

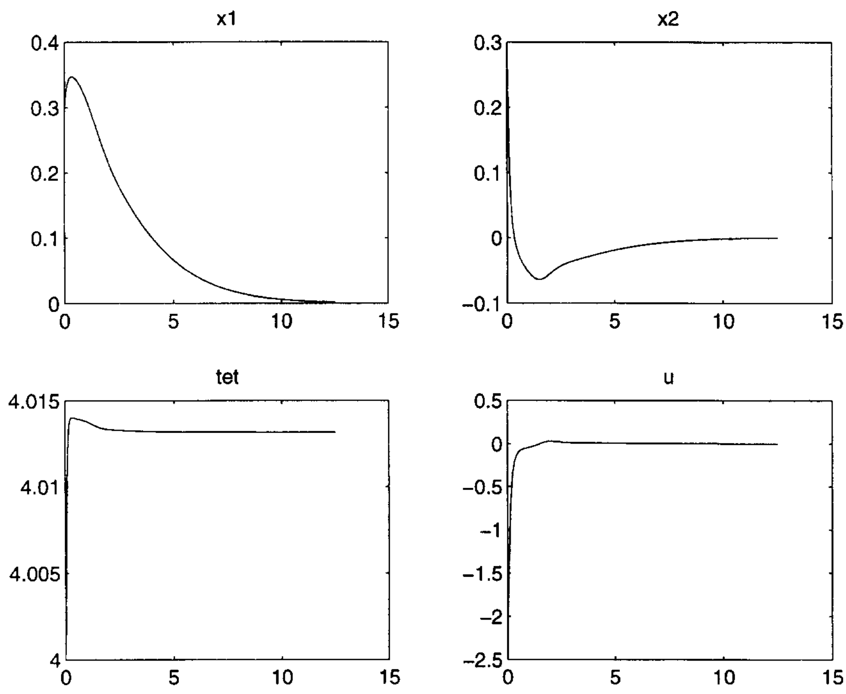

Fig. 2.

been given for this new extended class of systems. The results have also been validated through a simulation study. Moreover, in [6], such an adaptive control scheme has been successfully applied to cascaded reactions in stirred tank reactors, which are precisely in *nonlinearly parameterized pure feedback form.*

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