INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING, VOL. 4, 345–358 (1990)

INDIRECT ADAPTIVE STATE FEEDBACK CONTROL OF LINEARLY PARAMETRIZED NON-LINEAR SYSTEMS

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SUMMARY

Two indirect adaptive linearizing controllers are proposed, in continuous time, for the class of non-linear systems which are linearly parametrized and which can be linearized by state feedback through a parametrized diffeomorphism. In each case the stability of the closed loop is analysed in detail. It is also shown how these control laws can be extended to related situations, i.e. input–output feedback linearization and non-linear parametrizations.

KEY WORDS Adaptive control Non-linear systems.

1. INTRODUCTION

The issue of extending adaptive techniques to non-linear systems has recently stimulated a number of research studies and has given rise, among others, to a new investigation field which could be called 'adaptive feedback linearization of non-linear plants'. The main motivation arises from the fact that 'exact' feedback linearization of non-linear systems suffers from the drawback that it relies on a perfect cancellation (and consequently a perfect knowledge) of the plant non-linearities. Hence parameter adaptation obviously appears as an attractive way to robustify feedback-linearizing control in the case of parameter uncertainty.

To our knowledge, the earlier works which can be considered as belonging to this field were devoted to chemical and biochemical applications: adaptive pH control[1] and adaptive control of microbial growth,[2] the latter having led to a successful experimental validation.[3] Later on, adaptive linearization was studied for rigid link mechanical systems both in continuous time[4–7] and discrete time.[8] In each of these applications a crucial assumption was that the plant model must be linear with respect to some appropriate parametrization. From these specific applications it became apparent that the problem of adaptive feedback linearization of linearly parametrized non-linear systems was a relevant (and non-trivial) problem.

As in the linear case, both direct and indirect adaptive control schemes can be formulated. The difference lies in the way of desigining the parameter adaptation. In a direct scheme the parameter adaptation is driven by the control error itself, while in an indirect scheme it is driven by an auxiliary observation (or prediction) error. In contrast with the linear case, one specific difficulty arises from the fact that input state linearization requires a non-linear *parametrized* state transformation through an appropriate diffeomorphism (see Assumption 3 in Section 3).

This paper was recommended for publication by editor R. Bitmead

0890–6327/90/050345–14$07.00

Received 22 June 1989
Revised 15 May 1990

Direct adaptive state-space linearization has been studied by Taylor[9] and Taylor *et al.*[10] in the case where the linearizing diffeomorphism is restricted not to depend on the unknown parameters. In Reference 11 such a restriction is relaxed by requiring a structural condition called 'extended matching' which is *a priori* verifiable. Adaptive input–output linearization has been studied by Nam and Arapostathis[12] for systems with relative degree n and Lipschitz non-linearities. This was extended by Sastry and Isidori[13] to systems with stable zero dynamics and Lipschitz non-linearities. An important drawback of the scheme of Sastry and Isidori is that it requires model overparametrization. The same difficulty appears for direct adaptive state-space linearization with a parametrized diffeomorphism when the extended matching condition is not satisfied. This will be illustrated by an example in Section 3.

On the other hand, the indirect approach has been analysed in specific applications: robotics[6] and biotechnology.[14] It has also been studied by Pomet and Praly[15] under particular conditions on the 'growth at infinity' of the involved non-linearities.

Our contribution in this paper will be to present and discuss the main features of two indirect adaptive linearizing control schemes for a general class of linearly parametrized non-linear systems which can be fully state-linearized without restriction on the allowable diffeomorphism. An important advantage of the indirect approach is that there is no need for any 'over-reparametrization': it is the basic physical parametrization of the process model which is estimated on-line and used by the adaptive control algorithm. This feature is valid for state-space linearization as well as for input–output linearization.

The class of non-linear systems under consideration is described in Section 2. In Section 3, a weak and a strong condition for state feedback linearizability of the plant model are introduced and the related parametrized control laws are defined. The strong linearizability condition is implicitly equivalent to the extended matching condition of Kanellakopoulos *et al.*[11] In each case the adaptive control problem is to find a parameter adaptation law which guarantees the closed-loop stability. An observer-based parameter estimator is presented and analysed in Section 4. Then the *global* stability of the indirect adaptive control arising from the strong linearizability condition is shown in Section 5, while local closed-loop stability under the weak linearizability condition is analysed in Section 6. Finally, in Section 7 we briefly discuss how the approach can be extended to related situations, namely input–output adaptive linearization and the handling of some categories of non-linear parametrizations.

2. SYSTEM DESCRIPTION AND CONTROL OBJECTIVE

We consider non-linear parametrized plants, with parametric uncertainty, of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}^*) \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the control input vector and $\boldsymbol{\theta}^* \in \mathbb{R}^p$ is the parameter vector.

In this paper we shall be concerned with the design of controllers for the system (1) in order to track a desired trajectory $\mathbf{x}_d(t)$. The tracking error will be denoted

$$\tilde{\mathbf{x}} = \mathbf{x}_d - \mathbf{x} \tag{2}$$

By parametric uncertainty we mean that the value $\boldsymbol{\theta}^*$ of the parameter is unknown but that an estimate $\boldsymbol{\theta}$ is available. When the true parameter $\boldsymbol{\theta}^*$ is replaced in (1) by the estimate $\boldsymbol{\theta}$, we obtain a family of non-linear state-space models indexed by $\boldsymbol{\theta}$ and denoted

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) \tag{3}$$

Throughout the paper we shall use the same notation '**x**' for the actual state of the plant (1) and the state of the plant model (3). This should, however, not lead to any confusion.

We now state two fundamental assumptions regarding the desired trajectory and the model (3) which can be phrased as follows: the desired trajectory is bounded and smooth and the model is linearly parametrized.

Assumption 1. Desired trajectory

(a) $\mathbf{x}_d(t)$ is a C_1-function of time.
(b) $\mathbf{x}_d(t)$ and $\dot{\mathbf{x}}_d(t)$ are bounded $\forall t$.

Assumption 2. Linear parametrization

$\mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})$ is linearly parametrized as

$$\mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) = \boldsymbol{\Phi}^T(\mathbf{x}, \mathbf{u})\boldsymbol{\theta} + \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u}) \tag{4}$$

where $\boldsymbol{\Phi}(\mathbf{x}, \mathbf{u})$ and $\boldsymbol{\varphi}(\mathbf{x}, \mathbf{u})$ are respectively a $p \times n$ matrix and an n-vector of known smooth functions of $\mathbf{x}$ and $\mathbf{u}$.

3. INPUT STATE LINEARIZING CONTROL: BASIC DEFINITIONS AND STATEMENT OF THE ADAPTIVE CONTROL PROBLEM

We are concerned with 'indirect adaptive state-feedback-linearizing control' of the plant (1) in case of *full state measurement* and parameter uncertainty. The state feedback control law will be parametrized by the parameter estimate $\boldsymbol{\theta}$. To guarantee the existence of this control law, we state two alternative assumptions on the linearizability of the model (3), a weak one, Assumption 3, and a strong one, Assumption 4.

A weak linearizability condition

Assumption 3

There exist

(a) a Hurwitz matrix $\boldsymbol{\Lambda}$
(b) an open set $\mathbf{D}_\mathbf{x} \subseteq \mathbb{R}^n$ containing $\mathbf{x}_d(t)$ for all t
(c) an open set $\mathbf{D}_\theta \subseteq \mathbb{R}^p$ containing $\boldsymbol{\theta}^*$
(d) a family of parametrized diffeomorphisms indexed by $\boldsymbol{\theta} \in \mathbf{D}_\theta$,

$$\mathbf{W}\colon \mathbf{D}_\mathbf{x} \to \mathbb{R}^n\colon \mathbf{z} = \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) \tag{5}$$

such that the following implicit equation in the unknown $\mathbf{u}$,

$$\frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta})[\boldsymbol{\Phi}^T(\mathbf{x}, \mathbf{u})\boldsymbol{\theta} + \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u})] = \frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}_d, \boldsymbol{\theta})\dot{\mathbf{x}}_d - \boldsymbol{\Lambda}[\mathbf{W}(\mathbf{x}_d, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}, \boldsymbol{\theta})] \tag{6}$$

has a unique bounded solution $u = u_a(\mathbf{x}, \mathbf{x}_d, \boldsymbol{\theta})$ for all $\mathbf{x} \in \mathbf{D}_x$.

Comments

1. We explain why Assumption 3 is a condition of state feedback linearizability for the model (3). Suppose that the parameter estimate $\boldsymbol{\theta}$ is constant. Then the vector $\mathbf{z} = \mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$ is called the 'canonical' state of the model (3). The word 'canonical' refers to the fact that, under Assumption 3, $\mathbf{z}$ can be shown to be the state of an equivalent Brunovski canonical form for the model (3). We define the 'desired' canonical trajectory as

$$\mathbf{z}_\mathrm{d} = \mathbf{W}(\mathbf{x}_\mathrm{d}, \boldsymbol{\theta}) \tag{7}$$

Now let us suppose that the control law $u_\mathrm{a}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta})$ defined by (6) is applied to the model (3). Then straightforward calculations show that the closed loop is stable and that the canonical tracking error $(\mathbf{z}_\mathrm{d} - \mathbf{z})$ is governed by the stable and linear dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{z}_\mathrm{d} - \mathbf{z}) = \boldsymbol{\Lambda}(\mathbf{z}_\mathrm{d} - \mathbf{z}) \tag{8}$$

To summarize, we see that *if the parameter* $\boldsymbol{\theta}$ *is constant, the control law* $u_\mathrm{a}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta})$ *defined in Assumption 3 linearizes the canonical model obtained from the plant model through the diffeomorphism* $\mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$.

2. It is important to notice that in many practical applications the eigenvalues of the matrix $\boldsymbol{\Lambda}$ can be freely chosen by the user, which allows us to assign desired dynamics to the closed loop. This will be illustrated in Example 1.

Statement of the adaptive control problem

The problem is obviously not to control the plant model (3) but to control the actual plant (1). This can be done by using $u_\mathrm{a}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta})$ as an 'indirect adaptive control law' where

(a) $\mathbf{x}$ is the actual measured state of the plant
(b) $\boldsymbol{\theta}$ is a time-varying on-line estimate of the true parameter $\boldsymbol{\theta}^*$ provided by a recursive parameter estimator.

In such a case it is clear that the closed-loop plant is no longer described by the 'ideal' stable linear system (8). The problem is to find a suitable parameter estimation algorithm which ensures (at least in a neighbourhood of the desired trajectory) closed-loop stability. The parameter estimator will be presented in Section 4 and the stability analysis in Section 6.

Example 1

The following second-order system describes the dynamics of an autocatalysed chemical reaction:

$$\dot{x}_1 = \theta_1 x_1 x_2 - \theta_3 x_1 \tag{9a}$$

$$\dot{x}_2 = -\theta_2 x_1 x_2 - \theta_3 x_2 + u \tag{9b}$$

where x_1 is the autocatalyst concentration, x_2 is the reactant concentration and θ_i $(i = 1, 2, 3)$ are physical strictly positive parameters related to the stoichiometry, the specific reaction rate and the dilution rate of the process.

We first notice that this model can be written in the linear regression form (4) as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 & 0 & -x_1 \\ 0 & -x_1 x_2 & -x_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} \tag{10}$$

The open sets $\mathbf{D_x}$ and $\mathbf{D}_{\boldsymbol{\theta}}$ are defined as

$$\mathbf{D_x} = \mathbb{R}^2_+, \qquad \mathbf{D}_{\boldsymbol{\theta}} = \mathbb{R}^3_+ \tag{11}$$

The diffeomorphism $\mathbf{z} = \mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$ is defined as

$$z_1 = x_1, \qquad z_2 = \theta_1 x_1 x_2 - \theta_3 x_1 \tag{12}$$

The inverse of the diffeomorphism, $\mathbf{x} = \mathbf{W}^{-1}(\mathbf{x}, \boldsymbol{\theta})$, is

$$\mathbf{x}_1 = z_1, \qquad x_2 = \frac{\theta_3 z_1 + z_2}{\theta_1 z_1} \tag{13}$$

Notice that the diffeomorphism exists for all $\boldsymbol{\theta} \in \mathbf{D}_{\boldsymbol{\theta}}$.

Through the diffeomorphism (12), the model (10) is equivalent to the canonical form

$$\dot{z}_1 = z_2 \tag{14a}$$

$$\dot{z}_2 = \frac{z_2^2}{z_1} - \theta_2 z_1 z_2 - \theta_2\theta_3 z_1^2 - \theta_3^2 z_1 + \theta_3 z_2 + \theta_1 z_1 u \tag{14b}$$

This canonical form is clearly state-feedback-linearizable. We select the following matrix $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & 1 \\ -\lambda_1 & -\lambda_2 \end{pmatrix} \tag{15}$$

Then the following control law linearizes the canonical form (14):

$$u(\mathbf{z}, \mathbf{z}_\mathrm{d}, \boldsymbol{\theta}) = \frac{\dot{z}_{2\mathrm{d}} + \lambda_1(z_{1\mathrm{d}} - z_1) + \lambda_2(z_{2\mathrm{d}} - z_2) - z_2^2 z_1^{-1} + \theta_2 z_1 z_2 + \theta_2\theta_3 z_1^2 + \theta_3^2 z_1 - \theta_3 z_2}{\theta_1 z_1} \tag{16}$$

The linearizing control law $u_\mathrm{a}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta})$ defined by Assumption 3, which satisfies expression (6), is obtained by applying the diffeomorphism (12) to (16) and is written as

$$\begin{aligned} u_\mathrm{a}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta}) = (\theta_1 x_1)^{-1}[&\theta_1(\dot{x}_{1\mathrm{d}} x_{2\mathrm{d}} + x_{1\mathrm{d}}\dot{x}_{2\mathrm{d}}) - \theta_3 \dot{x}_{1\mathrm{d}} + (\lambda_1 - \lambda_2\theta_3)(x_{1\mathrm{d}} - x_1) \\ &+ \lambda_2\theta_1(x_{1\mathrm{d}}x_{2\mathrm{d}} - x_1 x_2) - x_1(\theta_1 x_2 - \theta_3)^2 + \theta_1\theta_2 x_1^2 x_2 + \theta_1\theta_3 x_1 x_2] \end{aligned} \tag{17}$$

It must be emphasized that the basic physical model (9) is linearly parametrized by $\boldsymbol{\theta}$ but that the canonical form (14) is *not*. However, it can be 'linearly *over*-reparametrized' by defining $\boldsymbol{\theta}' = (\theta_1, \theta_2, \theta_3, \theta_2\theta_3, \theta_3^2)$. This overparametrization can be a source of difficulty when designing *direct* adaptive controllers, but it will not cause any trouble in our case since we shall deal exclusively with the basic minimal parametrization $\boldsymbol{\theta}$ (see a related discussion in Reference 11). We notice also that in this example the eigenvalues of the linear reference model can be chosen freely by the user.

A strong linearizability condition

We now suppose that the parameter estimate $\boldsymbol{\theta}$ is time-varying and computed on-line with an adaptation mechanism of the form

$$\dot{\boldsymbol{\theta}} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}, t) \tag{18}$$

where $\mathbf{g}$ is a smooth time-varying function of $\mathbf{x}$, $\mathbf{u}$ and $\boldsymbol{\theta}$.

Consider the diffeomorphism $\mathbf{z} = \mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$ defined in Assumption 3, but with $\boldsymbol{\theta}$ time-varying and given by (18). More precisely, assume that

(a) for each fixed $\boldsymbol{\theta} \in \mathbf{D}_{\boldsymbol{\theta}}$, $\mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$ is the diffeomorphism of Assumption 3

(b) for each fixed $\mathbf{x} \in \mathbf{D}_{\mathbf{x}}$, $\mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$ is a C_1-function of $\boldsymbol{\theta}$,

$$\mathbf{W}\colon \mathbf{D}_{\boldsymbol{\theta}} \to \mathbb{R}^n\colon \ \mathbf{z} = \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) \tag{19}$$

Assumption 4

The adaptation function $\mathbf{g}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}, t)$ is such that the following implicit equation in the unknown $\mathbf{u}$,

$$\frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta})[\boldsymbol{\Phi}^{\mathrm{T}}(\mathbf{x}, \mathbf{u})\boldsymbol{\theta} + \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u})] + \left(\frac{\partial \mathbf{W}}{\partial \boldsymbol{\theta}}(\mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta}) - \frac{\partial \mathbf{W}}{\partial \boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta})\right)\mathbf{g}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}, t)$$
$$= \frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta})\dot{\mathbf{x}}_{\mathrm{d}} - \boldsymbol{\Lambda}[\mathbf{W}(\mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}, \boldsymbol{\theta})] \tag{20}$$

has a unique bounded solution $u = u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta})$ for all $\mathbf{x} \in \mathbf{D}_{\mathbf{x}}$ and all $\boldsymbol{\theta} \in \mathbf{D}_{\boldsymbol{\theta}}$. $\boldsymbol{\Lambda}$ is the Hurwitz matrix defined in Assumption 3.

Comments

3. With Assumption 3 we have considered that expression (3) represents a family of fixed models indexed by $\boldsymbol{\theta}$. Then the assumption guarantees that each model in that family can be linearized by state feedback. By contrast, with Assumption 4 we consider that (3) represents a *single* time-varying model parametrized by a time-varying parameter $\boldsymbol{\theta}(t)$ given by expression (18). Then, *the control law* $u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta})$ *linearizes the time-varying canonical model obtained from the plant model through the transformation* (18) *and the diffeomorphism* $\mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$.

4. Assumption 4 is implied by 'the extended matching condition' of Kanellakopoulos *et al.*[11] It is stronger than Assumption 3 in the sense that Assumption 4 implies Assumption 3 when $\mathbf{g} = 0$, because the matrix $\boldsymbol{\Lambda}$ and the diffeomorphism $\mathbf{W}$ are identical in both assumptions.

Statement of the adaptive control problem

Here also, the problem is obviously not to control the model (3) but to control the actual plant (1). This can be achieved by using the control law $u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta})$ as an indirect adaptive linearizing control law. The problem is to find a suitable parameter adaptation law ensuring closed-loop stability. A parameter estimation algorithm is proposed in Section 4 while the closed-loop stability is analysed in Section 5.

Example 1 (continued)

According to Assumption 4, for the system (9a), (9b) the control law has to satisfy the equation

$$\begin{aligned}\theta_1 x_1 u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta}) = {} & [\theta_1(\dot{x}_{1\mathrm{d}} x_{2\mathrm{d}} + x_{1\mathrm{d}}\dot{x}_{2\mathrm{d}}) - \theta_3 \dot{x}_{1\mathrm{d}} + (\lambda_1 - \lambda_2\theta_3)(x_{1\mathrm{d}} - x_1) \\ & + \lambda_2\theta_1(x_{1\mathrm{d}}x_{2\mathrm{d}} - x_1x_2) - x_1(\theta_1x_2 - \theta_3)^2 + \theta_1\theta_2x_1^2x_2 + \theta_1\theta_3x_1x_2] \\ & + (x_{1\mathrm{d}}x_{2\mathrm{d}} - x_1x_2)g_1(\mathbf{x}, u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta}), \boldsymbol{\theta}) - (x_{1\mathrm{d}} - x_1)g_3(\mathbf{x}, u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta}), \boldsymbol{\theta})\end{aligned} \tag{21}$$

We shall see that this equation has effectively a unique solution in $u_{\mathrm{b}}(\mathbf{x}, \mathbf{x}_{\mathrm{d}}, \boldsymbol{\theta})$ with the parameter estimator proposed in the next section.

4. A PARAMETER ESTIMATOR

We consider the diffeomorphism $\mathbf{z} = \mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$, but now we suppose that $\mathbf{x}$ *is the actual state of the plant* (i.e. not the state of the model) while $\boldsymbol{\theta}$ is the time-varying parameter provided by the parameter estimator we have to design. Taking the derivative of $\mathbf{z}$ we get

$$\dot{\mathbf{z}} = \frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta})[\boldsymbol{\Phi}^{\mathrm{T}}(\mathbf{x}, \mathbf{u})\boldsymbol{\theta}^* + \boldsymbol{\varphi}(\mathbf{x}, \mathbf{u})] + \frac{\partial \mathbf{W}}{\partial \boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{22}$$

or, equivalently,

$$\dot{\mathbf{z}} = \boldsymbol{\psi}_0(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) + \boldsymbol{\Psi}_1^{\mathrm{T}}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\boldsymbol{\theta}^* + \boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}, \boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{23}$$

where

$$\boldsymbol{\psi}_0(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) \equiv \frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\varphi}(\mathbf{x}, \mathbf{u}) \tag{24a}$$

$$\boldsymbol{\Psi}_1^{\mathrm{T}}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) \equiv \frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\Phi}^{\mathrm{T}}(\mathbf{x}, \mathbf{u}) \tag{24b}$$

$$\boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}, \boldsymbol{\theta}) \equiv \frac{\partial \mathbf{W}}{\partial \boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}) \tag{24c}$$

Then the basic idea behind the parameter estimator is to design a kind of adaptive observer for the system (23) as

$$\dot{\hat{\mathbf{z}}} = \boldsymbol{\psi}_0(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) + \boldsymbol{\Psi}_1^{\mathrm{T}}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\boldsymbol{\theta} + [\boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\Psi}_1(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\mathbf{P} - \boldsymbol{\Omega}]\,[\mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) - \hat{\mathbf{z}}] \tag{25a}$$

with $\hat{\mathbf{z}}(0) = \mathbf{W}(\mathbf{x}(0), \boldsymbol{\theta}(0))$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Psi}_1(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\mathbf{P}\,[\mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) - \hat{\mathbf{z}}] \tag{25b}$$

where $\boldsymbol{\Omega}$ is an arbitrary Hurwitz matrix and the positive definite symmetric gain matrix $\mathbf{P}$ is the solution of the Lyapunov equation

$$\boldsymbol{\Omega}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\boldsymbol{\Omega} = -\mathbf{Q} \tag{26}$$

with $\mathbf{Q}$ an arbitrary positive definite matrix.

Defining the 'observation' error $\mathbf{e}$ and the parametric error $\tilde{\boldsymbol{\theta}}$ as

$$\mathbf{e} \equiv \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) - \hat{\mathbf{z}}, \qquad \tilde{\boldsymbol{\theta}} \equiv \boldsymbol{\theta}^* - \boldsymbol{\theta} \tag{27}$$

the error system corresponding to (25) is written

$$\begin{pmatrix} \dot{\mathbf{e}} \\ \dot{\tilde{\boldsymbol{\theta}}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Omega} & \boldsymbol{\Psi}_1^{\mathrm{T}}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) \\ -\boldsymbol{\Psi}_1(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\mathbf{P} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix} \tag{28}$$

Irrespective of the control $\mathbf{u}$, we have the following stability properties.

Theorem 1

For the error system (28),

(a) $\mathbf{e}$ and $\tilde{\theta}$ are bounded as

$$\|\mathbf{e}(t)\| \leqslant \pi \|\tilde{\boldsymbol{\theta}}(0)\|, \qquad \|\tilde{\boldsymbol{\theta}}(t)\| \leqslant \|\tilde{\boldsymbol{\theta}}(0)\| \qquad \forall t \geqslant 0 \tag{29}$$

$$\pi \equiv \sqrt{\lambda_{\min}^{-1}(\mathbf{P})}$$

(b) $\mathbf{e}(t)$ is L_2.

Proof. Consider the positive definite function

$$V(\mathbf{e}, \tilde{\boldsymbol{\theta}}) = \mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \tilde{\boldsymbol{\theta}}^{\mathrm{T}}\tilde{\boldsymbol{\theta}} \tag{30}$$

The time derivative of V along the trajectories of (28) is

$$\dot{V} = -\mathbf{e}^{\mathrm{T}}\mathbf{Q}\mathbf{e} \tag{31}$$

Hence, since $\mathbf{e}(0) = 0$, we have

$$V[\mathbf{e}(t), \tilde{\boldsymbol{\theta}}(t)] \leqslant V[\mathbf{e}(0), \tilde{\boldsymbol{\theta}}(0)] \leqslant \| \tilde{\boldsymbol{\theta}}(0) \|^2 \tag{32}$$

Part (a) follows. Expressions (31) and (32) imply also that

$$\lambda_{\min}(\mathbf{Q}) \int_0^t \| \mathbf{e}(\tau) \|^2 \, \mathrm{d}\tau \leqslant V(0) - V(t) \leqslant V(0) \tag{33}$$

and part (b) follows.

Example 1 (continued)

For the system (9), the parameter adaptation equation (25b) specializes as

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \begin{pmatrix} x_1 x_2 & x_1 x_2(\theta_1 x_2 - \theta_3) \\ 0 & -\theta_1 x_1^2 x_2 \\ -x_1 & -2\theta_1 x_1 x_2 + \theta_3 x_1 \end{pmatrix} \mathbf{P} \begin{pmatrix} x_1 - \hat{z}_1 \\ \theta_1 x_1 x_2 - \theta_3 x_1 - \hat{z}_2 \end{pmatrix} \tag{34}$$

Since this expression is independent of the control $\mathbf{u}$, equation (21) has a unique solution as expected and Assumption 4 is satisfied.

5. INDIRECT ADAPTIVE CONTROL UNDER THE STRONG LINEARIZABILITY CONDITION: STABILITY ANALYSIS

We suppose that the control law $u_b(\mathbf{x}, \mathbf{x}_d, \boldsymbol{\theta})$ defined by Assumption 4 is applied to the plant (1) with $\mathbf{x}$ the *actual state of the plant* and $\boldsymbol{\theta}$ the parameter estimate computed with the parameter estimator (25). We define the 'canonical' tracking error $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon} = \mathbf{W}(\mathbf{x}_d, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) \tag{35}$$

Taking the derivative of (35) and using (1) we obtain

$$\dot{\boldsymbol{\varepsilon}} = \frac{\partial \mathbf{W}}{\partial \mathbf{x}} (\mathbf{x}_d, \boldsymbol{\theta})\dot{\mathbf{x}}_d - \boldsymbol{\psi}_0(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) - \boldsymbol{\Psi}_1^{\mathrm{T}}(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\boldsymbol{\theta}^* + [\boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}_d, \boldsymbol{\theta}) - \boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}, \boldsymbol{\theta})]\dot{\boldsymbol{\theta}} \tag{36}$$

Then, using the definition (20) of the control law and the definition (25) of the parameter estimator, it can be shown, after a few computations, that the tracking error $\boldsymbol{\varepsilon}$ is the output of a stable *linear* filter, driven by the 'observation error' $\mathbf{e}$, with the state representation

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\boldsymbol{\xi} + (\boldsymbol{\Omega} - \boldsymbol{\Lambda})\mathbf{e} \tag{37a}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\xi} - \mathbf{e} \tag{37b}$$

This linear filter is a key tool for the analysis of the closed-loop stability. Since the matrix $\boldsymbol{\Lambda}$ is Hurwitz, there exist two positive constants d_1 and d_2 such that

$$\| \exp(\boldsymbol{\Lambda}t) \| \leqslant d_1 \exp(-d_2 t) \tag{38}$$

It follows from Theorem 1 that the input $\mathbf{e}$ of the filter (37) is bounded by $\pi \| \tilde{\boldsymbol{\theta}}(0) \|$. It is then a standard result of systems stability theory[16] that the output $\boldsymbol{\varepsilon}$ is bounded as

$$\| \boldsymbol{\varepsilon}(t) \| \leqslant d_1 \| \boldsymbol{\varepsilon}(0) \| + \gamma \| \tilde{\boldsymbol{\theta}}(0) \| \tag{39a}$$

with

$$\gamma = \left(1 + \frac{d_1}{d_2} \| \boldsymbol{\Lambda} - \boldsymbol{\Omega} \| \right) \pi \tag{39b}$$

Define the two positive constants k_1 and k_2:

$$k_1 = \inf_{\boldsymbol{\theta} \in \mathbf{D}_\theta} \left(\inf_{\mathbf{x}_1 \in \partial \mathbf{D}_\mathrm{x} \text{ and } \mathbf{x}_2 \in \mathbf{x}_\mathrm{d}} (\| \mathbf{W}(\mathbf{x}_1, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}_2, \boldsymbol{\theta}) \|) \right) \tag{40a}$$

$$k_2 = \inf_{\boldsymbol{\theta} \in \partial \mathbf{D}_\theta} (\| \boldsymbol{\theta} - \boldsymbol{\theta}^* \|) \tag{40b}$$

where $\partial \mathbf{D}_\mathrm{x}$ and $\partial \mathbf{D}_\theta$ denote the boundaries of $\mathbf{D}_\mathrm{x}$ and $\mathbf{D}_\theta$. Define the open set $\mathbf{D}_0 \subseteq \mathbb{R}^n \times \mathbb{R}^p$:

$$\mathbf{D}_0 = \{(\mathbf{x}, \boldsymbol{\theta}) \mid d_1 \| \mathbf{W}(\mathbf{x}_\mathrm{d}, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) \| + \gamma \| \boldsymbol{\theta}^* - \boldsymbol{\theta} \| < k_1 \quad \text{and} \quad \| \boldsymbol{\theta}^* - \boldsymbol{\theta} \| < k_2\} \tag{41}$$

Then we have the following convergence result.

Theorem 2

Under Assumptions 1, 2 and 4, if the control law $u_\mathrm{b}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta})$ is applied to the system (1) with the parameter estimator (25) and if $(\mathbf{x}(0), \boldsymbol{\theta}(0)) \in \mathbf{D}_0$, then the tracking error converges to zero:

$$\lim_{t \to \infty} \| \mathbf{x}_\mathrm{d}(t) - \mathbf{x}(t) \| = 0 \tag{42}$$

Proof. $(\mathbf{x}(0), \boldsymbol{\theta}(0)) \in \mathbf{D}_0$, Theorem 1 and property (39) imply that

$$\begin{aligned} &\boldsymbol{\theta}(t) \in \mathbf{D}_\theta \quad \text{and bounded } \forall t \\ &\| \boldsymbol{\varepsilon}(t) \| \leqslant d_1 \| \boldsymbol{\varepsilon}(0) \| + \gamma \| \tilde{\boldsymbol{\theta}}(0) \| \leqslant k_1 \quad \forall t \end{aligned} \tag{43}$$

This in turn implies, by the definition (40a) of k_1, that $\mathbf{x}(t) \in \mathbf{D}_\mathrm{x}$ and is bounded for all t. Hence the control law $u_\mathrm{b}(\mathbf{x}, \mathbf{x}_\mathrm{d}, \boldsymbol{\theta})$ is well defined for all t, and by the definition of the diffeomorphism and the control law, $\boldsymbol{\Psi}_1(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})$ is bounded. Then, from (28), $\mathrm{d}\mathbf{e}/\mathrm{d}t$ is bounded, and since $\mathbf{e}$ is L_2, we have by Theorem 1

$$\lim_{t \to \infty} \mathbf{e}(t) = 0, \qquad \lim_{t \to \infty} \dot{\boldsymbol{\theta}}(t) = 0 \tag{44}$$

Then, since $\boldsymbol{\varepsilon}(t)$ is the output of the stable linear filter (37) with input $\mathbf{e}(t)$,

$$\lim_{t \to \infty} \| \boldsymbol{\varepsilon}(t) \| = \lim_{t \to \infty} \| \mathbf{W}(\mathbf{x}_\mathrm{d}, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) \| = 0 \tag{45}$$

Hence, since $\mathbf{W}(\mathbf{x}, \boldsymbol{\theta})$ is a diffeomorphism with respect to x, the result follows.

Global stability

It is important to notice that the linear filter (37) and the property (39) are independent of the plant description and rely only on the design parameters $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$. Hence, if Assumptions 1, 2 and 4 are global and hold for all $(\mathbf{x}, \boldsymbol{\theta}) \in \mathbb{R}^n \times \mathbb{R}^p$, then the stability of the adaptive control is also global and holds whatever $(\mathbf{x}(0), \boldsymbol{\theta}(0))$ in $\mathbb{R}^n \times \mathbb{R}^p$.

6. INDIRECT ADAPTIVE CONTROL UNDER THE WEAK LINEARIZABILITY CONDITION: STABILITY ANALYSIS

We suppose that the control law $u_a(\mathbf{x}, \mathbf{x}_d, \boldsymbol{\theta})$ defined by Assumption 3 is applied to the plant (1) with $\mathbf{x}$ the *actual state of the plant* and $\boldsymbol{\theta}$ the parameter estimate computed with the parameter estimator (25). In that case, using the definition (6) of the control law, it can be shown, after a few computations, that the tracking error $\boldsymbol{\varepsilon}$ (which is governed by the dynamics (36)) is the output of the time-varying linear filter

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\lambda}\boldsymbol{\xi} + [\boldsymbol{\Omega} - \boldsymbol{\Lambda} + (\boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}_d, \boldsymbol{\theta}) - \boldsymbol{\Psi}_2^{\mathrm{T}}(\mathbf{x}, \boldsymbol{\theta}))\boldsymbol{\Psi}_1(\mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\mathbf{P}]\,\mathbf{e} \tag{46a}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\xi} - \mathbf{e} \tag{46b}$$

In order to analyse locally the closed-loop stability around the desired trajectory $\mathbf{x}_d$, we introduce the closed sets $\mathbf{B}_x$ and $\mathbf{B}_\theta$.

Assumption 5

$$\mathbf{B}_x \subset \mathbf{D}_x, \qquad \mathbf{x}_d(t) \in \mathbf{B}_x, \qquad \mathbf{x}_d(t) \notin \partial\mathbf{B}_x \qquad \forall t \tag{47a}$$

$$\mathbf{B}_\theta \subset \mathbf{D}_\theta, \qquad \boldsymbol{\theta}^* \in \mathbf{B}_\theta, \qquad \boldsymbol{\theta}^* \notin \partial\mathbf{B}_\theta \tag{47b}$$

This assumption means that $\mathbf{B}_x$ and $\mathbf{B}_\theta$ are closed subsets of $\mathbf{D}_x$ and $\mathbf{D}_\theta$ respectively, that the desired trajectory is *strictly* contained in $\mathbf{B}_x$ and that $\boldsymbol{\theta}^*$ is *strictly* contained in $\mathbf{B}_\theta$.

We define the constants

$$\mu_0 = \sup_{\mathbf{x}_d, \dot{\mathbf{x}}_d, \boldsymbol{\theta} \in \mathbf{B}_\theta} \left\| \frac{\partial \mathbf{W}}{\partial \mathbf{x}} (\mathbf{x}_d, \boldsymbol{\theta})\dot{\mathbf{x}}_d \right\| \tag{48a}$$

$$\mu_1 = \sup_{\mathbf{x}_d, \mathbf{x} \in \mathbf{B}_x \text{ and } \boldsymbol{\theta} \in \mathbf{B}_\theta} \| \boldsymbol{\Psi}_1(\mathbf{x}, u_a(\mathbf{x}, \mathbf{x}_d, \boldsymbol{\theta}), \boldsymbol{\theta}) \| \tag{48b}$$

$$\mu_2 = \sup_{\mathbf{x} \in \mathbf{B}_x \text{ and } \boldsymbol{\theta} \in \mathbf{B}_\theta} \| \boldsymbol{\Psi}_2(\mathbf{x}, \boldsymbol{\theta}) \| \tag{48c}$$

$$C_1 = \inf_{\boldsymbol{\theta} \in \mathbf{B}_\theta} \left(\inf_{\mathbf{x}_1 \in \partial\mathbf{B}_x \text{ and } \mathbf{x}_2 \in \mathbf{x}_d} (\| \mathbf{W}(\mathbf{x}_1, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}_2, \boldsymbol{\theta}) \|) \right) \tag{49a}$$

$$C_2 = \inf_{\boldsymbol{\theta} \in \partial\mathbf{B}_\theta} (\| \boldsymbol{\theta} - \boldsymbol{\theta}^* \|) \tag{49b}$$

We notice that $C_1 > 0$ and $C_2 > 0$ by Assumption 5. Let δ be an arbitrarily small strictly positive constant, strictly smaller than C_1:

$$0 < \delta < C_1 \tag{50}$$

We have the following preliminary lemmas.

Lemma 1

If $\| \boldsymbol{\varepsilon}(t_1) \| \leqslant C_1 - \delta$ and if $\| \tilde{\boldsymbol{\theta}}(0) \| \leqslant C_2$, then there exists a time interval $\Delta t > 0$, independent of t_1, such that

$$\| \boldsymbol{\varepsilon}(\tau) \| < C_1 \quad \forall \tau, \quad t_1 < \tau < t_1 + \Delta t$$

and

$$\| \boldsymbol{\varepsilon}(t_1 + \Delta t) \| \leqslant C_1$$

Proof. We use a proof by contradiction. We first notice that by Theorem 1

$$\| \tilde{\boldsymbol{\theta}}(0) \| \leqslant C_2 \text{ implies that } \boldsymbol{\theta}(t) \in \mathbf{B}_\theta \quad \forall t \tag{51}$$

We define the time interval Δt as

$$\mu^* \equiv \mu_0 + \mu_1 \| \boldsymbol{\theta}^* \| + 2\mu_1\mu_2\pi \| \mathbf{P} \| \, \| \tilde{\boldsymbol{\theta}}(0) \| \tag{52}$$

$$\Delta t = \frac{\delta}{\mu^* \sqrt{n}} \qquad (\text{or } \delta^2 = n(\mu^* \Delta t)^2) \tag{53}$$

Now suppose that the statement of the lemma is false and that there exists t^*, $t_1 < t^* < t_1 + \Delta t$, such that

$$\forall \tau, \quad t_1 \leqslant \tau < t^*, \quad \| \boldsymbol{\varepsilon}(\tau) \| < C_1 \quad \text{and} \quad \| \boldsymbol{\varepsilon}(t^*) \| = C_1 \tag{54}$$

From (36) and (49) this implies that

$$\mathbf{x}(\tau) \in \mathbf{B}_\mathbf{x} \quad \text{and} \quad \| \dot{\boldsymbol{\varepsilon}}(\tau) \| \leqslant \mu^* \quad \forall \tau, \quad t_1 \leqslant \tau \leqslant t^* \tag{55}$$

Hence for each component ε_i of $\boldsymbol{\varepsilon}$ we have

$$| \varepsilon_i(t^*) - \varepsilon_i(t) | \leqslant \mu^*(t^* - t_1) \tag{56}$$

Then from (54) and (56) we find

$$\delta^2 \leqslant \| \boldsymbol{\varepsilon}(t^*) - \boldsymbol{\varepsilon}(t_1) \|^2 = \sum_{i=1}^{n} | \varepsilon_i(t^*) - \varepsilon_i(t_1) |^2 \leqslant n(\mu^*(t^* - t_1))^2 < n(\mu^* \Delta t)^2 \tag{57}$$

This contradicts (53) and the lemma follows.

Lemma 2

If $\| \tilde{\boldsymbol{\theta}}(0) \| \leqslant C_2$, if $\| \boldsymbol{\varepsilon}(\sigma) \| \leqslant C_1 - \delta$, $\forall \sigma$, $0 \leqslant \sigma \leqslant t_1$, and if $d_1 \| \varepsilon(0) \| + \gamma_1 \| \tilde{\boldsymbol{\theta}}(0) \| \leqslant C_1 - \delta$, with

$$\gamma_1 = \left(1 + \frac{d_1}{d_2} (\| \boldsymbol{\Lambda} - \boldsymbol{\Omega} \| + 2\mu_1\mu_2 \| \mathbf{P} \|)\right) \pi$$

then

$$\| \boldsymbol{\varepsilon}(\tau) \| \leqslant C_1 - \delta \quad \forall \tau, \quad t_1 < \tau \leqslant t_1 + \Delta t$$

with Δt defined by (53).

Proof. We first recall that by Theorem 1, $\boldsymbol{\theta}(t) \in \mathbf{B}_\theta$ for all $t \geqslant 0$. It also follows from Lemma 1 that

$$\| \boldsymbol{\varepsilon}(\tau) \| \leqslant C_1 \quad \forall \tau, \, 0 \leqslant \tau \leqslant t_1 + \Delta t \tag{58}$$

and hence by (49)

$$\mathbf{x}(\tau) \in \mathbf{B}_\mathbf{x} \quad \forall \tau, \quad 0 \leqslant \tau \leqslant t_1 + \Delta t$$

Then it is a standard result of systems theory[16] that for the linear time-varying system (46) we have (compare also with property (39))

$$\| \boldsymbol{\varepsilon}(\tau) \| \leqslant d_1 \| \boldsymbol{\varepsilon}(0) \| + \gamma_1 \| \tilde{\boldsymbol{\theta}}(0) \| \quad \forall \tau, \quad 0 \leqslant \tau \leqslant t_1 + \Delta t \tag{59}$$

The lemma follows.

Define the closed set $\mathbf{B}_0 \subseteq \mathbb{R}^n \times \mathbb{R}^p$:

$$\mathbf{B}_0 = \{(\mathbf{x}, \boldsymbol{\theta}) \mid d_1 \| \mathbf{W}(\mathbf{x}_d, \boldsymbol{\theta}) - \mathbf{W}(\mathbf{x}, \boldsymbol{\theta}) \| + \gamma_1 \| \boldsymbol{\theta}^* - \boldsymbol{\theta} \| \leqslant C_1 - \delta \quad \text{and} \quad \boldsymbol{\theta}^* - \boldsymbol{\theta} \| \leqslant C_2\} \tag{60}$$

We have the following convergence result.

Theorem 3

Under Assumptions 1–3 and 5, if the control law $u_a(\mathbf{x}, \mathbf{x}_d, \boldsymbol{\theta})$ is applied to the system (1) with the parameter estimator (25) and if $(\mathbf{x}(0), \boldsymbol{\theta}(0)) \in \mathbf{B}_0$, then the tracking error converges to zero:

$$\lim_{t \to \infty} \| \mathbf{x}_d(t) - \mathbf{x}(t) \| = 0 \tag{61}$$

Proof. $(\mathbf{x}(0), \boldsymbol{\theta}(0)) \in \mathbf{B}_0$ and Theorem 1 imply that

$$\boldsymbol{\theta}(t) \in \mathbf{B}_\theta \quad \forall t \quad \text{and} \quad \| \boldsymbol{\varepsilon}(0) \| \leqslant C_1 - \delta \tag{62}$$

This allows the initialization of the induction mechanism of Lemma 2, which in turn implies that

$$\| \boldsymbol{\varepsilon}(t) \| \leqslant C_1 - \delta \quad \text{and} \quad \mathbf{x}(t) \in \mathbf{B}_x \quad \forall t \geqslant 0 \tag{63}$$

Hence $\boldsymbol{\varepsilon}$, $\mathbf{x}$ and $\boldsymbol{\theta}$ are bounded for all t and the theorem follows by using exactly the same argumentation as in Theorem 2.

7. EXTENSIONS

So far our search for adaptive controllers has been restricted to a specific class of non-linear systems and to a specific set of assumptions, Assumptions 1–5. However, it is evident that many variants and extensions are feasible for other classes of systems and/or when one of the the assumptions is violated. We discuss two such extensions.

Input–Output adaptive linearization

We consider SISO non-linear systems, linearly parametrized and linear in the control input, of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}^*) + \mathbf{u}\mathbf{g}(\mathbf{x}, \boldsymbol{\theta}^*) \tag{64a}$$

$$y = h(\mathbf{x}) \tag{64b}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}$ is the control input vector, $\boldsymbol{\theta}^* \in \mathbb{R}^p$ is the parameter vector and $y = h(\mathbf{x}) \in \mathbb{R}$ is the process output. The system is supposed to be linearly parametrized, i.e. one can define $\boldsymbol{\varphi}(\mathbf{x}, \mathbf{u})$ such that

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\theta}^*) + \mathbf{u}\mathbf{g}(\mathbf{x}, \boldsymbol{\theta}^*) \triangleq \boldsymbol{\varphi}^{\mathrm{T}}(\mathbf{x}, \mathbf{u})\boldsymbol{\theta}^* \tag{65}$$

Assume that there exists a state feedback control law $u(\mathbf{x}, y_d, \boldsymbol{\theta}^*)$ able to stabilize the system and to achieve a *linear input–output* behaviour (this is an alternative to Assumptions 3 and 5).

If system (64) has a relative degree σ, it can be rewritten

$$\frac{\mathrm{d}^\sigma y}{\mathrm{d}t^\sigma} = L_f^\sigma h + u L_g L_f^{\sigma - 1} h \tag{66}$$

where L_f and L_g denote Lie derivatives. The linearizing control law is then written

$$u(\mathbf{x}, y_d, \boldsymbol{\theta}) = (L_g L_f^{\sigma-1} H)^{-1}\left(-L_f^{\sigma} h + \frac{d^{\sigma}}{dt^{\sigma}} y_d + \sum_{j=0}^{\sigma-1} \lambda_j \frac{d^j}{dt^j}(y_d - h(\mathbf{x}))\right) \tag{67}$$

with y_d the desired output and λ_j the design parameters defining the linear closed-loop dynamics. In the case of parametric uncertainty, the design of a *direct* adaptive controller for the system (64) is not a trivial task because the input–output model (66) is not linearly parametrized, though it can be reparametrized with a *very* large *over*-reparametrization (the issue is extensively discussed in Reference 13).

By contrast, the design of an *indirect* adaptive controller is immediate: it is sufficient to use the linearizing controller (67) with $\boldsymbol{\theta}^*$ replaced by a current estimate $\boldsymbol{\theta}$ provided by the parameter adaptation law (25) proposed in Section 4.

Non-linear parametrization

So far we have restricted ourselves to systems with a linearly parametrized state-space representation (Assumption 2). Actually, this assumption can be relaxed and the indirect adaptive controller extended to the class of *non-linearly* parametrized state-space models (see also Reference 15 for a related discussion):

$$\dot{\mathbf{x}} = \mathbf{A}^{-1}(\mathbf{x}, \boldsymbol{\theta}^*)[\mathbf{f}(\mathbf{x}, \boldsymbol{\theta}^*) + \mathbf{G}(\mathbf{x}, \boldsymbol{\theta}^*)\mathbf{u}]$$

under the following alternative assumptions.

(a) $\mathbf{f}(\mathbf{x}, \boldsymbol{\theta})$ and $\mathbf{G}(\mathbf{x}, \boldsymbol{\theta})$ are linearly parametrized.
(b) $\mathbf{A}(\mathbf{x}, \boldsymbol{\theta})$ is an $n \times n$ non-singular matrix for all $(\mathbf{x}, \boldsymbol{\theta}) \in \mathbf{B}_x \times \mathbf{B}_\theta$.
(c) $\mathbf{A}(\mathbf{x}, \boldsymbol{\theta})$ is linearly parametrized:

$$\mathbf{A}(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{A}_0(\mathbf{x}) + \sum_{i=1}^{p} \theta_i \mathbf{A}_i(\mathbf{x})$$

(d) There exist a sequence of primitives $\mathbf{Q}_i(\mathbf{x})$ such that

$$\mathbf{A}_i(\mathbf{x}) = \frac{\partial \mathbf{Q}_i(\mathbf{x})}{\partial \mathbf{x}} \quad \forall i$$

This class of systems may seem restrictive but is of primary importance because it involves the articulated rigid link mechanical systems (e.g. References 6 and 8).

8. CONCLUSIONS

Two *indirect* adaptive linearizing controllers have been proposed, in continuous time, for the class of non-linear systems which are linearly parametrized and state-feedback-linearizable. In each case the stability of the closed loop was analysed in detail. These control laws can be extended to other related situations, i.e. input–output feedback linearization and non-linear parametrizations. They can also be transposed to the case of discrete time control of continuous time systems through sampling and ZOH control action. This issue is discussed in Reference 17.

ACKNOWLEDGEMENTS

Professor Michel Gevers is gratefully acknowledged for helpful suggestions for improving the readability of the paper.

The results presented in this paper have been obtained within the framework of the Belgian Program on Concerted Research Actions and on Interuniversity Attraction Poles initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

The scientific responsibility rests with its authors.

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