Stable Adaptive Observers for Nonlinear Time-Varying Systems

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Abstract—**We describe an adaptive observer/identifier for single input-single output observable nonlinear systems that can be transformed to a certain observable canonical form. We provide sufficient conditions for stability of this observer. These conditions are in terms of the structure of the system and its canonical form, the boundedness of the parameter variations, and the sufficient richness of some signals. We motivate the scope of our canonical form and the use of our observer/identifier by presenting applications to time-invariant bilinear systems, nonlinear systems in phase-variable form, a biotechnological process, and a robot manipulator. In each case we present the specific stability conditions.**

I. Introduction

A GOAL in many practical applications is to combine *a priori* knowledge about a physical system with experimental data to provide on-line estimation of states or parameters of that system. A common situation is where one has a single input-single output (SISO) nonlinear time-varying deterministic system described as follows:

$$\begin{cases} \dot{z}=f(z, u, p) \\ y=z_1 \end{cases} \tag{1.1}$$

where $u(t) \in D_u \subseteq \mathbb{R}$ is a measurable input, possibly constrained to a subspace D_u of $\mathbb{R}$, $y(t) \in \mathbb{R}$ is a measurable output, $z(t) \in \mathbb{R}^n$ is a state vector, $p(t) \in D_p \subseteq R^q$ is a vector of unknown bounded possibly time-varying parameters, and $f(\cdot)$ is a smooth vector field on a smooth n-dimensional manifold. The parameters $p(t)$ can be (possibly unknown) functions of $z(t)$, as in the example of Section VII, but they will be treated as unknown possibly time-varying parameters. *A priori* knowledge may constrain $p(t)$ to be in a subspace D_p of $\mathbb{R}^q$. The structure of the system [i.e., the function $f(\cdot)$] is known from physical laws or from the user's experience, i.e., from *a priori* knowledge. Most often also, the states $z_i(t)$ and some of the unknown parameters $p_i(t)$ in (1.1) have a clear physical significance. Therefore, throughout this paper, we shall call (1.1) the given physical system, abbreviated GPS.

Now the user may want to combine this *a priori* knowledge with on-line measurements of $u(t)$ and $y(t)$ to solve one of the following three problems.

Problem 1: The on-line estimation of the nonmeasured states $z_i(t)$ of the GPS from input-output (I/O) data. This is called adaptive state estimation.

Problem 2: The on-line estimation of some of the physical parameters $p_i(t)$ of the GPS from I/O data. This is called adaptive parameter identification.

Problem 3: The design of an adaptive observer for the on-line estimation of the states, possibly in an equivalent state-space model. This is called adaptive observer design. It is to be distinguished from Problem 1 in that the states here need not be the physical z_i of the GPS; their estimates might be needed for a state-feedback controller, say.

Problem 2 makes sense only if the GPS is parameter identifiable, while Problems 1 and 3 require that, in addition, for all $u(t) \in D_u$ and all $p(t) \in D_p$ the GPS be locally observable: see [1]. We shall, therefore, make these assumptions throughout the paper.

One commonly used method to solve these three problems is to augment the state $z(t)$ with the parameter vector $p(t)$ and to implement an extended Kalman filter (EKF): see, e.g., [2]. In our opinion, such an approach has several important drawbacks.

1) A stability analysis for the EKF applied to the parameter estimation of a nonlinear system is very difficult and, to our knowledge, has never been performed. Even for a linear system, the EKF can diverge or lead to biased estimates; see [3].

2) The EKF is very expensive in computations and can be numerically ill-conditioned.

3) The use of the EKF requires an *a priori* choice of a stochastic model for the time variations of the parameter vector $p(t)$. This model may have no connection whatsoever with the physical reality.

There is therefore a clear incentive to search for simpler adaptive observers/identifiers that can be guaranteed stable. For linear time-invariant systems, stable adaptive observers have been proposed by, e.g., Lüders and Narendra [4]-[6], Narendra [7], and Kreisselmeier [8], [9]. The robustness of these observers in the case of unmodeled fast parasitic modes has been analyzed by Ioannou and Kokotovic [10].

Even in the case where $f(\cdot)$ is a known function of z, the design of asymptotically stable observers for general nonlinear systems is a very hard task; see [11]. The purpose of this paper is to show that, for many nonlinear systems of the form (1.1), Problem 3, and to a lesser extent, Problems 1 and 2, can be solved using a special adaptive observer/identifier, presented in Section III, which alleviates some of the disadvantages of the brute force EKF approach. This adaptive observer/identifier is an extension to nonlinear time-varying systems of the observer of Lüders and Narendra [5], which is known to be exponentially asymptotically stable (EAS) when applied to linear time-invariant systems; see [12]. The main advantages of our observer over the EKF are that:

1) its stability can be proved under reasonable conditions on the GPS and, in particular, for arbitrarily fast parameter variations with the proviso, however, that some signals must be sufficiently rich;

2) it is computationally much simpler than the EKF and, in particular, does not require the solution of a Riccati equation;

3) it does not need any dynamical model of the parameter variations (although if such a model were given, it could easily be incorporated).

A major feature of our approach is to transform the nonlinear GPS into a time-varying observable canonical form (called

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AOCF) which has the property that it is linear in the unknown quantities. These can include states, parameters, or combinations thereof. An adaptive observer is then derived for this canonical form and the main issue is to prove its global stability. This involves conditions on the structure of the GPS and on the input that guarantee the boundedness of these unknown quantities and their time variations, the persistence of excitation of certain regressors, the output reachability of certain auxiliary filters, etc. The proofs use mostly standard arguments on adaptive systems analysis and persistence of excitation, and extensions of these.

The outline of the paper is as follows. In Section II we describe the canonical form (AOCF) mentioned above, and motivate its use, while in Section III we show how an adaptive observer/identifier can be derived from this form. In Section IV we give a precise and complete set of sufficient conditions on the GPS and on the signals for the global stability of the observer/identifier. Our adaptive observer can be applied to all GPS for which a transformation to the AOCF exists. In order to show that this includes a very large number of observable and parameter identifiable nonlinear systems, we illustrate this with a number of examples:

- the class of time-invariant observable bilinear systems in Section V;
- the class of observable second-order nonlinear systems in "phase variable form" in Section VI;
- a nonlinear biotechnological process in Section VII;
- a nonlinear robotics application in Section VIII.

In each case, we will specify which of Problems 1, 2, or 3 can be solved and we will give conditions on the system (or the classes of systems) that guarantee the global stability of the adaptive observer and/or identifier.

II. Transformation to a Canonical Representation

A. The Adaptive Observer Canonical Form

From now on we consider the nonlinear systems of the form (1.1) which can be transformed, by a time-invariant possibly nonlinear smooth transformation

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = T(z, p, c_2, \cdots, c_n) \tag{2.1}$$

into the following equivalent form, which we shall call for convenience the *adaptive observer canonical form* (AOCF):

$$\begin{cases} \dot{x}(t) = Rx(t) + \Omega(\omega(t))\theta(t) + g(t) \\ y(t) = x_1(t). \end{cases} \tag{2.2}$$

In (2.1) and (2.2):

- $x(t) \in \mathbb{R}^n$ is a state-vector of the same dimension as $z(t)$;
- $\theta(t) \in \mathbb{R}^m$ is a vector of *unknown* time-varying parameters, which will be estimated on-line;
- $\omega(t) \in \mathbb{R}^s$ is a vector of *known* functions of $u(t)$ and $y(t)$, e.g., $\omega(t) = [u(t), y(t), y^2(t), \sin y(t)]$;
- $\Omega(\omega(t))$ is an $n \times m$ matrix whose elements are all of the form $\Omega_{ij}(\omega(t)) = \alpha_{ij}^T\omega(t)$ for *known* constant, possibly zero, vectors $\alpha_{ij} \in \mathbb{R}^s$;
- R is a *known* constant $n \times m$ matrix of the following form:

$$R = \left[\begin{array}{c|c} 0 & k^T \\ \hline 0 & \\ \cdot & F(c_2, c_3, \cdots, c_n) \\ \cdot & \\ 0 & \end{array}\right], \quad k^T \triangleq [k_2, \cdots, k_n] \tag{2.3}$$

where $k_2, \cdots, k_n$ are known constants and $F(c_2, \cdots, c_n)$ is a $(n-1) \times (n-1)$ constant matrix whose eigenvalues can be freely assigned by a proper choice of the constant design parameters $c_2, \cdots, c_n$. Typically, $F = \text{diag}(-c_2, \cdots, -c_n)$ with $c_i > 0$;

- $g(t) \in \mathbb{R}^n$ is a vector of *known* functions of time;
- $T(\cdot) \in \mathbb{R}^{n+m}$ is a continuous smooth transformation from (z, p) to (x, θ) parametrized by $n-1$ parameters $c_2, \cdots, c_n$.

For the system (2.2) we shall describe an adaptive observer and provide sufficient conditions on the GPS (1.1) to guarantee its global stability. This will provide a solution to Problem 3. If the transformation T in (2.1) is such that the inverse transformation

$$z = H_1(x, \theta, c_2, \cdots, c_n) \tag{2.4}$$

exists, is unique, and is continuous for all $u \in D_u$, then this will simultaneously solve Problem 1. If the inverse transformation

$$p = H_2(x, \theta, c_2, \cdots, c_n) \tag{2.5}$$

exists, is unique, and is continuous for all $u \in D_u$, this will also provide a solution to Problem 2. The applications in Sections VII and VIII will illustrate these points.

B. Discussion and Motivation

The structure of the AOFC (2.2) might appear very strange. Its crucial feature is its linearity in the unknown quantities $x(t)$ and $\theta(t)$; notice that $\Omega(\omega(t))$ is a possibly nonlinear or time-varying but *known* function of the data $u(t)$ and $y(t)$.

The motivation for introducing the AOFC is twofold.

- First its linear structure in $x(t)$ and $\theta(t)$ allows us to derive a globally stable adaptive observer/identifier for (2.2), which we describe in Section III. This observer is closely related to one initially derived by Lüders and Narendra [5] for linear time-invariant systems. An important new issue in our extension of the Lüders–Narendra observer is that of identifiability of $\theta(t)$ in the structure (2.2); this is related to the persistence of excitation of the regression vector that will appear in the adaptive observer. Conditions on R and Ω that guarantee this persistence of excitation (and are needed for global stability of the observer) will be derived in Section IV-C. They are one of the contributions of our paper.
- Another major contribution is to show that large numbers of SISO nonlinear systems of practical interest can be transformed into AOCF, even though some effort may be needed to find the transformation T: this will be illustrated in Sections V–VIII. The systems that can be transformed to AOCF include all time varying observable linear systems, all time-invariant observable bilinear systems, as well as second-order nonlinear systems in phase variable form (such as many mechanical systems).

III. The Adaptive Observer

For the system described by (2.2) we propose the following adaptive observer.

State Estimation:

$$\begin{cases} \dot{\hat{x}}(t) = R\hat{x}(t) + \Omega(\omega(t))\hat{\theta}(t) + g(t) + \begin{bmatrix} c_1\tilde{y}(t) \\ V(t)\dot{\hat{\theta}}(t) \end{bmatrix} & (3.1\text{a}) \\ \hat{y}(t) = \hat{x}_1(t), \quad \tilde{y}(t) \triangleq y(t) - \hat{y}(t) & (3.1\text{b}) \end{cases}$$

where c_1 is an arbitrary positive constant, and $c_2, \cdots, c_n$ are chosen such that the eigenvalues of $F(c_2, \cdots, c_n)$ are in the open left-half plane.

Parameter Adaptation:

$$\dot{\hat{\theta}}(t) = \Gamma\varphi(t)\tilde{y}(t) \tag{3.1c}$$

where Γ is an arbitrary positive definite matrix, normally chosen as $\Gamma = \text{diag}(\gamma_1, \cdots, \gamma_m)$, $\gamma_i > 0$.

Auxiliary Filter: $V(t)$ is an $(n-1)\times m$ matrix and $\varphi(t)$ is an m-vector; they are the solution of the following auxiliary filter:

$$\dot{V}(t)=FV(t)+\bar{\Omega}(\omega(t)),\ \ V(0)=0 \tag{3.1d}$$

$$\varphi(t)=V^T(t)k+\Omega_1^T(\omega(t)) \tag{3.1e}$$

where Ω_1 is the first row of $\Omega(\omega(t))$ and $\bar{\Omega}$ are the remaining rows, i.e.,

$$\Omega \triangleq \begin{bmatrix}\Omega_1\\ \bar{\Omega}\end{bmatrix}. \tag{3.2}$$

Recall that F and k are submatrices of R defined by (2.3), that $\Omega(\omega(t))$ and $g(t)$ are known functions and that $y = x_1$ is measured. It is worth noting that, most often, $\bar{\Omega}(\omega(t))$ contains a number of zero elements. If, in addition, F is diagonal, then the corresponding elements of $V(t)$ are identically zero, and the solution of (3.1d), (3.1e) simplifies considerably.

IV. Stability Conditions for the Adaptive Observer

In this section we derive a complete set of sufficient conditions on the GPS and on the signals for the global stability of the observer (3.1). To do this, we first have to derive stability conditions for the error system. Then we shall transfer these stability conditions to conditions on the GPS; this will involve analyzing the output reachability of the auxiliary filter.

A. The Error System

We define $\tilde{x} \triangleq x - \hat{x}$, $\tilde{\theta} \triangleq \theta - \hat{\theta}$ and we introduce the following auxiliary error vector:

$$\tilde{x}^* \triangleq \tilde{x} - \begin{bmatrix}0\\ V\tilde{\theta}\end{bmatrix}. \tag{4.1}$$

Using (2.2) and (3.1) we can then write, after some lengthy but straightforward manipulations, the following error system:

$$\begin{bmatrix}\dot{\tilde{x}}^*\\ \dot{\tilde{\theta}}\end{bmatrix} = \left[\begin{array}{c|c} R^* & \begin{array}{c}\varphi^T\\ 0\end{array}\\ \hline \begin{array}{cc}-\Gamma\varphi & 0\end{array} & 0\end{array}\right]\begin{bmatrix}\tilde{x}^*\\ \tilde{\theta}\end{bmatrix} + \left[\begin{array}{c}0\cdots 0\\ -V\\ \hline I_m\end{array}\right]\dot{\theta} \tag{4.2}$$

where

$$R^* \triangleq \left[\begin{array}{c|c} -c_1 & k^T\\ \hline \begin{array}{c}0\\ \cdot\\ \cdot\\ 0\end{array} & F(c_2,\cdots,c_n)\end{array}\right]. \tag{4.3}$$

Note that $\dim V(t) = (n-1)\times m$. Recall also that F is a constant matrix whose eigenvalues are completely determined by the parameters $c_2,\cdots,c_n$, which are at the designer's disposal in the transformation (2.1) that leads to the AOCF (2.2). As mentioned before, F will often be diag $(-c_2,\cdots,-c_n)$ with $c_i > 0$ and all different. It is then immediately clear from the error system that, if $\hat{\theta} = \theta$, the error $\tilde{x}$ is the solution of a linear time-invariant equation whose poles are entirely determined by the design parameters $c_1,\cdots,c_n$.

B. Stability Conditions on the Error System

We describe a set of sufficient conditions that guarantee:

i) that the homogeneous part of the error system is exponentially asymptotically stable (EAS);

ii) that the error system is therefore BIBS stable;

iii) that $\tilde{x}$ and $\tilde{\theta}$ are therefore bounded if $\dot{\theta}$ is bounded.

We denote

$$e(t) \triangleq \begin{bmatrix}\tilde{x}(t)\\ \tilde{\theta}(t)\end{bmatrix} \tag{4.4}$$

and S_Δ any set of signals $r(t)$ such that:

1) $r(t)$ is bounded $\forall t \geqslant 0$;

2) $\dot{r}(t)$ is bounded $\forall t \geqslant 0$ except possibly at a countable number of points $\{t_i\}$ such that $\min|t_i - t_j| \geqslant \Delta > 0$ for some arbitrary fixed Δ.

Theorem 4.1: If

i) $c_1 > 0$ and $c_2,\cdots,c_n$ are chosen such that $\text{Re}\,[\lambda_i(F)] < 0\ \forall i$

ii) $\varphi(t) \in S_\Delta$

iii) there exist positive constants α and T such that $\forall t \geqslant 0$

$$0<\alpha I \leqslant \int_t^{t+T}\varphi(\tau)\varphi^T(\tau)\,d\tau. \tag{4.5}$$

iv) there exists a positive constant M_1 such that $\forall t \geqslant 0$

$$|V(t)\dot{\theta}(t)| \leqslant M_1<\infty \tag{4.6}$$

then there exist finite constants K_1, K_2, and K_3 such that

$$1)\ |e(t)| \leqslant K_1|e(0)|+K_2 \qquad \forall t \geqslant 0 \tag{4.7a}$$

$$2)\ \limsup_{t\to\infty}|e(t)| \leqslant K_3M_1. \tag{4.7b}$$

Proof: We first consider the homogeneous part of (4.2). By eliminating $\tilde{x}^*$ we can write (with a slight, but by now standard abuse of notation)

$$\dot{\tilde{\theta}}(t) = -\Gamma\varphi(t)H(s)\{\varphi^T(t)\tilde{\theta}(t)\} \tag{4.8}$$

where $H(s) = e_1^T(sI - R^*)^{-1}e_1 = 1/s + c_1$ [with $e_1^T \triangleq (1\ 0 \cdots 0)$].

We note that $H(s)$ is strictly positive real (SPR) since $c_1 > 0$. Using Theorem 2.3 of [13], it now follows from assumptions i)–iii) that the homogeneous part of the error system (4.2) is exponentially asymptotically stable. The result (4.7) then follows from Theorem 3.1, p. 105 of [14], using assumption iv) and the relation (4.1) between $\tilde{x}^*$ and $\tilde{x}$.

The remainder of this section is concerned with transferring the conditions of Theorem 4.1 to stability conditions on the GPS and its representation in AOCF.

C. An Output Reachability Condition

We first give a structural condition on the AOCF which guarantees the output reachability of the auxiliary filter (3.1d), (3.1e). This in turn will ensure that $\varphi(t)$ is persistently exciting [cfr. condition iii)] when $\omega(t)$ is sufficiently rich. We define the following $m \times m$ matrix $S(\omega(t))$:

$$S(\omega(t)) \triangleq \begin{bmatrix}s_1(\omega(t))\\ \vdots\\ s_m(\omega(t))\end{bmatrix} \tag{4.9a}$$

where [see (3.1d), (3.1e)]

$$s_1=\Omega_1,\ s_j=k^TF^{j-2}\bar{\Omega},\qquad j=2,\cdots,m. \tag{4.9b}$$

Theorem 4.2: The auxiliary filter (3.1d), (3.1e) is output reachable from $\omega(t)$ if and only if $S(\omega(t))$ has full column rank over $\mathbb{R}$, i.e., iff there exists no constant m-vector $\beta \neq 0$ such that $S(\omega(t))\beta \equiv 0$.

Proof: Denote by $\xi(t)$ the vector made up of all nonzero elements of $V(t)$, arranged in arbitrary order, and let $\dim \xi = q$. Since each element of $\bar{\Omega}(\omega(t))$ and $\Omega_1(\omega(t))$ can be written as

$\alpha^T\omega(t)$, it follows that (3.1d), (3.1e) is equivalent with

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + B\omega(t) \\ \varphi(t) = C\xi(t) + D\omega(t) \end{cases} \tag{4.10}$$

where A, B, C, D are constant matrices of dimensions $q \times q$, $q \times s$, $m \times q$, and $m \times s$, respectively. Therefore, (3.1d), (3.1e) is output reachable from $\omega(t)$ if and only if (4.10) is so, i.e., if and only if the following $m \times ms$ output reachability matrix for (4.10) has full rank:

$$M = [D \ CB \ CAB \ \cdots \ CA^{m-2}B].$$

Now, because (3.1d), (3.1e), and (4.10) are equivalent, it follows that

$$s_1^T(\omega(t)) = \Omega_1^T(\omega(t)) = D\omega(t)$$

$$s_2^T(\omega(t)) = \bar{\Omega}^T(\omega(t))k = CB\omega(t)$$

$$s_3^T(\omega(t)) = \bar{\Omega}^T(\omega(t))F^Tk = CAB\omega(t)$$

etc... .

Equivalently

$$S(\omega(t)) = \begin{bmatrix} \omega^T & 0 & \cdots & 0 \\ 0 & \omega^T & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \omega^T \end{bmatrix} M^T. \tag{4.11}$$

The system is output reachable if and only if there exists no $\beta \neq 0$ such that $M^T\beta = 0$. The result follows immediately from (4.11).

Comment 4.1: Note that Theorem 4.2 is a condition on the structure of the canonical form (2.2), since k, F, Ω_1, and $\bar{\Omega}$ are all defined by R and Ω in (2.2). Hence, the output reachability of the auxiliary filter of our adaptive observer can be checked right from the start.

D. Conditions on the GPS for the Stability of the Adaptive Observer

Using Theorems 4.1 and 4.2, we can now spell out conditions on the GPS, the transformation T of (2.1), and the signals that will guarantee global stability of our adaptive observer.

We distinguish between conditions on the system (which can be checked beforehand) and conditions on the signals.

Structural Conditions (On the GPS and the Transformation T):

S.1: The GPS (1.1) is BIBO stable.

S.2: The GPS and the transformation T are such that

S.2.1: the elements of $\omega(t)$ are bounded functions of $u(t)$ and $y(t)$

S.2.2: R and Ω in (2.2) make $S(\omega(t))$ in (4.9) of full column rank over $\mathfrak{R}$ (i.e., the auxiliary filter (3.1d), (3.1e) is output reachable).

S.3: The parameter variation $\dot{p}(t)$ and the transformation T are such that

$$|\dot{\theta}(t)| \leqslant M_1 < \infty \qquad \text{for all } t \geqslant 0.$$

Conditions on the Signals:

SI.1: $u(t) \in S_\Delta$ for some $\Delta > 0$.

SI.2: There exist positive constants γ and T such that

$$\int_t^{t+T} W(\tau)W^T(\tau)\,d\tau \geqslant \gamma I > 0 \qquad \forall t \geqslant 0 \tag{4.12a}$$

with

$$W^T(\tau) = \frac{1}{(s+\delta)^q}[\omega^T(\tau) \ \dot{\omega}^T(\tau) \ \cdots \ \omega^{(q)T}(\tau)] \tag{4.12b}$$

where $\delta > 0$ but otherwise arbitrary, and q is the number of elements of $V(t)$ which are not identically zero.

Theorem 4.3: If the conditions S.1–S.3 and SI.1, SI.2 are satisfied, and if the design parameters $c_1, \cdots, c_n$ are chosen such that $c_1 > 0$ and Re $[\lambda_i(F(c_2, \cdots, c_n))] < 0 \ \forall i$, then there exist positive constants K_1 and K_2 such that (4.7) is satisfied.

Proof: The proof consists of checking that the conditions i)–iv) of Theorem 4.1 are satisfied. i) is obvious. $\varphi(t)$ is the output of the BIBO filter (3.1d), (3.1e) driven by elements of $\Omega(\omega(t))$ which are all of the form $\alpha^T\omega(t)$ for some real α. Therefore, $\varphi(t) \in S_\Delta$ by S.2.1, SI.1, and S.1, and hence ii) is satisfied. Since $V(t)$ in (3.1) contains q nonzero elements, the auxiliary filter can be modeled by a vector differential equation such as (4.10) with dim $\xi = q$. By S.2.2 the auxiliary filter is output reachable. Condition iii) then follows from SI.2, using Theorem 4.2 of Mareels and Gevers [15]. Finally, iv) follows from S.3 and the stability assumption on F.

In the remaining sections we shall apply our observer to a number of nonlinear systems. In each case we shall specialize the structural stability conditions to the specific application.

As for the stability conditions on the signals: condition SI.1 can of course always be met; SI.2 is a condition on the sufficient richness of $u(t)$ and on the unknown GPS. Explicit conditions can only be given in specific cases. Using some recent results of [15] we shall derive conditions on the GPS and on $u(t)$ which will entail SI.2 for three out of our four applications. The purpose in presenting these applications is twofold: first to show that many realistic applications can be transformed to the AOCF (2.2); second to show that for these nonlinear applications all the required stability conditions can be satisfied provided the parameter variations are not too fast.

V. Application to Bilinear Systems

Consider that the GPS is a time-invariant observable bilinear system described by

$$\begin{cases} \dot{z}(t) = M(p_M)z(t) + u(t)N(p_N)z(t) + K(p_K)u(t) \\ y(t) = z_1(t) \end{cases} \tag{5.1}$$

where M and N are constant $n \times m$ matrices and K is a constant n-vector, which depend on the constant but unknown vectors of physical parameters p_M, p_N, and p_K, respectively, and where $u(t) \in S_\Delta$ for some Δ. Then it was shown by Williamson [16] that there exists a constant nonsingular matrix T_1 such that, with $\zeta = T_1 z$, the GPS (5.1) is equivalent with

$$\begin{cases} \dot{\zeta}(t) = \left[\begin{array}{c|c} -a_1 & I_{n-1} \\ \vdots & \\ -a_n & 0 \cdots 0 \end{array}\right] \zeta(t) + \begin{bmatrix} b_1(\zeta) \\ \vdots \\ b_n(\zeta) \end{bmatrix} u(t) \\ \qquad = A\zeta(t) + b(\zeta)u(t) \\ y(t) = \zeta_1(t) \end{cases} \tag{5.2a}$$

where

$$A = T_1MT_1^{-1}, \ b(\zeta) = B\zeta + T_1K \tag{5.2b}$$

and

$$B = T_1NT_1^{-1} = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ b_{n1} & \cdots & \cdots & b_{nn} \end{bmatrix}. \tag{5.2c}$$

Note that (5.2) is a special case of the following "observer form"

$$\dot{z} = \left[\begin{array}{c|c} -a_1(z,t) & I_{n-1} \\ \vdots & \\ -a_n(z,t) & 0 \cdots 0 \end{array}\right] z + \begin{bmatrix} -b_1(z,t) \\ \vdots \\ -b_n(z,t) \end{bmatrix} u$$

$$y = z_1. \tag{5.3}$$

Therefore, by applying the transformation described in the Appendix, (5.2a) will be transformed to the AOCF (2.2) with R and Ω as in (A.10) and (A.5), $g(t) = 0$ and θ and x given by

$$\theta(t)=\bar{T}^{-1}\begin{bmatrix}\begin{bmatrix}t_1\\0\end{bmatrix}-a\\ b(x)\end{bmatrix},\quad x(t)=\bar{T}^{-1}\zeta(t) \tag{5.4}$$

where $\bar{T}$ and t_1 are defined in the Appendix, $a \triangleq (a_1, \cdots a_n)^T$ and, using (5.2b),

$$b(x)=B\bar{T}x+T_1K. \tag{5.5}$$

We can now apply the adaptive observer to this AOCF; the stability conditions are given in the following theorem.

Theorem 5.1: Let the GPS be given by (5.1). Then (5.1) can be transformed into AOCF by a constant transformation T. The adaptive observer (3.1) for this system is then globally stable (i.e., (4.7) is satisfied) if the following conditions hold.

B.1: The GPS (5.1) is BIBS stable.

B.2: The coefficients $c_1, \cdots, c_n$ are all positive and $c_2, \cdots, c_n$ are all different.

B.3: There exists $T > 0$ such that for all t and for all $s, \tau \in (t, t+T)$, $\|Gx(s) - Gx(\tau)\| < \epsilon$ for $\epsilon > 0$ and sufficiently small.

B.4: $u(t) \in S_\Delta$ for some $\Delta > 0$ and $u(t)$ is sufficiently rich of order $2n$, i.e., there exist constants t_1, $\alpha > 0$ and $T > 0$ such that, for any $\delta > 0$, the vector

$$\psi(t)=\frac{1}{(s+\delta)^{2n-1}}[1\ s\ s^2\ \cdots\ s^{2n-1}]^T u(t)$$

satisfies

$$\frac{1}{T}\int_t^{t+T}\psi(\tau)\psi^T(\tau)\,d\tau \geqslant \alpha I \qquad \forall t \geqslant t_1.$$

Proof: The existence of the transformation T follows from the discussion above. B.1 implies S.1. The structure of Ω and R, together with B.2, imply S.2. It follows from (5.4) and (5.5) that

$$\dot{\theta}=\begin{bmatrix}0\\ \bar{T}^{-1}B\bar{T}\dot{x}\end{bmatrix}=\begin{bmatrix}0\\ \bar{T}^{-1}B\dot{\zeta}\end{bmatrix}=\begin{bmatrix}0\\ \bar{T}^{-1}BT_1\dot{z}\end{bmatrix} \tag{5.6}$$

where B and T_1 are constant. Now, B.1 and B.4 imply that there exists γ such that

$$|\dot{z}| \leqslant \|M\||z|+|u|\|N\||z|+\|K\||u| \leqslant \gamma<\infty. \tag{5.7}$$

Hence, S.3 is satisfied. SI.1 follows from B.4. Instead of proving SI.2, we prove directly that $\varphi(\tau)$ in (3.1e) is persistently exciting: see condition iii) of Theorem 4.1. We note that in this case the regression vector $\varphi(t)$ takes the special form

$$\varphi^T(t)=\left[y\ \frac{y}{s+c_2}\ \cdots\ \frac{y}{s+c_n}\ u\ \frac{u}{s+c_2}\ \cdots\ \frac{u}{s+c_n}\right].$$

Condition (4.5) now follows from B.3 and B.4 using Corollary 4.2 and Theorem 6.2 of [15].

Comment 5.1: Conditions B.3 and B.4 essentially tell us that the regressor $\varphi(t)$ will be persistently exciting if $u(t)$ is sufficiently rich and if $\|G\dot{x}(t)\|$ is uniformly sufficiently small, i.e., if the bilinear system does not deviate too much from a linear one.

Comment 5.2: Theorem 5.1 tells us under what conditions the adaptive observer of Section III is globally stable for the AOCF obtained from the GPS (5.1). This gives a complete solution to Problem 3: it provides bounded estimates $\hat{x}$ and $\hat{\theta}$. By (5.4) and (5.5), this yields bounded $\hat{\zeta}$, $\hat{a}$, and $\hat{b}$. Whether Problems 1 and 2 can also be solved therefore depends on whether the constant transformation T_1 has a unique inverse for z and/or p; see Section II-A.

VI. Application to Second-Order Nonlinear Systems

Suppose the GPS has the following form:

$$\ddot{y}(t)+\alpha_1(\dot{y},y,p,t)\dot{y}(t)+\alpha_2(\dot{y},y,p,t)y(t)=b(\dot{y},y,p,t)u(t) \tag{6.1}$$

where α_1, α_2, and b are functions of $\dot{y}$, y, t and possibly of an unknown parameter $p(t)$, and where $u(t) \in D_u \subseteq S_\Delta$ for some $\Delta > 0$. The important point is that we shall treat a_1, a_2, and b as unknown time-varying parameters. Systems of the form (6.1) have applications in mechanics and robotics.

We observe that the GPS can be written in the "observer form" (5.3)

$$\begin{cases}\dot{z}=\begin{bmatrix}-a_1(t) & 1\\ -a_2(t) & 0\end{bmatrix}z+\begin{bmatrix}0\\ b(t)\end{bmatrix}u\\ y=z_1\end{cases} \tag{6.2}$$

where

$$a_1=\alpha_1,\ a_2=\alpha_2-\dot{\alpha}_1$$

$$z_1=y,\ z_2=\dot{y}+\alpha_1 y. \tag{6.3}$$

The form (6.2) can now be transformed into AOCF (see the Appendix) and the adaptive observer of Section III can be applied. The following theorem states the stability conditions for the observer.

Theorem 6.1: Let the GPS be given by (6.1). Then this system can be transformed to AOCF. The corresponding adaptive observer (3.1) is then globally stable if the following conditions hold.

P.1: The system (6.1) is BIBO stable.

P.2: $|\dot{a}_1| \leqslant K$, $|\dot{a}_2| \leqslant K$, $|\dot{b}| \leqslant K$, $|\ddot{a}_1| \leqslant K$ for all t and for K sufficiently small.

P.3: $u(t)$ and $y(t)$ belong to S_Δ for some $\Delta > 0$.

P.4: There exist $\delta > 0$, $t_0 > 0$ and $\alpha_1 > 0$ such that $\forall t \geqslant t_0$

$$\int_t^{t+\delta} W(\tau)W^T(\tau)\,d\tau \geqslant \alpha_1 I$$

where

$$W^T(\tau)=\frac{1}{(s+\gamma)^3}[u\ su\ s^2u\ s^3u]$$

for some arbitrary $\gamma > 0$.

Proof: See [17].

Comment 6.1: Sufficient conditions for BIBO stability (P.1) can be expressed in terms of bounds on the parameters $a_1(t)$, $a_2(t)$, and $b(t)$; see [17]. Conditions P.2 and P.4 will guarantee that the regression vector $\varphi(t)$ of the adaptive observer is persistently exciting; condition P.2 states that the parameters must vary slowly enough.

Comment 6.2: Theorem 6.1 provides a complete solution to Problems 3 and 1. In particular, it yields on-line estimates of $\dot{y}(t)$ using the transformations $z(t) = \bar{T}x(t)$, (A.6), (A.11), and (6.3). For systems of order higher than 2, the relationships (6.3) between the derivatives of y and the states z_i will depend upon derivatives of the a_i, and therefore only Problem 3 can be solved for such systems.

VII. Application to a Nonlinear Biotechnological System

A fermentation is a process of growth of a biomass by the consumption of an appropriate substrate under suitable environmental conditions. A critical issue in controlling fermentation processes is that cheap and reliable sensors for on-line measurement of the main biological variables (i.e., biomass, substrate, or byproducts concentrations), are most often not available. The use of adaptive observers as "software sensors" for some of these variables can therefore constitute a valuable alternative. Here we apply our adaptive observer to one such problem. A more complete overview of several different adaptive observers applied to a variety of biotechnological problems can be found in [18], [19].

The growth of biomass in a continuous stirred tank reactor is most often described by the following second-order model (with a unit flow rate):

$$\begin{cases} \dot{z}_1 = -\left[\dfrac{p_1(z_1, z_2, t)}{p_2}\right] z_1 z_2 - z_1 - p_3 z_2 + u_1 \\ \dot{z}_2 = [p_1(z_1, z_2, t)]\, z_1 z_2 - z_2 + u_2 \end{cases} \tag{7.1}$$

where z_1, z_2, u_1, u_2, p_2, p_3 are, respectively, the substrate concentration, the biomass concentration, the substrate feed rate, the biomass feed rate, the yield parameter, and the maintenance parameter. The time-varying parameter $p_0 = p_1 z_1$ is known as the "specific growth rate." It has been described by many different analytical expressions in the literature; among the most commonly used expressions are

$$\text{the Monod law: } p_0(z_1, z_2, t) = \frac{\mu^* z_1}{K_m + z_1} \tag{7.2}$$

$$\text{the Contois law: } p_0(z_1, z_2, t) = \frac{\mu^* z_1}{K_c z_2 + z_1} \tag{7.3}$$

where K_m, K_c are positive constants and μ^* is the maximum growth rate (which depends on temperature, pH, $\cdots$).

One problem of practical interest is to design an adaptive observer/identifier for the on-line estimation of $z_2(t)$, $p_1(t)$, p_2, and p_3 from on-line measurements of $u_1(t)$, $u_2(t)$, and $z_1(t)$; this is Problems 1 and 2 as described in Section II. We shall now show that our adaptive observer can solve these problems without making any assumption on a particular structure for $p_0(z_1, z_2, t)$.

Transformation to AOCF: We first transform the GPS (7.1) into AOCF as follows:

$$x_1 = z_1 \tag{7.4a}$$

$$x_2 = (1 - c_2) z_1 - p_3 z_2 \tag{7.4b}$$

$$\theta_1 = -\frac{p_1}{p_2} z_2 + (c_2 - 2) \tag{7.4c}$$

$$\theta_2 = \left[-(1 - c_2)\frac{p_1}{p_2} - p_1 p_3\right] z_2 \tag{7.4d}$$

$$\theta_3 = -p_3. \tag{7.4e}$$

This leads to the following AOCF:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y & 0 & 0 \\ 0 & y & u_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\ \qquad + \begin{bmatrix} u_1 \\ (1 - c_2) u_1 - (1 - c_2)^2 y \end{bmatrix} \\ y = x_1. \end{cases} \tag{7.5}$$

Note that the transformation (7.4) is uniquely invertible as follows:

$$p_3 = -\theta_3 \tag{7.6a}$$

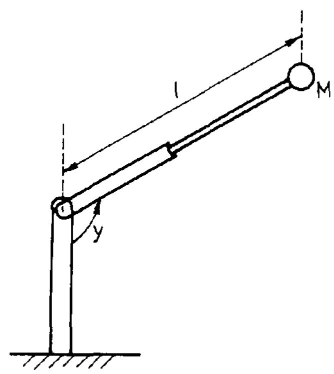

Fig. 1.

$$z_2 = \frac{-x_2 + (1 - c_2) y}{\theta_3} \tag{7.6b}$$

$$p_1 = \frac{(c_2 - 2 - \theta_1)(1 - c_2) + \theta_2}{\theta_3 z_2} \tag{7.6c}$$

$$p_2 = \frac{p_1 z_2}{(c_2 - 2) - \theta_1}. \tag{7.6d}$$

Therefore, on-line estimates of z_2, p_1, p_2, p_3 can be recovered from on-line estimates of x_2, θ_1, θ_2, θ_3, thereby solving Problems 1 and 2.

Stability of the Observer: On the basis of physical considerations, the following assumptions are quite realistic.

F.1: The specific growth rate is positive and bounded:

$$0 \leqslant p_0(z_1, z_2, t) \leqslant \bar{p}_0 \;\forall\; z_1, z_2, t.$$

F.2: p_2, p_3 constant; $p_2 > 0$; $p_2 p_3 \ll 1$.

F.3: The derivative of p_1 is bounded

$$\left|\frac{dp_1}{dt}\right| \leqslant M < \infty \;\forall\; t.$$

F.4: The biomass and substrate feed rates fulfill the following conditions:

a) $p_2 p_3 u_{\max} \leqslant u_1 + u_2/p_2 \leqslant u_{\max}$

b) $\dot{u}_1$ and $\dot{u}_2 \in S_\Delta$ for some Δ

c) $u = (u_1\ u_2)$ is sufficiently rich in the sense of assumption B.4.

F.5: The initial conditions $z_1(0)$ and $z_2(0)$ are such that

$$z_1(0) \leqslant u_{\max} \quad z_2(0) \leqslant p_2 u_{\max}.$$

Then, under these assumptions, the adaptive observer (3.1) applied to the AOCF (7.5) can be shown to be globally stable (i.e., the conditions S.1–S.3 and SI1, SI2 hold) by combining appropriately:

- a trivial extension of Lemma 1 in [20]
- Theorem 3.2 of chapter 4 in [14]
- Corollary 4.2 and Theorem 6.2 of [15]
- Theorem 4.2 of this paper.

VIII. Application to a Robot Manipulator

We consider an application to a telescopic arm in a vertical plane which performs a "pick and place" operation; see Fig. 1. We call M the mass of the load, $l(t)$ the variable length of the arm, $y(t)$ the angle with vertical axis, α_F and k_F the viscous friction coefficients, α_s and k_s the stiffness coefficients, u_1 and u_2 the voltages applied to the electrical motors in the joint and the arm, respectively. Assuming that the time constants of these motors are negligible, the torque in the joint and the longitudinal force in the arm are $T_1 = \alpha_m u_1$ and $T_2 = k_m u_2$, respectively, where α_m and k_m are unknown constants. We assume that the arm mass is negligible w.r.t. the load.

Then, the equations of motion are as follows:

$$Ml^2\ddot{y} + 2Ml\dot{l}\dot{y} + \alpha_F \dot{y} + \alpha_s y + Mlg \sin y = \alpha_m u_1 \tag{8.1}$$

$$M\ddot{l} + k_F \dot{l} + k_s l - Mg \cos y - \dot{y}^2 Ml = k_m u_2. \tag{8.2}$$

We consider an application where the angular position $y(t)$, the length $l(t)$, and the voltages $u_1(t)$ and $u_2(t)$ are measured on line, where the load M is known, and where it is desired to estimate the angular speed $\dot{y}(t)$, the longitudinal speed $\dot{l}(t)$, the coefficients α_F, α_S, k_F, k_s, and the motor parameters α_m and k_m. The idea of performing an experiment to estimate the mechanical parameters of a robot is typical in robotics applications.

We now rewrite (8.1) and (8.2) as follows:

$$\ddot{y} + \alpha_1 \dot{y} + \alpha_2 y + \frac{g}{l} \sin y = \alpha_m \frac{u_1}{Ml^2} \tag{8.3}$$

$$\ddot{l} + \alpha_3 \dot{l} + \alpha_4 l - g \cos y = k_m \frac{u_2}{M} \tag{8.4}$$

where

$$\alpha_1(t) = \frac{\alpha_F}{Ml^2} + 2\frac{\dot{l}}{l} \qquad \alpha_2(t) = \frac{\alpha_s}{Ml^2} \tag{8.5}$$

$$\alpha_3 = \frac{k_F}{M} \qquad \alpha_4(t) = \frac{k_s}{M} - \dot{y}^2. \tag{8.6}$$

We now apply the following transformation T:

$$x_1 = y$$

$$x_2 = \dot{y} + (\alpha_1 - c_2) y$$

$$x_3 = l$$

$$x_4 = \dot{l} + (\alpha_3 - c_4) l$$

$$\theta_1 = c_2 - \alpha_1$$

$$\theta_2 = \dot{\alpha}_1 - (\alpha_2 + c_2\theta_1)$$

$$\theta_3 = \alpha_m$$

$$\theta_4 = c_4 - \alpha_3$$

$$\theta_5 = \alpha_4 + c_4\theta_4$$

$$\theta_6 = k_m.$$

Equations (8.3) and (8.4) can then be written in the following two AOCF:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y & 0 & 0 \\ 0 & y & \frac{u_1}{Ml^2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{l} \sin y \end{bmatrix} \tag{8.7}$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -c_4 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} l & 0 & 0 \\ 0 & -l & \frac{u_2}{M} \end{bmatrix} \begin{bmatrix} \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} + \begin{bmatrix} 0 \\ g \cos y \end{bmatrix}. \tag{8.8}$$

Using the adaptive observer of Section III on both AOCF (8.7) and (8.8), we obtain the following on line estimates:

$$\hat{\dot{y}} = \hat{x}_2 + \hat{\theta}_1 y$$

$$\hat{\dot{l}} = \hat{x}_4 + \hat{\theta}_4 l$$

$$\hat{\alpha}_F = Ml^2(c_2 - \hat{\theta}_1) - 2Ml\hat{\dot{l}}$$

$$\hat{\alpha}_m = \hat{\theta}_3$$

$$\hat{k}_F = M(c_4 - \hat{\theta}_4)$$

$$\hat{k}_s = M(\hat{\theta}_5 - c_4\hat{\theta}_4) + \hat{\dot{y}}^2$$

$$\hat{k}_m = \hat{\theta}_6.$$

Therefore, the adaptive observer provides a solution to Problems 1 and 3 and, partially, to Problem 2. We shall not derive the explicit stability conditions here.

IX. Conclusions

We have shown how a large number of observable nonlinear SISO systems can be transformed to a "canonical form" that has the crucial property of being "linear in the unknown quantities." We have then shown how an adaptive observer, inspired by an earlier observer for linear time-invariant systems, can be applied to this transformed system. Our main contribution, besides this canonical form, has been to establish a precise set of sufficient conditions for global stability of our observer. In conclusion, we should like to point out two limitations of our present theory, which may also be avenues for further research.

First, it is not clear how general our AOCF is. We have worked with numerous observable nonlinear models, originating from practical applications, for which a transformation to AOCF could be found. In this paper, we have tried to convince the reader of the wide applicability of this canonical form by presenting four examples of very diverse nonlinear models or classes of models. However, we have not been able to prove that any observable system can be transformed to AOCF and it would be interesting to find the exact conditions on the GPS that make it equivalent to an AOCF.

Secondly, it appears from our stability theorems that stability of the adaptive observer will be guaranteed for arbitrarily fast parameter variations, as long as they are bounded. This is an important feature, which contrasts with the more classical result on time-varying systems (see [21]), which roughly states that if an error system, say, is exponentially stable for all values of a parameter $p(t)$ in a compact set, then there exists a δ such that the system remains exponentially stable if $|\dot{p}(t)| < \delta$. This δ could be arbitrarily small, while our bounds on $|\dot{p}|$ or $|\dot{\theta}|$ can be arbitrarily large. However, there is a condition on sufficient richness of $u(t)$ in all our theorems which introduces an upper bound on the allowable speed of parameter variation.

Appendix

Transformation from (5.3) to the AOCF (2.2)

First observe that (5.3) can be rewritten as

$$\dot{z} = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 \cdot & \cdot \cdot \cdot & \cdot 0 \end{bmatrix} z + \Omega(\omega)\bar{\theta}$$

$$y = z_1 \tag{A.1}$$

where

$$\bar{\theta} \triangleq [-a_1(z,t) \cdots -a_n(z,t) \; b_1(z,t) \cdots b_n(z,t)]^T \tag{A.2}$$

$$= [-a^T(z,t) \; b^T(z,t)]^T \tag{A.3}$$

with obvious definitions of $a(z, t)$ and $b(z, t)$ and where

$$\omega^T = [u \;\; y] \tag{A.4}$$

$$\Omega(\omega) = \begin{bmatrix} y & 0 & \cdots & 0 & u & 0 & \cdots & 0 \\ 0 & y & & \vdots & 0 & u & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & & \cdots & 0 \;\; y & 0 & & \cdots & 0 \;\; u \end{bmatrix}. \tag{A.5}$$

The transformation from (A.1) to AOCF proceeds in two steps.

Step 1: We apply a similarity transformation $z = \bar{T}x$ initially proposed by Lüders and Narendra [4] in their derivation of an adaptive observer for linear time-invariant systems. Let $\bar{T}$ be a constant $n \times n$ matrix defined as follows:

$$\bar{T} \triangleq \begin{bmatrix} 1 & 0 \cdots\cdots 0 \\ t_1 & t_2 \cdots\cdots t_n \end{bmatrix} \tag{A.6}$$

where $t_1 \in \mathbb{R}^{n-1}$. The column vector

$$\begin{bmatrix} 1 \\ t_1 \end{bmatrix}$$

is made up of the coefficients of $\Pi_{i=2}^{n} (s + c_i)$, while the column vector t_j is made up of the coefficients of $\Pi_{\substack{i=2 \\ i \neq j}}^{n} (s + c_i)$ for $j = 2, \cdots, n$. The coefficients $c_2, \cdots, c_n$ are those appearing in the transformation T of (2.1); they are all different (which implies that $\bar{T}$ is nonsingular), but otherwise arbitrary. Then, with $z = \bar{T}x$, (A.1) is equivalent with (see [4])

$$\dot{x} = \left[\begin{array}{c|cccc} m_1 & 1 & \cdots & \cdots & 1 \\ \hline m_2 & -c_2 & & & 0 \\ \vdots & & & \ddots & \\ m_n & 0 & & & -c_n \end{array}\right] x + \bar{\Omega}(\omega)\bar{\theta}$$

$$y = x_1 \tag{A.7}$$

where $\bar{\Omega}(\omega) = \bar{T}^{-1}\Omega(\omega)$ and the constant vector $m^T \triangleq [m_1 \cdots m_n]$ is uniquely defined from $(c_2, \cdots, c_n)$ by

$$\bar{T}_m = \begin{bmatrix} t_1 \\ 0 \end{bmatrix}. \tag{A.8}$$

Step 2: It follows from (A.3) and (A.5) that

$$\bar{\Omega}(\omega)\bar{\theta} \triangleq \bar{T}^{-1}\Omega(\omega)\bar{\theta} = \Omega(\omega)\begin{bmatrix} -\bar{T}^{-1}a(z, t) \\ \bar{T}^{-1}b(z, t) \end{bmatrix}. \tag{A.9}$$

Finally, using (A.9) and (A.5), it is easy to rewrite (A.7) as

$$\dot{x} = \left[\begin{array}{c|cccc} 0 & 1 & \cdots & \cdots & 1 \\ \hline 0 & -c_2 & & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & & & -c_n \end{array}\right] x + \Omega(\omega)\theta = Rx + \Omega(\omega)\theta$$

$$y = x_1 \tag{A.10}$$

with

$$\theta(t) \triangleq \begin{bmatrix} m - \bar{T}^{-1}a(z, t) \\ \bar{T}^{-1}b(z, t) \end{bmatrix} = \bar{T}^{-1}\begin{bmatrix} \begin{bmatrix} t_1 \\ 0 \end{bmatrix} - a(z, t) \\ b(z, t) \end{bmatrix}. \tag{A.11}$$

Notice that the transformations from z to x, and from $\bar{\theta}$ to θ are invertible, since $\bar{T}$ and m depend only on the known constants $c_2, \cdots, c_n$. This means that if the GPS is given in the form (5.3), any solution to Problem 3 also solves Problems 1 and 2.

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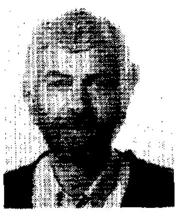

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