Structural Properties and Classification of Kinematic and Dynamic Models of Wheeled Mobile Robots

Guy Campion, Georges Bastin, and Brigitte D'Andréa-Novel

Abstract—**The structure of the kinematic and dynamic models of wheeled mobile robots is analyzed. It is shown that, for a large class of possible configurations, they can be classified into five types, characterized by generic structures of the model equations. For each type of model the following questions are addressed: (ir)reducibility and (non)holonomy, mobility and controllability, configuration of the motorization, and feedback equivalence.**

I. Introduction

WHEELED Mobile Robots (WMR) constitute a class of mechanical systems characterized by kinematic constraints that are not integrable and cannot therefore be eliminated from the model equations. The consequence is that the standard planning and control algorithms developed for robotic manipulators without constraints are no more applicable. This has given rise recently to an abundant literature dealing with the derivation of planning and control algorithms especially dedicated to specific simplified kinematic models of "trailer-like" or "car-like" rigid WMR (see, for instance and among many other relevant publications, [1]–[8]). However, commercial wheeled mobile robots available on the market have generally a constructive structure which is much more complex than the simple models usually considered (for instance, robots with three or four motorized steering wheels) and for which the modeling issue (which is often a prerequisite to motion planning and control design) is still a relevant question.

The aim of the present paper is to give a general and unifying presentation of the modeling issue of WMR. Several examples of derivation of kinematic and/or dynamic models for WMR are available in the literature, for particular prototypes of mobile robots (see, for instance, [9]–[11] and [1]), as well as for general robots equipped with wheels of several types. A systematic procedure for model derivation can be found in [12] and [13]. In this paper we also consider a general WMR, with an arbitrary number of wheels of various types and various motorizations. Our purpose is to point out the structural properties of the kinematic and dynamic models, taking into account the restriction to the robot mobility induced by the constraints. By introducing the concepts of *degree of mobility* and of *degree of steeribility*, we show that, notwithstanding the variety of possible robot constructions and wheel configurations, the set of WMR can be partitioned in 5 classes. This analysis is carried out in Section II and illustrated in Section III with practical examples of robots belonging to the five classes.

We then introduce four different kinds of state space models that are of interest for the understanding of the behavior of WMR.

- The *posture kinematic model* (Section IV) is the simplest state space model able to give a global description of WMR. It is shown that within each of the five classes, this model has a particular generic structure which allows to understand the maneuverability properties of the robot. The reducibility, the controllability, and the stabilizability of this model are also analyzed.
- The *configuration kinematic model* (Section V) allows to analyze the behavior of WMR within the framework of the theory of nonholonomic systems.
- The *configuration dynamical model* (Section VI) is the more general state space model. It gives a complete description of the dynamics of the system including the generalized forces provided by the actuators. In particular, the issue of the configuration of the motorization is addressed: a criterion is proposed to check whether the motorization is sufficient to fully exploit the kinematic mobility.
- The *posture dynamical model* (Section VII) which is feedback equivalent to the configuration dynamical model and useful to analyze its reducibility, its controllability, and its stabilizability properties.

II. Kinematics of Wheeled Mobile Robots

A. Robot Position

A wheeled mobile robot is a wheeled vehicle which is capable of an autonomous motion (without external human driver) because it is equipped, for its motion, with motors that are driven by an embarked computer. We assume that the mobile robots under study in this paper are made up of a rigid frame equipped with nondeformable wheels and that they are moving on a horizontal plane. The position of the robot on the plane is described as follows (see Fig. 1). An arbitrary orthonormal inertial basis $\{0, \vec{I_1}, \vec{I_2}\}$ is fixed in the plane of the motion. An arbitrary reference point P on the frame and an arbitrary basis $\{\vec{x_1}, \vec{x_2}\}$ attached to the frame are defined. The position of the robot is then completely specified by the 3 variables x, y, θ:

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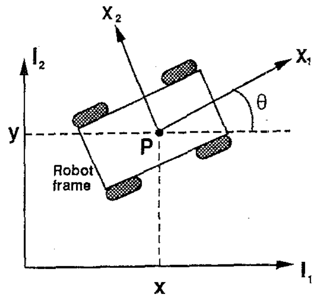

Fig. 1. Posture definition.

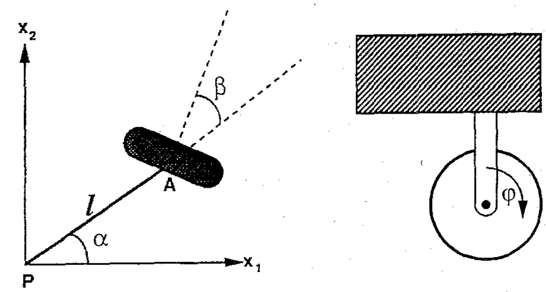

Fig. 2. Fixed and conventional centered orientable wheels.

- x, y are the coordinates of the reference point P in the inertial basis, i.e.,

$$\vec{OP} = x\vec{I_1} + y\vec{I_2};$$

- θ is the orientation of the basis $\{\vec{x_1}, \vec{x_2}\}$ with respect to the inertial basis $\{\vec{I_1}, \vec{I_2}\}$.

We define the 3-vector ξ describing the robot posture:

$$\xi \triangleq \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}. \qquad (1)$$

We also define the following orthogonal rotation matrix:

$$R(\theta) \triangleq \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (2)$$

B. Wheels Description

We assume that, during the motion, the plane of each wheel remains vertical and the wheel rotates around its (horizontal) axle whose orientation with respect to the frame can be fixed or varying. We distinguish between two basic classes of idealized wheels: the *conventional* wheels and the *swedish* wheels. In each case, it is assumed that the contact between the wheel and the ground is reduced to a single point of the plane.

For a *conventional wheel*, the contact between the wheel and the ground is supposed to satisfy the pure rolling without slipping condition. This means that the velocity of the contact point is equal to zero and implies that the components of this velocity parallel and orthogonal to the plane of the wheel are equal to zero.

For a *swedish* wheel, only *one* component of the velocity of the contact point of the wheel with the ground is supposed to be equal to zero along the motion. The direction of this zero component of the velocity is a priori arbitrary but is fixed with respect to the orientation of the wheel.

We now derive explicitly the expressions of the constraints for conventional and swedish wheels.

1) Conventional Wheels:

Fixed wheels: The center of the wheel, denoted A, is a fixed point of the frame (Fig. 2). The position of A in the basis $\{\vec{x_1}, \vec{x_2}\}$ is characterized using polar coordinates by the distance $PA = l$ and the angle α. The orientation of the plane of the wheel with respect to PA is represented by the constant angle β. The rotation angle of the wheel around its (horizontal) axle is denoted $\varphi(t)$ and the radius of the wheel is denoted r.

The position of the wheel is thus characterized by 4 constants, α, β, l, r, and its motion by a time varying angle $\varphi(t)$. With this description, the components of the velocity of the contact point are easily computed and we can deduce the 2 following constraints:

- *along the wheel plane*

$$[-\sin(\alpha+\beta) \quad \cos(\alpha+\beta) \quad l\cos\beta]R(\theta)\dot{\xi} + r\dot{\varphi} = 0 \qquad (3)$$

- *orthogonal to the wheel plane*

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]R(\theta)\dot{\xi} = 0. \qquad (4)$$

Centered orientable wheels: A *centered orientable wheel* is such that the motion of the wheel plane with respect to the frame is a rotation around a vertical axle passing through the center of the wheel (Fig. 2). The description is the same as for a fixed wheel, except that now the angle $\beta(t)$ is not constant but time varying. The position of the weel is characterized by 3 constants, l, α, r, and its motion with respect to the frame by 2 time-varying angles $\beta(t)$ and $\varphi(t)$. The constraints have the same form as above:

$$[-\sin(\alpha+\beta) \quad \cos(\alpha+\beta) \quad l\cos\beta]R(\theta)\dot{\xi} + r\dot{\varphi} = 0 \qquad (5)$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]R(\theta)\dot{\xi} = 0. \qquad (6)$$

Off-centered orientable wheels ("Castor wheels"): An off-centered orientable wheel is also a wheel which is orientable with respect to the frame, but the rotation of the wheel plane is around a vertical axle which does *not* pass through the center of the wheel (Fig. 3). In this case, the description of the wheel configuration requires more parameters. The center of the wheel is now denoted B and is connected to the frame by a rigid rod AB of constant length d which can rotate around a fixed vertical axle at point A. This point A is itself a fixed point of the frame and its position is specified by the 2 polar coordinates l and α as above. The plane of the wheel is aligned along AB.

The position of the wheel is described by 4 constants, α, l, r, d, and its motion by 2 time varying angles $\beta(t)$ and $\varphi(t)$. With these notations, the constraints have the following form:

$$[-\sin(\alpha+\beta) \quad \cos(\alpha+\beta) \quad l\cos\beta]R(\theta)\dot{\xi} + r\dot{\varphi} = 0 \qquad (7)$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad d + l\sin\beta]R(\theta)\dot{\xi} + d\dot{\beta} = 0. \qquad (8)$$

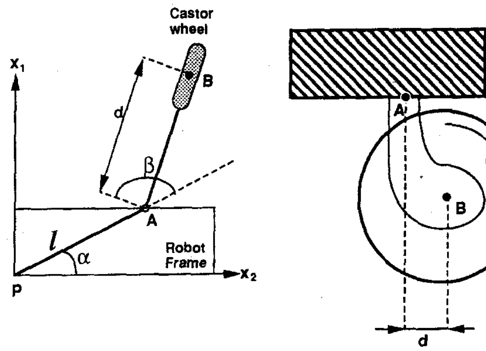

Fig. 3. Conventional off-centered orientable wheels.

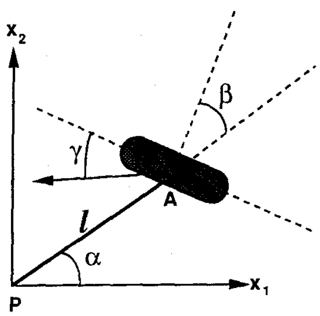

Fig. 4. Swedish wheels.

2) Swedish Wheels: The position of the wheel with respect to the frame is described, as for the conventional fixed wheel, by the 3 constant parameters, α, β, l. An additional parameter is required to characterize the direction, with respect to the wheel plane, of the zero component of the velocity of the contact point represented by the angle γ (Fig. 4). The motion constraint is expressed as

$$[-\sin(\alpha+\beta+\gamma) \quad \cos(\alpha+\beta+\gamma) \quad l\cos(\beta+\gamma)]R(\theta)\dot{\xi} + r\cos\gamma\dot{\varphi} = 0. \tag{9}$$

C. Restrictions to the Robot Mobility

We now consider a general mobile robot, equipped with N wheels of the 4 above described categories. We use the 4 following subscripts to identify quantities relative to these 4 classes: f for conventional fixed wheels, c for conventional centered orientable wheels, oc for conventional off-centered orientable wheels, and sw for swedish wheels. The numbers of wheels of each type are denoted N_f, N_c, N_{oc}, N_{sw} with $N_f + N_c + N_{oc} + N_{sw} = N$.

The configuration of the robot is fully described by the following vectors of coordinates.

- Posture coordinates: $\xi(t) \triangleq \begin{pmatrix} x(t) \\ y(t) \\ \theta(t) \end{pmatrix}$ for the position coordinates in the plane.
- Angular coordinates: $\beta_c(t)$for the orientation angles of the centered orientable wheels and $\beta_{oc}(t)$ for the orientation angles of the off-centered orientable wheels.
- Rotation coordinates: $\varphi(t) \triangleq \begin{pmatrix} \varphi_f(t) \\ \varphi_c(t) \\ \varphi_{oc}(t) \\ \varphi_{sw}(t) \end{pmatrix}$ for the rotation angles of the wheels around their horizontal axle of rotation.

The whole set of posture, angular, and rotation coordinates ξ, β_c, β_{oc}, and φ is called the set of *configuration coordinates* in the sequel. The total number of configuration coordinates is clearly$N_f + 2N_c + 2N_{oc} + N_{sw} + 3$.

With these notations the constraints can be written under the general matrix form:

$$J_1(\beta_c, \beta_{oc})R(\theta)\dot{\xi} + J_2\dot{\varphi} = 0 \tag{10}$$

$$C_1(\beta_c, \beta_{oc})R(\theta)\dot{\xi} + C_2\dot{\beta}_{oc} = 0 \tag{11}$$

with the following definitions.

a)

$$J_1(\beta_c, \beta_{oc}) \triangleq \begin{pmatrix} J_{1f} \\ J_{1c}(\beta_c) \\ J_{1oc}(\beta_{oc}) \\ J_{1sw} \end{pmatrix}$$

where $J_{1f}, J_{1c}, J_{oc}, J_{sw}$ are respectively an $(N_f \times 3)$, an $(N_c \times 3)$, an $(N_{oc} \times 3)$, and an $(N_{sw} \times 3)$ matrix whose forms derive readily from the constraints (3), (5), (7), and (9). J_{1f} and J_{sw} are constant, while J_{1c} and J_{1oc} are time varying respectively through $\beta_c(t)$ and $\beta_{oc}(t)$. J_2 is a constant $(N \times N)$ matrix whose diagonal entries are the radii of the wheels, except for the radii of the swedish wheels which are multiplied by $\cos\gamma$.

b)

$$C_1(\beta_c, \beta_{oc}) \triangleq \begin{pmatrix} C_1 f \\ C_{1c}(\beta_c) \\ C_{1oc}(\beta_{oc}) \end{pmatrix}, \quad C_2 \triangleq \begin{pmatrix} 0 \\ 0 \\ C_{2oc} \end{pmatrix}$$

where C_{1f}, C_{1c}, C_{1oc} are 3 matrices respectively of dimension $(N_f \times 3)$, $(N_c \times 3)$, $(N_{oc} \times 3)$ whose rows derive from the constraints (4), (6), and (8). C_{1f} is constant while C_{1c} and C_{1oc} are time varying. C_{2oc} is a diagonal matrix whose diagonal entries are equal to d for the N_{oc} off-centered orientable wheels.

We introduce the following assumption concerning the configuration of the swedish wheels.

A1: For each swedish wheel: $\gamma \neq \frac{\pi}{2}$. The value $\gamma = \frac{\pi}{2}$ would correspond to the direction of the zero component of the velocity being orthogonal to the plane of the wheel. Such a wheel would be subjected to a constraint identical to the nonslipping constraint of conventional wheels, hence loosing the benefit of implementing a swedish wheel.

Consider now the $(N_f + N_c)$ first constraints from (11) and written explicitly as

$$C_{1f}R(\theta)\dot{\xi} = 0 \tag{12}$$

$$C_{1c}(\beta_c)R(\theta)\dot{\xi} = 0. \tag{13}$$

These constraints imply that the vector $R(\theta)\dot{\xi}$ belong to the null space of the following matrix $C_1^*(\beta_c)$:

$$C_1^*(\beta_c) = \begin{pmatrix} C_{1f} \\ C_{1c}(\beta_c) \end{pmatrix} \tag{14}$$

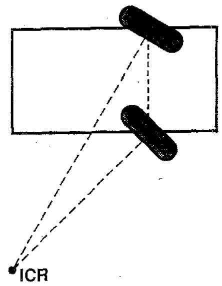

Fig. 5. Instantaneous center of rotation.

$$R(\theta)\dot{\xi} \in \mathcal{N}[C_1^*(\beta_c)]. \tag{15}$$

Obviously rank $[C_1^*(\beta_c)] \leq 3$. If rank $[C_1^*(\beta_c)] = 3$, then $R(\theta)\dot{\xi} = 0$ and any motion in the plane is impossible! More generally, the limitations of the mobility of the robot are related to the rank of C_1^*. This point will be discussed in detail hereafter.

Before that, it is important to notice that conditions (12) and (13) have an interesting geometrical interpretation. At each instant the motion of the robot can be viewed as an instantaneous rotation around the instantaneous center of rotation (ICR) whose position with respect to the frame can be time-varying. Hence at each instant the velocity vector of any point of the frame is orthogonal to the straight line joining this point and the ICR. In particular this is true for the centers of the conventional fixed and centered orientable wheels. This implies that at each time instant, the horizontal rotation axles of all the conventional fixed and centered orientable wheels are concurrent at the ICR. This fact is illustrated in Fig. 5 and is equivalent to the condition that rank $[C_1^*(\beta_c)] \leq 2$.

Obviously the rank of the matrix $C_1^*(\beta_c)$ depends on the design of the mobile robot. We define the *degree of mobility* δ_m of a mobile robot as

$$\delta_m = \dim \mathcal{N}[C_1^*(\beta_c)] = 3 - \text{rank}[C_1^*(\beta_c)].$$

Let us now examine the case rank$[C_{1f}] = 2$ which implies that the robot has at least 2 fixed wheels and, if there are more than 2, that their axles are concurrent to the ICR whose position with respect to the frame is *fixed*. In such a case, it is clear that the only possible motion is a rotation of the robot around a fixed ICR. Obviously this limitation is not acceptable in practice and we thus assume that rank$[C_{1f}] \leq 1$. We assume moreover that the robot is nondegenerate in the following sense.

A2: A mobile robot is nondegenerate if:

i) rank$C_{1f} \leq 1$;

ii) rank$[C_1^*(\beta_c)]$ =rankC_{1f}+rank$C_{1c}(\beta_c) \leq 2$.

This assumption is equivalent to the following conditions.

1) If the robot has more than one conventional fixed wheel (i.e., $N_f > 1$), then they are all on a single common axle.
2) The centers of the conventional centered orientable wheels do not belong to this common axle of the fixed wheels.
3) The number rank$C_{1c}(\beta_c) \leq 2$ is the number of conventional centered orientable wheels that can be oriented independently in order to steer the robot. We call this number the *degree of steeribility* δ_s:

$$\delta_s = \text{rank}C_{1c}(\beta_c).$$

The number and the choice of these δ_s steering wheels is obviously a privilege of the robot designer. If a mobile robot is equipped with more than δ_s conventional centered orientable wheels (i.e., $N_c > \delta_s$), the motion of the extra wheels must be coordinated to guarantee the existence of the Instantaneous Center of Rotation at each time instant.

It follows that only *five* nonsingular structures are of practical interest and such that:

1) the degree of mobility δ_m satisfies the following inequalities:

$$1 \leq \delta_m \leq 3. \tag{16}$$

(The upper bound is obvious. The lower ound means that we consider only the case where a motion is possible, i.e., $\delta_m \neq 0$.);

2) the degree of steeribility δ_s satisfies the following inequalities:

$$0 \leq \delta_s \leq 2. \tag{17}$$

(The upper bound can be reached only for robots without fixed wheels ($N_f = 0$), the lower bound corresponds to robots without centered orientable wheel ($N_c = 0$).); and

3) the following inequalities are satisfied:

$$2 \leq \delta_m + \delta_s \leq 3. \tag{18}$$

(The case $\delta_m + \delta_s = 1$ is not acceptable because it corresponds to the rotation of the robot around a **fixed** ICR as we have seen above. The cases ($\delta_m \geq 2, \delta_s = 2$) are excluded because, according to Assumption A2-ii, $\delta_s = 2$ implies that $\delta_m = 1$).

Hence, there exist only five types of wheeled mobile robots, corresponding to the five pairs of values of δ_m and δ_s that satisfy inequalities (16), (17), and (18) according to the following table:

δ_m	3	2	2	1	1
δ_s	0	0	1	1	2

In the sequel, we shall designate these types of structures by using a denomination of the form: "mobile robot of **Type** (δ_m, δ_s)."

The main design characteristics of each type of mobile robot are now briefly presented.

- **Type** (3,0). $\delta_m = \dim \mathcal{N}(C_1^*(\beta_c)) = 3, \delta_s = 0$. These robots have *no* conventional fixed wheels ($N_f = 0$) and *no* conventional centered orientable wheels ($N_c = 0$). Such robots are called *omnidirectional* because they have a full mobility in the plane which means that they can move at each instant in any direction without any reorientation. In contrast, the other four types of robots have a restricted mobility (degree of mobility less than 3). Examples of omnidirectional robots are the URANUS robot [9] and the UCL robot [1].

- **Type** (2,0). $\delta_m = \dim \mathcal{N}(C_1^*(\beta_c)) = \dim \mathcal{N}(C_{1f}) = 2, \delta_s = 0$. These robots have *no* conventional centered orientable wheels ($N_c = 0$). They have either one conventional fixed wheel or several conventional fixed wheels but with a single common axle (otherwise Rank $[C_{1f}]$ would be greater than 1). The mobility of the robot is restricted in the sense that, for any admissible trajectory $\xi(t)$, the velocity $\dot{\xi}(t)$ is constrained to belong to the two-dimensional distribution spanned by the vector fields $R^T(\theta)s_1$ and $R^T(\theta)s_2$, where s_1 and s_2 are two *constant* vectors spanning $\mathcal{N}(C_{1f})$. The well known robot HILARE [4] belongs to this class.
- **Type** (2,1). $\delta_m = \dim(C_1^*(\beta_c)) = \dim \mathcal{N}(C_{1c}(\beta_c)) = 2, \delta_s = 1$. These robots have *no* conventional fixed wheel ($N_f = 0$) and at least *one* conventional centered orientable wheel ($N_c \geq 1$). If there are more than one centered wheel, their orientations must be coordinated in such a way that rank $C_{1c}(\beta_c) = \delta_s = 1$. The velocity $\dot{\xi}(t)$ is constrained to belong to the two-dimensional distribution spanned by the vector fields $R^T(\theta)s_1(\beta_c)$ and $R^T(\theta)s_2(\beta_c)$ where $s_1(\beta_c)$ and $s_2(\beta_c)$ are two vectors spanning $\mathcal{N}(C_{1c}(\beta_c))$ and parametrized by the angle β_c of one arbitrary chosen conventional centered orientable wheel.
- **Type** (1,1). $\delta_m = \dim \mathcal{N}(C_1^*(\beta_c)) = 1, \delta_s = 1$. These robots have one or several conventional fixed wheels with a single common axle. They have also one or several conventional centered orientable wheels, with the condition that the center of one of them is *not* located on the axle of the conventional fixed wheels (otherwise the structure would be singular) and that their orientations are coordinated in such a way that rank $C_{1c}(\beta_c) = \delta_s = 1$. The velocity $\dot{\xi}(t)$ is constrained to belong to a one-dimensional distribution parametrized by the orientation angle of one arbitrarily chosen conventional centered orientable wheel. Mobile robots that are built on the model of a conventional car (often called *car-like* robots) belong to this class. Examples from the literature are the HERO 1 and the AVATAR robots (see [14] and [15]).
- **Type** (1,2). $\delta_m = \dim \mathcal{N}(C_1^*(\beta_c)) = \dim \mathcal{N}(C_{1c}(\beta_c)) = 1, \delta_s = 2$. These robots have *no* conventional fixed wheels ($N_f = 0$). They have at least *two* conventional centered orientable wheels ($N_c \geq 2$). If there are more than 2 centered wheels, their orientations must be coordinated in such a way that rank $C_{1c}(\beta_c) = \delta_s = 2$. The velocity $\dot{\xi}(t)$ is constrained to belong to a one-dimensional distribution parametrized by the orientation angles of two arbitrarily chosen conventional centered orientable wheels of the robot. A typical example is the KLUDGE robot ([16]).

III. EXAMPLES

We present in this section six concrete examples of mobile robots to illustrate the five types of nonsingular structures that have been presented above. We restrict our attention to robots with *three* wheels.

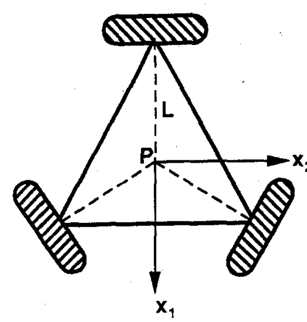

Fig. 6. Omnidirectionnal robot—Type (3,0). Three sweedish wheels.

As we have shown in Section II-B, the wheels of a mobile robot are described by (at most) six characteristic constants:

1) three angles α, β, γ; and
2) three lengths l, r, d.

For each example, we give successively a table with the numerical values of these characteristic constants and a presentation of the various matrices J and C involved in the mathematical expresions (10) and (11) of the constraints.

However, we shall assume that the radii r and the distances d are identical for all the wheels of all the examples. Hence, we will specify only the values of α, β, γ, l.

Example 1: Omnidirectional Robots with Swedish Wheels (Type (3,0)): The considered robot (see Fig. 6) has three swedish wheels located at the vertices of the frame that has the form of an equilateral triangle (see for instance, [1]):

Wheels	α	β	γ	l
$1sw$	$\pi/3$	0	0	L
$2sw$	π	0	0	L
$3sw$	$5\pi/3$	0	0	L

The constraints have the form (10) where

$$J_1 = [J_{1sw}] = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & L \\ 0 & -1 & L \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & L \end{pmatrix}, \quad J_2 = \mathrm{diag}(r).$$

Example 2: Omnidirectional Robots with Off-Centered Orientable Wheels (Type (3,0)): The robot has three conventional off-centered orientable wheels as shown in Fig. 7.

The constraints have the form (10) and (11) where

$$\begin{aligned} J_1 &= [J_{1oc}(\beta_{oc})] \\ &= \begin{pmatrix} -\sin\beta_{oc1} & \cos\beta_{oc1} & L\cos\beta_{oc1} \\ \sin\beta_{oc2} & -\cos\beta_{oc2} & L\cos\beta_{oc2} \\ \cos\beta_{oc3} & \sin\beta_{oc3} & L\cos\beta_{oc3} \end{pmatrix} \\ J_2 &= \mathrm{diag}(r) \\ C_1 &= [C_{1oc}(\beta_{oc})] \\ &= \begin{pmatrix} \cos\beta_{oc1} & \sin\beta_{oc1} & d + L\sin\beta_{oc1} \\ -\cos\beta_{oc2} & -\sin\beta_{oc2} & d + L\sin\beta_{oc2} \\ \sin\beta_{oc3} & -\cos\beta_{oc3} & d + L\sin\beta_{oc3} \end{pmatrix} \\ C_2 &= [C_{2oc}] = \mathrm{diag}\ (d). \end{aligned}$$

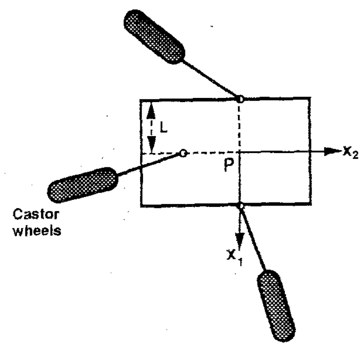

Fig. 7. Omnidirectional robot—Type (3,0). Three off-centered orientable wheels.

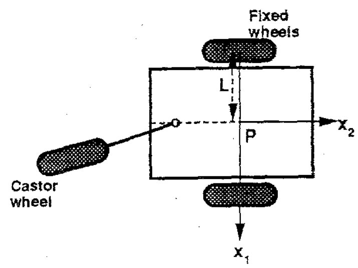

Fig. 8. Robot of Type (2,0).

Wheels	α	β	l
$1oc$	0	–	L
$2oc$	π	–	L
$3oc$	$3\pi/2$	–	L

Example 3: Type (2,0): Robot with two conventional fixed wheels on the same axle and one conventional off-centered orientable wheel, Fig. 8.

Wheels	α	β	l
$1f$	0	0	L
$2f$	π	0	L
$3oc$	$3\pi/2$	–	L

The constraints have the form (10) and (11) where

$$\begin{aligned} J_1 &= \begin{pmatrix} J_{1f} \\ J_{1oc} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & L \\ 0 & -1 & L \\ \cos\beta_{oc3} & \sin\beta_{oc3} & L\cos\beta_{oc3} \end{pmatrix} \\ J_2 &= \operatorname{diag}(r) \\ C_1 &= \begin{pmatrix} C_{1f} \\ C_{1oc} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ \sin\beta_{oc3} & -\cos\beta_{oc3} & d+L\sin\beta_{oc3} \end{pmatrix} \end{aligned} \tag{19}$$

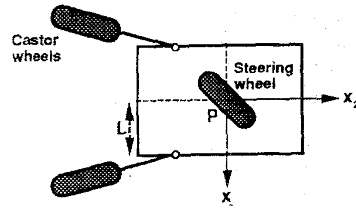

Fig. 9. Robot of Type (2,1).

$$C_2 = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}. \tag{20}$$

We note that the nonslipping constraints of to the 2 fixed wheels are equivalent (see the first 2 rows of C_1). Hence, the matrix C_1^* has a rank equal to 1 as expected.

Example 4: Type (2,1): Robot with one conventional centered orientable wheel and two conventional off-centered orientable wheels, Fig. 9.

Wheels	α	β	l
$1c$	0	–	0
$2oc$	$\frac{5\pi}{4}$	–	$\sqrt{2}L$
$3oc$	$\frac{7\pi}{4}$	–	$\sqrt{2}L$

The constraints have the form (10) and (11) where

$$\begin{aligned} J_1 &= \begin{pmatrix} J_{1c}(\beta_{c1}) \\ J_{1oc}(\beta_{oc2}, \beta_{oc3}) \end{pmatrix} \\ &= \begin{pmatrix} -\sin\beta_{c1} & \cos\beta_{c1} & l\cos\beta_{c1} \\ \sin(\beta_{oc2}+\frac{\pi}{4}) & -\cos(\beta_{oc2}+\frac{\pi}{4}) & l\cos\beta_{oc2} \\ -\sin(\beta_{oc3}-\frac{\pi}{4}) & \sin(\beta_{oc3}-\frac{\pi}{4}) & l\cos\beta_{oc3} \end{pmatrix} \\ J_2 &= \operatorname{diag}(r) \\ C_1 &= \begin{pmatrix} C_{1c}(\beta_{c1}) \\ C_{1oc}(\beta_{oc2}, \beta_{oc3}) \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta_{c1} & \sin\beta_{c1} & l\sin\beta_{c1} \\ -\cos(\beta_{oc2}+\frac{\pi}{4}) & -\sin(\beta_{oc2}+\frac{\pi}{4}) & d+l\sin\beta_{oc2} \\ \cos(\beta_{oc3}-\frac{\pi}{4}) & \sin(\beta_{oc3}-\frac{\pi}{4}) & d+l\sin\beta_{oc3} \end{pmatrix} \\ C_2 &= \begin{pmatrix} 0 \\ \operatorname{diag} d \end{pmatrix}. \end{aligned}$$

Example 5: Type (1,1): Robot with two conventional fixed wheels on the same axle and one conventional centered orientable wheel (like the tricycles of the kids), Fig. 10.

Wheels	α	β	l
$1f$	0	0	L
$2f$	π	0	L
$3c$	$3\pi/2$	–	L

The constraints have the form (10) and (11) where

$$\begin{aligned} J_1 &= \begin{pmatrix} J_{1f} \\ J_{1c} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & L \\ 0 & -1 & L \\ \cos\beta_{c3} & \sin\beta_{c3} & L\cos\beta_{c3} \end{pmatrix} \end{aligned}$$

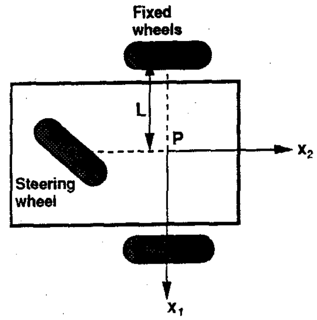

Fig. 10. Robot of Type (1,1).

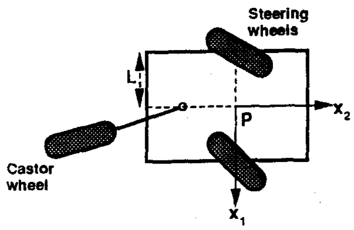

Fig. 11. Robot of Type (1,2).

$$J_2 = \mathrm{diag}(r)$$
$$C_1 = \begin{pmatrix} C_{1f} \\ C_{1c} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ \sin\beta_{c3} & -\cos\beta_{c3} & L\sin\beta_{c3} \end{pmatrix}$$
$$C_2 = 0.$$

Example 6: Type (1,2): Robot with two conventional centered orientable wheels and one conventional off-centered orientable wheel, Fig. 11.

Wheels	α	β	l
$1c$	0	$-$	L
$2c$	π	$-$	L
$3oc$	$3\pi/2$	$-$	L

The constraints have the form (10) and (11) where

$$J_1 = \begin{pmatrix} J_{1c}(\beta_{c1},\beta_{c2}) \\ J_{1oc}(\beta_{oc3}) \end{pmatrix} = \begin{pmatrix} -\sin\beta_{c1} & \cos\beta_{c1} & L\cos\beta_{c1} \\ \sin\beta_{c2} & -\cos\beta_{c2} & L\cos\beta_{c2} \\ \cos\beta_{oc3} & \sin\beta_{oc3} & L\cos\beta_{oc3} \end{pmatrix}$$
$$J_2 = \mathrm{diag}\ (r) \tag{21}$$
$$C_1 = \begin{pmatrix} C_{1c}(\beta_{c1},\beta_{c2}) \\ C_{1oc}(\beta_{oc3}) \end{pmatrix} = \begin{pmatrix} \cos\beta_{c1} & \sin\beta_{c1} & L\sin\beta_{c1} \\ -\cos\beta_{c2} & -\sin\beta_{c2} & L\sin\beta_{c2} \\ \sin\beta_{oc3} & -\cos\beta_{oc3} & d+L\sin\beta_{oc3} \end{pmatrix}$$
$$C_2 = \begin{pmatrix} 0 \\ C_{2oc} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}. \tag{22}$$

IV. The Posture Kinematic Model

In this section, the analysis of the mobility, as discussed in Section II-C, is reformulated into a state space form which will be useful for our subsequent developments.

We have shown (see (15)) that, whatever the type of mobile robot, the velocity $\dot{\xi}(t)$ is restricted to belong to a distribution Δ_c defined as

$$\dot{\xi}(t) \in \Delta_c \triangleq \text{ span }\{\text{col } R^T(\theta)\Sigma(\beta_c)\} \qquad \forall t$$

where the columns of the matrix $\Sigma(\beta_c)$ form a basis of $\mathcal{N}J[C_1^*(\beta_c)]$:

$$\mathcal{N}J[C_1^*(\beta_c)] = \text{span}\{\text{col}\Sigma(\beta_c)\}.$$

This is trivially equivalent to the following statement: *for all t, there exists a time varying vector $\eta(t)$ such that*

$$\dot{\xi} = R^T(\theta)\Sigma(\beta_c)\eta. \tag{23}$$

The dimension of the distribution Δ_c and hence of the vector $\eta(t)$ is the degree of mobility δ_m of the robot. Obviously, in the case where the robot has no conventional centered orientable wheels ($\delta_s = 0$), the matrix Σ is constant and the expression (23) reduces to

$$\dot{\xi} = R^T(\theta)\Sigma\eta. \tag{24}$$

In the opposite case ($\delta_s \geq 1$), the matrix Σ explicitely depends on the angular coordinates β_c and the expression (23) can be augmented as follows:

$$\dot{\xi} = R^T(\theta)\Sigma(\beta_c)\eta \tag{25}$$
$$\dot{\beta}_c = \zeta. \tag{26}$$

This representation (24) or (25) and (26) can be regarded as a state space representation of the system (called the *posture kinematic model*), with the posture coordinates ξ and (possibly) the angular coordinates β_c as state variables while η and ζ (that are homogeneous to velocities) can be interpreted as control inputs entering the model linearly. This interpretation must however be taken with some care since the true physical control inputs of a mobile robot are the torques provided by the embarked motors: the kinematic state space model is in fact only a subsystem of the general dynamical model that will be presented in Section VI.

In the next subsections we derive the posture kinematic models corresponding to the examples of Section III, and show that these models are generic and irreducible. This will allow to discuss the controllability and the stabilizability of the posture kinematic models of *all* WMR.

A. The Five Generic Models of Wheeled Mobile Robots

We rewrite the posture kinematic model in the following compact form:

$$\dot{z} = B(z)u \tag{27}$$

with either (when $N_c = 0$)

$$z \triangleq \xi \quad B(z) \triangleq R^T(\theta)\Sigma \quad u \triangleq \eta$$

TABLE I
THE 5 GENERIC POSTURE KINEMATIC MODELS OF WHEELED MOBILE ROBOTS

Type	Example	z	$\Sigma(\beta_c)$ or Σ	Equations of the posture kinematic model $\dot{z} = B(z)u$
(3,0)	1 & 2	x y θ	Identity Matrix	$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}$
(2,0)	3	x y θ	$\begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}-\sin\theta & 0\\ \cos\theta & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$
(2,1)	4	x y θ β_{c1}	$\begin{pmatrix}-\sin\beta_{c1} & 0\\ \cos\beta_{c1} & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_{c1}\end{pmatrix} = \begin{pmatrix}-\sin(\theta+\beta_{c1}) & 0 & 0\\ \cos(\theta+\beta_{c1}) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \zeta_1\end{pmatrix}$
(1,1)	5	x y θ β_{c3}	$\begin{pmatrix}0\\ L\sin\beta_{c3}\\ \cos\beta_{c3}\end{pmatrix}$	$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_{c3}\end{pmatrix} = \begin{pmatrix}-L\sin\theta\sin\beta_{c3} & 0\\ L\cos\theta\sin\beta_{c3} & 0\\ \cos\beta_{c3} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}$
(1,2)	6	x y θ β_{c1} β_{c2}	$\begin{pmatrix}-2L\sin\beta_{c1}\sin\beta_{c2}\\ L\sin(\beta_{c1}+\beta_{c2})\\ \sin(\beta_{c2}-\beta_{c1})\end{pmatrix}$	$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_{c1}\\ \dot{\beta}_{c2}\end{pmatrix} = \begin{pmatrix}-L(\sin\beta_{c1}\sin(\theta+\beta_{c2})+\sin\beta_{c2}\sin(\theta+\beta_{c1})) & 0 & 0\\ L(\sin\beta_{c1}\cos(\theta+\beta_{c2})+\sin\beta_{c2}\cos(\theta+\beta_{c1})) & 0 & 0\\ \sin(\beta_{c2}-\beta_{c1}) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \zeta_1\\ \zeta_2\end{pmatrix}$

or (when $N_c \geq 0$)

$$z \triangleq \begin{pmatrix}\xi\\ \beta_c\end{pmatrix} \quad B(z) \triangleq \begin{pmatrix}R^T(\theta)\Sigma(\beta_c) & 0\\ 0 & I\end{pmatrix} \quad u \triangleq \begin{pmatrix}\eta\\ \zeta\end{pmatrix}.$$

We give in Table I the particular form of $z, \Sigma(\beta_c)$ or Σ and the equations of the posture kinematic models for the examples that have been presented in Section III.

A natural question arises: for each class of WMR, is this posture kinematic model *generic* for all the robots belonging to this class ? The answer is affirmative: for any nondegenerate robot it is always possible to select the reference point P and the basis $\{\vec{x_1}, \vec{x_2}\}$ attached to the robot frame in such way that the posture kinematic model takes exactly the form corresponding to the type of the robot, as given in Table I, as follows.

- For a Type (3,0) robot, the reference point P and the basis $\{\vec{x_1}, \vec{x_2}\}$ can be chosen arbitrarily.
- For a Type (2,0) robot, P is chosen as a point of the axle of the fixed wheels, with $\vec{x_1}$ aligned along this axle. See, for example, Fig 8.
- For a Type (2,1) robot, we select one of the centered wheels and P is choosen as the center of this wheel, with $\{\vec{x_1}, \vec{x_2}\}$ chosen arbitrarily. See, for example, Fig 9.
- For a Type (1,1) robot we select one of the centered wheels. P is the foot of the perpendicular drawn, from the center of the selected centered wheel onto the common axle of the fixed wheels. L is the length of this perpendicular. See, for example, Fig 10.
- For a Type (1,2) robot, we select 2 centered orientable wheels. P is chosen as the mid-distance point between the centers of these 2 wheels, with $\vec{x_1}$ aligned along the line joining their centers; L is the half distance between these centers. See, for example, Fig 11.

B. Mobility, Steeribility, and Maneuverability

This kinematic posture model allows to discuss further the maneuverability of WMR. The *degree of mobility* δ_m is a first criterion of the maneuverability: it is equal to the number of degrees of freedom that can be **directly** manipulated from the inputs η, without reorientation of the centered wheels. Intuitively, it corresponds to how many "degrees of freedom" the robot could have instantaneously from its current configuration, without steering any of its wheels. This δ_m is not equal to the overall number of "degrees of freedom" of the robot that can be manipulated from the inputs η and ζ. In fact this number is equal to the sum $\delta_M = \delta_m + \delta_s$ that we could call *degree of maneuverability*. It includes the δ_s additional degrees of freedom that are accessible from the inputs ζ. But the action of ζ on the posture coordinates ξ is indirect, since it is achieved only through the coordinates β_c, that are related to ζ by an integral action. This reflects the fact that the modification of the orientation of a centered wheel can not be achieved instantaneously.

The maneuverability of a WRM depends on δ_M, but also on the way how these δ_M degrees of freedom are partitionned into δ_m and δ_s. Therefore 2 indices are needed to characterize the maneuverability: δ_M and δ_m, or, equivalently, δ_m and δ_s, which are the 2 indices identifying the five classes of robots in Table I.

Two robots with the same value of δ_M, but different δ_m, are not equivalent. For robots with $\delta_M = 3$, it is possible to freely assign the position of the ICR, either directly from η, for the Type (3,0) robots, or by orientation of 1 or 2 centered wheels for the Type (2,1) and (1,2) robots. For robots with $\delta_M = 2$, the ICR is constrained to belong to a straight line (the axle of the fixed wheel). Its position on this line is assigned either directly for Type (2,0) robots, or by the orientation of a centered wheel for Type (1,1) robots.

Similarly, two WRM with the same value of δ_m, but different δ_M, are not equivalent: the robot with the largest δ_M is more manoeuvrable. Compare, for instance, a Type (1,1) and a Type (1,2) robots, with $\delta_m = 1$, and, respectively, $\delta_M = 2$ and $\delta_M = 3$. The position of the ICR for the Type (1,2) robot can be assigned freely in the plane, just by orienting 2 centered wheels, while for the Type (1,1), the ICR is constrained to belong to the axle of the fixed wheels, its position on this axle being specified by the orientation of the centered wheel. Since the steering directions of the centered wheels can usually be changed very quickly, especially for small indoor robots, it results, from a practical veiwpoint, that the Type (1,2) robot is more manoeuvrable than the Type (1,1).

Obviously, the ideal situation is that of omnidirectional robots where $\delta_m = \delta_M = 3$.

C. Irreducibility of the Posture Kinematic Model

In this section, we address the question of the reducibility of the posture kinematic model (27). A state space model is reducible if there exists a change of coordinates such that some of the new coordinates are identically zero along the motion of the system. For a nonlinear dynamical system wihout drift like (27), the reducibility is related to the involutive closure $\bar{\Delta}$ of the following distribution Δ, expressed in local coordinates as:

$$\Delta(z) \triangleq \text{span}\{\text{col}(B(z))\}.$$

It is a well known consequence of Frobenius Theorem that the system is reducible only if $\dim \bar{\Delta} < \dim z$.

In this section, we shall prove that the posture kinematic model of nondegenerate mobile robots (see Assumption A2) is always irreducible. To establish this result, we proceed by first analyzing in details the particular case of the robot of Type (1,1) whose posture kinematic model is as follows (see Table I):

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\beta}_{c3} \end{pmatrix}}_{\dot{z}} = \underbrace{\begin{pmatrix} -L\sin\theta\sin\beta_{c3} & 0 \\ L\cos\theta\sin\beta_{c3} & 0 \\ \cos\beta_{c3} & 0 \\ 0 & 1 \end{pmatrix}}_{B(z)} \underbrace{\begin{pmatrix} \eta_1 \\ \zeta_1 \end{pmatrix}}_{u}. \tag{28}$$

In this particular case, a basis of $\bar{\Delta}(z)$ is as follows:

$$\bar{\Delta}(z) = \text{span}\{b_1(z), b_2(z), b_3(z), b_4(z)\}$$

with $b_1(z), b_2(z)$ the columns of $B(z)$:

$$b_1(z) = \begin{pmatrix} -L\sin\theta\sin\beta_{c3} \\ L\cos\theta\sin\beta_{c3} \\ \cos\beta_{c3} \\ 0 \end{pmatrix} \quad b_2(z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$b_3(z) = [b_1(z), b_2(z)] = \begin{pmatrix} L\sin\theta\cos\beta_{c3} \\ -L\cos\theta\cos\beta_{c3} \\ \sin\beta_{c3} \\ 0 \end{pmatrix}$$

$$b_4(z) = [b_1(z), b_3(z)] = \begin{pmatrix} L\cos\theta \\ L\sin\theta \\ 0 \\ 0 \end{pmatrix}.$$

We see that rank $B(z) = \delta_m + \delta_s = 2$ and $\dim \bar{\Delta}(z) = \dim z = 4$ everywhere in the state space. It follows that the kinematic state space model (28) of a robot of Type (1,1) is irreducible.

Furthermore, the same line of reasoning that has been followed for Type (1,1) can be followed easily for any Type of mobile robot. It appears that all posture kinematic models of Table I are irreducible. We summarize this analysis in a Property.

Property 1:

1) For the posture kinematic model $\dot{z} = B(z)u$ of wheeled mobile robots:

   a) the input matrix $B(z)$ has full rank:

   $$\text{rank } B(z) = \delta_m + \delta_s \quad \text{for all } z.$$

   b) the involutive distribution $\bar{\Delta}(z) \triangleq$ inv span $\{\text{col } B(z)\}$ has constant maximal dimension:

   $$\dim \bar{\Delta}(z) = 3 + \delta_s \quad \text{for all } z.$$

2) Consequently, the posture kinematic model of wheeled mobile robots is irreducible. This is a coordinate free property. ■

D. Controllability, Feedback Linearizability, and Stabilizability

In this section, we summarize the main controllability and feedback stabilizability properties of the posture kinematic model of wheeled mobile robots. We first examine the linear approximation around an arbitrary equilibrium state $\bar{z} \triangleq (\bar{\xi}, \bar{\beta}_c)$. Equilibrium means that the robot is at rest somewhere, with a given constant posture $\bar{\xi}$ and a given constant orientation $\bar{\beta}_c$ of the orientable centered wheels. Obviously, the velocities are zero: $\bar{u} = 0$.

Property 2: The controllability rank of the linear approximation of the posture kinematic model $\dot{z} = B(z)u$ around an equilibrium state is $\delta_m + \delta_s$.

Indeed, the linear approximation around $(\bar{z}, \bar{u} = 0)$ is written:

$$\frac{d}{dt}(z - \bar{z}) = B(\bar{z})u.$$

It follows that the controllability matrix reduces to $B(\bar{z})$ whose rank is $\delta_m + \delta_s$ for all $\bar{z}$ by Property 1. ■

This implies that the linear approximation of the posture kinematic model of *omnidirectional* robots (Type (3,0)) is completely controllable since δ_m is precisely the state dimension in this case while it is *not* controllable for *restricted mobility* robots (Types (2,0), (2,1), (1,1), (1,2), $\delta_m \leq 2$) since $\delta_m + \delta_s < 3 + \delta_s = \dim z$.

This property, however, does not prevent restricted mobility robots from being **controllable**, in accordance with the physical intuition.

Property 3: The posture kinematic model $\dot{z} = B(z)u$ of wheeled mobile robots is controllable.

This follows from the fact that the involutive distribution $\bar{\Delta}(z)$ has constant maximal rank which implies controllability for driftless systems (see e.g., [17], ch. 3). ■

Practically, this property means that a mobile robot can always be driven from any initial posture ξ_0 to any final one ξ_f, in a finite time, by manipulating the velocity control input $u = (\eta, \zeta)^T$.

Let us now consider the question of the existence of a feedback control $u(z)$ able to **linearize** a mobile robot at a particular state z^*.

For omnidirectional robots, the answer to that question is obvious. For example:

$$u(z) = B(z)^{-1}A(z - z^*)$$

with A an arbitrary Hurwitz matrix, is clearly a *linearizing* smooth feedback control law that drives the robot exponentially to z^*. Indeed, the closed loop is described by the freely assignable linear dynamics:

$$\frac{d}{dt}(z - z^*) = A(z - z^*).$$

Hence, omnidirectional mobile robots are full state feedback linearizable and therefore quite similar to fully actuated robotic manipulators.

For restricted mobility robots, the situation is less favorable since it appears from Property 2 that they are certainly not full state linearizable (the controllability of the linear approximation is necessary for that, see e.g., [18], ch. 4). But we can address the question of the dimension of the largest subsystem which is linearizable by static state feedback or the question whether the posture kinematic model is full state linearizable by dynamic feedback.

We have the following properties.

Property 4:

a) The dimension of the largest feedback linearizable subsystem of the posture kinematic model $\dot{z} = B(z)u$ by static state feedback is $\delta_m + \delta_s$. This result is a straightforward application of the algorithm of Marino [19].

b) The posture kinematic model $\dot{z} = B(z)u$ of restricted mobility robots is a "differentially flat system" in the sense of [7] and [20]. This implies that it is full state linearizable by dynamic state feedback provided the η-part of the input vector u is nonzero. A discussion of this property including an explicit derivation of the linearizing outputs for each type of robot can be found in [21]. ■

Another interesting problem is that one of designing a smooth feedback control driving the robot from any state $z_0 = (\xi_0, \beta_{c0})^T$ to a given equilibrium state $\bar{z} = (\bar{\xi}, \bar{\beta})^T$, and making this equilibrium stable and attractive. Here again, for omnidirectional robots the solution is trivial while for restricted mobility robots we have the following properties.

Property 5:

a) For restricted mobility robots the posture kinematic model $\dot{z} = B(z)u$ is not stabilizable by a continuous static time invariant state feedback $u(z)$.

Indeed the necessary condition of Brockett [22] is not satisfied: the map $(z, u) \to B(z)u$ is not onto on a neighborhood of the equilibrium $\bar{z} = (\bar{\xi}, \bar{\beta}_c)^T, \bar{u} = 0$.

b) The posture kinematic model is stabilizable by a continuous *time varying* static state feedback $u(z, t)$.

This result is a special case of a general stabilizability result for driftless systems presented by Coron in [23]. A systematic procedure for the design of such stabilizing time varying feedback controls has been proposed by Pomet [24]. It is applicable to all the posture kinematic models because in each case (see Table I) one column of the matrix $B(z)$ is of the form $(0, \ldots, 0, 1)^T$. ■

V. The Configuration Kinematic Model

So far, we have used only a subset of the constraints (10) and (11): namely that part of the constraints which is relative to the fixed and centered orientable wheels, expressed by (12) and (13). The remaining constraints are now used to derive the equations of the evolution of the angular and rotation velocities $\dot{\beta}_{oc}$ and $\dot{\varphi}$ not involved in the posture kinematic model (25) and (26).

It results directly from (10) and (11) that

$$\dot{\beta}_{oc} = -C_{2oc}^{-1}C_{1oc}(\beta_{oc})R(\theta)\dot{\xi} \tag{29}$$

$$\dot{\varphi} = -J_2^{-1}J_1(\beta_c, \beta_{oc})R(\theta)\dot{\xi}. \tag{30}$$

Combining these equations with the posture kinematic model (25) the state equations for β_{oc} and φ are written as

$$\dot{\beta}_{oc} = D(\beta_{oc})\Sigma(\beta_c)\eta \tag{31}$$

$$\dot{\varphi} = E(\beta_c, \beta_{oc})\Sigma(\beta_c)\eta \tag{32}$$

with the following definition of $D(\beta_{oc})$ and $E(\beta_c, \beta_{oc})$:

$$D(\beta_{oc}) \triangleq -C_{2oc}^{-1}C_{1oc}(\beta_{oc})$$

$$E(\beta_c, \beta_{oc}) \triangleq -J_2^{-1}J_1(\beta_c, \beta_{oc}).$$

We note also that these matrices satisfy the following equations:

$$J_1(\beta_c, \beta_{oc}) + J_2E(\beta_c, \beta_{oc}) = 0 \tag{33}$$

$$C_{1oc}(\beta_{oc}) + C_{2oc}D(\beta_{oc}) = 0. \tag{34}$$

Defining q as the vector of the configuration coordinates, i.e.,

$$q \triangleq \begin{pmatrix} \xi \\ \beta_c \\ \beta_{oc} \\ \varphi \end{pmatrix} \tag{35}$$

the evolution of the configuration coordinates can be described by the following compact equation, resulting from (25), (26), (31), and (32) called the *configuration kinematic model*

$$\dot{q} = S(q)u \tag{36}$$

where

$$S(q) \triangleq \begin{pmatrix} R^T(\theta)\Sigma(\beta_c) & 0 \\ 0 & I \\ D(\beta_{oc})\Sigma(\beta_c) & 0 \\ E(\beta_c, \beta_{oc})\Sigma(\beta_c) & 0 \end{pmatrix} \text{ and } u \triangleq \begin{pmatrix} \eta \\ \zeta \end{pmatrix}. \tag{37}$$

Equation (36) has the standard form of the kinematic model of a system subjected to independent velocity constraints. We now connect this formulation with the standard theory of nonholonomic mechanical systems (see, e.g., [25]–[27]).

The reducibility of (36) is directly related to the dimension of the involutive closure of the distribution Δ_1 spanned in local coordinates q by the columns of the matrix $S(q)$, i.e.,

$$\Delta_1(q) \triangleq \text{span}\{\text{col } (S(q)\}. \tag{38}$$

It results immediately that

$$\begin{aligned}\delta_m + N_c &= \dim(\Delta_1) \leq \dim(\text{inv}(\Delta_1)) \leq \dim(q)\\ &= 3 + N_c + N_{oc} + N.\end{aligned}$$

We define the *degree of nonholomy* M of a mobile robot as

$$M = \dim(\text{inv}(\Delta_1)) - (\delta_m + N_c).$$

This number M represents the number of velocity constraints that are not integrable and cannot therefore be eliminated whatever the choice of the generalized coordinates. It must be pointed out that this number depends on the particular structure of the robot and therefore has not the same value for all the robots belonging to a given class.

On the other hand, for a particular choice of generalized coordinates, the number of coordinates that can be eliminated by integration of the constraints is equal to the difference between $\dim(q)$ and $\dim(\text{inv}\Delta_1)$.

Property 6:

a) The configuration kinematic model $\dot{q} = S(s)u$ of WMR is nonholonomic i.e., $M > 0$ for all types of mobile robots.
b) The configuration kinematic model $\dot{q} = S(q)u$ of wheeled mobile robots is reducible, i.e., $\dim(q) > \dim(\text{inv}(\Delta_1))$ for all types of mobile robots. ■

This property is not contradictory with the irreducibility of the posture kinematic state space model (25) and (26), as discussed in Section IV-A: the reducibility of (36) means that there exists at least one smooth function of $(\xi, \beta_c, \beta_{oc}, \varphi)$, involving explicitely at least one of the variables (β_{oc}, φ) that is constant along the trajectories of the system compatible with all the constraints (10) and (11).

This discussion is illustrated with two examples.

Example 1: Type (3,0): Omnidirectional robot with 3 swedish wheels (Fig. 6.).

For this robot, $\delta_m = 3$ and the configuration coordinates are

$$q = (x\ y\ \theta\ \varphi_1\ \varphi_2\ \varphi_3)^T.$$

The configuration model is characterized by $S(q)$, defined as

$$S(q) = \begin{pmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\\ -\frac{\sqrt{3}}{2r} & \frac{1}{2r} & \frac{L}{r}\\ 0 & -\frac{1}{r} & \frac{L}{r}\\ \frac{\sqrt{3}}{2r} & \frac{1}{2r} & \frac{L}{r}\end{pmatrix}.$$

It is easy to check that

$$\dim(\Delta_1) = 3 \text{ and } \dim(\text{inv } (\Delta_1)) = 5.$$

It results that the degree of nonholonomy is equal to $5-3=2$, while the number of coordinates that can be eliminated is equal to $6-5=1$. In fact, the structure of the configuration model implies that

$$\dot{\varphi}_1 + \dot{\varphi}_2 + \dot{\varphi}_3 = -\frac{3L}{r}\dot{\theta}.$$

This means that $(\varphi_1 + \varphi_2 + \varphi_3 + \frac{3L}{r}\theta)$ is constant along any trajectory compatible with the constraints. It is then possible to eliminate one of the four variables $\varphi_1, \varphi_2, \varphi_3, \theta$.

Example 3: Type (2,0): Robot with 2 fixed wheels and one off-centered orientable wheel (Fig. 8).

For this robot, $\delta_m = 2$ and the configuration coordinates are

$$q = (x\ y\ \theta\ \beta_{oc3}\ \varphi_1\ \varphi_2\ \varphi_3)^T.$$

The matrix $S(q)$ is as follows:

$$S(q) = \begin{pmatrix} -\sin\theta & 0\\ \cos\theta & 0\\ 0 & 1\\ \frac{1}{d}\cos\beta_{oc3} & -\frac{1}{d}(d + L\sin\beta_{oc3})\\ -\frac{1}{r} & -\frac{L}{r}\\ \frac{1}{r} & -\frac{L}{r}\\ -\frac{1}{r}\sin\beta_{oc3} & -\frac{L}{r}\cos\beta_{oc3}\end{pmatrix}.$$

It can be checked that

$$\dim(\Delta_1) = 2 \text{ and } \dim(\text{inv } \Delta_1) = 6.$$

It results that the degree of nonholonomy is equal to $6-2=4$, and the number of coordinates that can be eliminated is equal to $7-6=1$. It results from the configuration model that

$$\dot{\varphi}_1 + \dot{\varphi}_2 = \frac{-2L}{r}\dot{\theta}.$$

This means that the variable $(\varphi_1 + \varphi_2 + \frac{2L}{r}\theta)$ has a constant value along any trajectory compatible with the constraints.

VI. The Configuration Dynamical Model

The aim of this section is the derivation of a general dynamical state space model of wheeled mobile robots describing the dynamical relations between the configuration coordinates $\xi, \beta_{oc}, \varphi, \beta_c$ and the torques developed by the embarked motors.

This general state space model will be called "configuration dynamical model" and is made up of six kinds of state equations: one for each of the coordinates $\xi, \beta_{oc}, \varphi, \beta_c$ and one for each of the internal coordinates η and ζ that were introduced in Section IV. The state equations for ξ, β_c, β_{oc}, and φ have been derived in Section V under the form of the configuration kinematic model. The state equation for η and ζ will be established in Section VI-A using the Lagrange formalism. The configuration of the motorisation will be discussed in Section VI-B.

A. Derivation of the Configuration Dynamical Model

We assume that the robot is equipped with motors that can force either the orientation of the orientable wheels (angular coordinates β_c and β_{oc}) or the rotation of the wheels (rotation coordinates φ). The torques provided by the motors are denoted as follows:

- τ_φ for the rotation of the wheels;
- τ_{oc} for the orientation of the off-centered wheels; and
- τ_c for the orientation of the centered wheels.

Using the Lagrange formalism, the dynamics of wheeled mobile robots are described by the following $(3 + N_{oc} + N + N_c)$ Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) - \frac{\partial T}{\partial \xi} = R^T(\theta) J_1^T(\beta_c, \beta_{oc})\lambda + R^T(\theta) C_1^T(\beta_c, \beta_{oc})\mu \quad (39)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}_{oc}}\right) - \frac{\partial T}{\partial \beta_{oc}} = C_2^T \mu + \tau_{oc} \quad (40)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = J_2^T \lambda + \tau_\varphi \quad (41)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}_c}\right) - \frac{\partial T}{\partial \beta_c} = \tau_c \quad (42)$$

where T represents the kinetic energy and λ, μ are the Lagrange coefficients associated with the constraints (10) and (11), respectively.

In order to eliminate the Lagrange coefficients, we proceed as follows. The first three Lagrange equations (39)–(41) are premultiplied by the matrices $\Sigma^T(\beta_c)R(\theta), \Sigma^T(\beta_c)D(\beta_{oc})$ and $\Sigma^T(\beta_c)E(\beta_c, \beta_{oc})$, respectively, and then summed up. This leads to the two following equations, from which the Lagrange coefficients have disappeared owing to (33) and (34):

$$\Sigma^T(\beta_c)R(\theta)[T]_\xi + D(\beta_{oc})[T]_{\beta_{oc}} + E(\beta_c, \beta_{oc})[T]_\varphi = \Sigma^T(\beta_c)\{D^T(\beta_{oc})\tau_{oc} + E^T(\beta_c, \beta_{oc})\tau_\varphi\} \quad (43)$$

$$[T]_{\beta_c} = \tau_c \quad (44)$$

with the compact notation

$$[T]_\psi \triangleq \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi}.$$

The kinetic energy of wheeled mobile robots can be expressed as follows:

$$T = \dot{\xi}^T R^T(\theta)[M(\beta_{oc})R(\theta)\dot{\xi} + 2V(\beta_{oc})\dot{\beta}_{oc} + 2W\dot{\beta}_c] + \dot{\beta}_{oc} I_{oc} \dot{\beta}_{oc} + \dot{\varphi} I_\varphi \dot{\varphi} + \dot{\beta}_c I_c \dot{\beta}_c$$

with appropriate definitions of the matrices $M(\beta_{oc})$, $V(\beta_{oc}), W, I_{oc}, I_\varphi, I_c$ which are dependent of the mass distribution and the inertia moments of the various rigid bodies (frame and wheels) that constitute the robot. The state equations for η and ζ are then obtained (after rather lenghty calculations) by substituting this expression of T in the dynamical equations (43) and (44) and eliminating the velocities $\dot{\xi}, \dot{\beta}_{oc}, \dot{\varphi}, \dot{\beta}_c$ and the accelerations $\ddot{\xi}, \ddot{\beta}_{oc}, \ddot{\varphi}, \ddot{\beta}_c$ with the aid of the kinematic equations (25), (26), (31), (32), and their derivatives.

The general dynamical state space model of wheeled mobile robots then takes the following general form:

General configuration dynamical model:

$$\dot{\xi} = R^T(\theta)\Sigma(\beta_c)\eta \quad (45)$$

$$\dot{\beta}_c = \zeta \quad (46)$$

$$\dot{\beta}_{oc} = D(\beta_{oc})\Sigma(\beta_c)\eta \quad (47)$$

$$H_1(\beta_c, \beta_{oc})\dot{\eta} + \Sigma^T(\beta_c)V(\beta_{oc})\dot{\zeta} + f_1(\beta_c, \beta_{oc}, \eta, \zeta) = \Sigma^T(\beta_c)[D^T(\beta_{oc})\tau_{oc} + E^T(\beta_c, \beta_{oc})\tau_\varphi] \quad (48)$$

$$V^T(\beta_{oc})\Sigma(\beta_c)\dot{\eta} + I_c\dot{\zeta} + f_2(\beta_c, \beta_{oc}, \eta, \zeta) = \tau_c \quad (49)$$

$$\dot{\varphi} = E(\beta_{oc}, \beta_c)\Sigma(\beta_c)\eta \quad (50)$$

with

$$H_1(\beta_c, \beta_{oc}) \triangleq \Sigma^T(\beta_c)[M(\beta_{oc}) + D^T(\beta_{oc})V^T(\beta_{oc}) + V(\beta_{oc})D(\beta_{oc}) + D^T(\beta_{oc})I_{oc}D(\beta_{oc}) + E^T(\beta_c, \beta_{oc})I_\varphi E(\beta_c, \beta_{oc})]\Sigma(\beta_c).$$

B. Configuration of the Motorization

In the general configuration dynamical model (45)–(50), the vectors τ_φ, τ_{oc}, and τ_c represent all the torques that can *potentially* be applied for the rotation and the orientation of the wheels of the robot. In practice, however, only a limited number of motors will be used, which means that many components of τ_φ, τ_{oc}, and τ_c are identically zero.

Our concern in this section is to explicit the configurations of the motorization that allow a full maneuverability of the robots while requiring a number of motors as limited as possible.

First, it is clear that all the centered orientable wheels must be provided with a motor for their orientation (otherwise, these wheels would just play the role of fixed wheels).

Moreover, to ensure a full robot mobility, $N_m$additional motors (with $N_m \geq \delta_m$) must be implemented for either the rotation of some wheels or the orientation of some off-centered orientable wheels. The vector of the torques developed by these motors is denoted τ_m and we have

$$\begin{pmatrix} \tau_{oc} \\ \tau_\varphi \end{pmatrix} = P\tau_m \quad (51)$$

where P is a $(N_{oc}+N) \times N_m$ elementary matrix which selects the components of $(\tau_{oc}, \tau_\varphi)$ that are effectively used as control inputs.

Using (51) we see that (48) of the general dynamical model is rewritten as

$$H_1(\beta_c, \beta_{oc})\dot{\eta} + \Sigma^T(\beta_c)V(\beta_{oc})\dot{\zeta} + f_1(\beta_c, \beta_{oc}, \eta, \zeta) = B(\beta_c, \beta_{oc})P\tau_m$$

with $B(\beta_c, \beta_{oc}) \triangleq \Sigma^T(\beta_c)[D^T(\beta_{oc}) \;\; E^T(\beta_c, \beta_{oc})]$.

We introduce the following assumption.

A3: The configuration of the motorization is such that the matrix

$$B(\beta_c, \beta_{oc})P$$

has full rank for all $(\beta_c, \beta_{oc}) \in \mathbb{R}^{N_c+N_{oc}}$. ∎

We now present the minimal admissible motorizations for the various types of mobile robots that have been described in the Section III.

- **Type** (3,0)—Example 1 (Fig. 6). Omnidirectionnal robot with three swedish wheels.

In this case, the matrix B is constant and reduces to

$$B = \Sigma^T E^T = -J_3^{-1} J_1 = -\frac{1}{r}\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & L \\ O & -1 & L \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & L \end{pmatrix}$$

which is nonsingular. We conclude that the only admissible motorization is to equip each wheel with a motor.

- **Type** (3,0)—Example 2 (Fig. 7). Omnidirectionnal robot with three off-centered orientable wheels.

  In this case the matrix $B(\beta_{oc})$ is written as follows:

$$B(\beta_{oc}) = \Sigma^T [D^T(\beta_{oc}) \;\; E^T(\beta_{oc})] \text{ with } \Sigma^T D^T(\beta_{oc}) = -\frac{1}{d}\begin{pmatrix} \cos\beta_{oc1} & -\cos\beta_{oc2} & \sin\beta_{oc3} \\ \sin\beta_{oc1} & -\sin\beta_{oc2} & -\cos\beta_{oc3} \\ d + L\sin\beta_{oc1} & d + L\sin\beta_{oc2} & d + L\sin\beta_{oc3} \end{pmatrix}$$

with $\Sigma^T E^T(\beta_{oc}) =$

$$-\frac{1}{r}\begin{pmatrix} -\sin\beta_{oc1} & \sin\beta_{oc2} & \cos\beta_{oc3} \\ \cos\beta_{oc1} & \cos\beta_{oc2} & \sin\beta_{oc3} \\ L\cos\beta_{oc1} & L\cos\beta_{oc2} & L\cos\beta_{oc3} \end{pmatrix}.$$

It appears that there is no set of 3 columns of $B(\beta_{oc})$ which are independent for any $(\beta_{oc1}, \beta_{oc2}, \beta_{oc3})$. It is therefore necessary to use (at least) $N_m = 4$ motors. An admissible configuration is as follows: 2 motors (one for the orientation and one for the rotation) on 2 of the 3 wheels (the third one being not motorized and hence self-aligning). For instance, if the wheels 1 and 2 are motorized in this way, the selection matrix P is as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix $B(\beta_{oc})P$ has full rank (= 3) for any configuration of the robot.

- **Type** (2,0)—Example 3 (Fig. 8).

  Robot with two fixed wheels and one off-centered orientable wheel.

  The matrix $B(\beta_{oc})$ is written

$$B(\beta_{oc3}) = \begin{pmatrix} \frac{1}{d}\cos\beta_{oc3} & -\frac{1}{r} & \frac{1}{r} & -\frac{1}{r}\sin\beta_{oc3} \\ -\frac{1}{d}(d + L\sin\beta_{oc3}) & \frac{L}{r} & -\frac{L}{r} & -\frac{L}{r}\cos\beta_{oc3} \end{pmatrix}.$$

Several configurations with 2 motors are admissible:

a) 2 rotation motors on wheels 1 and 2;

b) 1 motor for the orientation of wheel 3 and one for the rotation of wheel 2 (or 3), provided $d > L\frac{\sqrt{2}}{2}$; and

c) 2 motors (orientation and rotation) on the off-centered wheel 3, provided $d < L$.

The corresponding selection matrices P are as follows:

$$\text{a) } P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{b) } P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{c) } P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- **Type** (2,1)—Example 4 (Fig. 9).

  Robot with one centered orientable wheel and two off-centered orientable wheels.

  In this case, we first need a first orientation motor for the centered orientable wheel. The matrix $BJ(\beta_c, \beta_{oc})$ is then written as follows:

$$B(\beta_c, \beta_{oc}) = \Sigma^T(\beta_c)[D^T(\beta_{oc}) \;\; E^T(\beta_c, \beta_{oc})]$$

$$\text{with } \Sigma^T(\beta_c) = \begin{pmatrix} 0 & L\sin\beta_{c3} & \cos\beta_{c3} \\ L & 0 & -1 \end{pmatrix}$$

$$D^T(\beta_{oc}) = \begin{pmatrix} \cos\beta_{oc1} & -\cos\beta_{oc2} \\ \sin\beta_{oc1} & \sin\beta_{oc2} \\ d + L\sin\beta_{oc1} & d + L\sin\beta_{oc2} \end{pmatrix}$$

$$E^T(\beta_c, \beta_{oc}) = -\frac{1}{r}\begin{pmatrix} \cos\beta_{c3} & -\sin\beta_{oc1} & L\sin\beta_{oc2} \\ \sin\beta_{c3} & \cos\beta_{oc1} & -\cos\beta_{oc2} \\ L\cos\beta_{c3} & L\cos\beta_{oc1} & L\cos\beta_{oc2} \end{pmatrix}.$$

The columns 1 and 3 of $B(\beta_c, \beta_{oc})$ are independent if $d > L\sqrt{2}$. The same holds for columns 2 and 3. Hence two admissible motorizations are obtained by using a second motor for the rotation of the centered wheel (number 3) and a third motor for the orientation of either wheel 1 or wheel 2.

The two corresponding matrices P are:

$$1) \; P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 2) \; P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- **Type** (1,1)—Example 5 (Fig. 10). Robot with two fixed wheels and one centered orientable wheel.

  A first orientation motor for the centered orientable wheel is needed. The matrix $B(\beta_c)$ reduces to the vector

$$B = [\sin\beta_{c3} + \cos\beta_{c3} \;\; -\sin\beta_{c3} + \cos\beta_{c3} \;\; 1].$$

Since $\delta_m = 1$, A3 will be satisfied if a second motor is provided for the rotation of the third wheel. The matrix P is written:

$$P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- **Type** (1,2)—Example 6 (Fig. 11).

  Robot with two centered orientable wheels and one off-centered orientable wheel.

  Two motors are required for the orientation of the two centered wheels. The matrix $B(\beta_c, \beta_{oc})$ is written as

$$B(\beta_c, \beta_{oc}) = \Sigma^T(\beta_c)[D^T(\beta_{oc}) \;\; E^T(\beta_c, \beta_{oc})]$$

$$\text{with } \Sigma(\beta_c) = \begin{pmatrix} -2L \sin\beta_{c1} \sin\beta_{c2} \\ L\sin(\beta_{c1} + \beta_{c2}) \\ 2\sin\beta_{c2}\cos\beta_{c1} - \sin(\beta_{c1} + \beta_{c2}) \end{pmatrix}$$

$$D^T(\beta_{oc}) = \begin{pmatrix} -d^{-1}\sin\beta_{oc3} \\ d^{-1}\cos\beta_{oc3} \\ -1 - Ld^{-1}\sin\beta_{oc3} \end{pmatrix}$$

$$E^T(\beta_c, \beta_{oc}) = -\frac{1}{r}\begin{pmatrix} -\sin\beta_{c1} & \sin\beta_{c2} & \cos\beta_{oc3} \\ \cos\beta_{c1} & -\cos\beta_{c2} & \sin\beta_{oc3} \\ L\cos\beta_{c1} & L\cos\beta_{c2} & L\cos\beta_{oc3} \end{pmatrix}.$$

  Since $\delta_m = 1$, it would be sufficient to have one column of $B(\beta_c, \beta_{oc})$ being nonzero for all the possible configurations. However, there is no such column. It is therefore necessary to use 2 additional motors, for instance for the rotation of wheels 1 and 2 giving the matrix P

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Finally the following table summarizes the results:

Type	Number of motors
(δ_m, δ_s)	$N_m + N_c$
Type $(3,0)$	3 or 4
Type $(2,0)$	2
Type $(2,1)$	3
Type $(1,1)$	2
Type $(1,2)$	4

VII. The Posture Dynamical Model

The configuration dynamical model of wheeled mobile robots can be rewritten in the following more compact form:

$$\dot{q} = S(q)u \quad (52)$$

$$H(\beta)\dot{u} + f(\beta, u) = F(\beta)\tau_0 \quad (53)$$

with the following definitions:

$$\beta \triangleq \begin{pmatrix} \beta_c \\ \beta_{oc} \end{pmatrix}$$

$$q \triangleq \begin{pmatrix} \xi \\ \beta \\ \varphi \end{pmatrix}$$

$$u \triangleq \begin{pmatrix} \eta \\ \zeta \end{pmatrix}$$

$$H(\beta) \triangleq \begin{pmatrix} H_1(\beta_c, \beta_{oc}) & \Sigma^T(\beta_c)V(\beta_{oc}) \\ V^T(\beta_{0c})\Sigma(\beta_c) & I_c \end{pmatrix}$$

$$f(\beta, u) \triangleq \begin{pmatrix} f_1(\beta_c, \beta_{oc}, \eta, \zeta) \\ f_2(\beta_c, \beta_{oc}, \eta, \zeta) \end{pmatrix}$$

$$F(\beta) \triangleq \begin{pmatrix} B(\beta_c, \beta_{oc})P & 0 \\ 0 & I \end{pmatrix}$$

$$\tau_0 \triangleq \begin{pmatrix} \tau_m \\ \tau_c \end{pmatrix}.$$

It results from Assumption A3 that the matrix $F(\beta)$ has full rank for all β. This property is important to analyze the behavior of wheeled mobile robots and the design of feedback controllers. It is first used to transform the general state space model into a simpler and more convenient form, by a smooth static state feedback.

Property 7: The configuration dynamical model of wheeled mobile robots ((52) and (53)) is feedback equivalent (by a smooth static time-invariant state feedback) to the following system:

$$\dot{q} = S(q)u \quad (54)$$

$$\dot{u} = v \quad (55)$$

where v represents a set of δ_m auxiliary control inputs.

Indeed it follows readily from Property 1 that the following smooth static time invariant state feedback is well defined everywhere in the state space:

$$\tau_0 = F^{\dagger}(\beta)[H(\beta)u - f(\beta, u)] \quad (56)$$

where $F^{\dagger}$ denotes an arbitrary left inverse of $F(\beta, u)$. ■

We would like to emphasize that a further simplification is of interest from an operational viewpoint. In a context of trajectory planning or feedback control design, it is clear that the user will be essentially concerned by controlling the posture of the robot (namely the coordinate $\xi(t)$) by using the control input v. We observe that this implies that we can ignore deliberately the coordinates β_{oc} and φ and restrict our attention to the following posture dynamical model.

Posture dynamical model:

$$\dot{z} = B(z)u \quad (57)$$

$$\dot{u} = v \quad (58)$$

where we recall that $z \triangleq (\xi, \beta_c)^T$ and $u \triangleq (\eta, \zeta)^T$.

This posture dynamical model fully describes the system dynamics between the control input v and the posture ξ. The coordinates β_{oc} and φ have apparently disappeared but it is important to notice that they are in fact hidden in the feedback (56).

The difference with the posture kinematic model is that now the variables u are part of the state vector. This implies the existence of a drift term and the fact that the input vector fields are constant.

The posture dynamical model inherits of the structural properties of the posture kinematic model discussed in Section IV-C.

Property 8:

a) The posture dynamical model is generic and irreducible.
b) The posture dynamical model is Small-Time-Locally-Controllable.
c) For restricted mobility robots, the posture dynamical model is not stabilizable by a continuous static time

invariant state feedback, but is stabilizable by a time varying static state feedback.

d) The dimension of the largest feedback linearizable subsystem of the posture dynamical model by static state feedback is $2(\delta_m + \delta_s)$. Omnidirectional robots are therefore state feedback linearizable.

e) The posture dynamical model is a differentially flat system.

■

VIII. Conclusion

It has been shown that, according to the restriction to the mobility induced by the kinematic constraints, all WMR can be classified into 5 categories, with particular structures of the corresponding kinematic and dynamic models.

Four models have been introduced.

- The *posture kinematic model* (Section IV), which is sufficient to describe the global motion of the robot. This model is generic in the sense that all the robots belonging to the same class are described by the same posture kinematic model.
- The *configuration kinematic model* (Section V) describing the evolution of all the configuration variables.
- The *configuration dynamical model* (Section VI) taking account the dynamics of the robot, including the torques provided by the motors
- The *posture dynamical model* (Section VII), which is feedback equivalent to the configuration dynamical model. The equations of this model can be obtained from the posture kinematic model just by adding one integrator on each input.

The inputs of the kinematic models are homogeneous to velocities, while the inputs of the dynamical models are either torques or accelerations. The posture models are generic, irreducible, and controllable; they are sufficient for posture control purpose. The configuration models are nongeneric, reducible, and not controllable; they depend on the particular structure of the robot and allow to describe the evolution of all the configuration variables.

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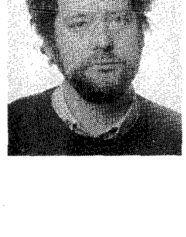

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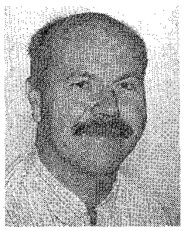

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