The usefulness of viscosity for the robustness of boundary feedback control of an unstable fluid flow system ⋆

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Abstract

The potential lack of robustness to delays and characteristic velocities is a well known feature of boundary feedback control of hyperbolic systems. We consider the case of a one-dimensional fluid system with the flow rate as control input located at one boundary and the density as measured output located at the other boundary. Using a simple model where friction and viscosity are neglected, the system is open-loop unstable but it can be stabilized by a dynamic controller that involves a delayed output feedback. However this control is not robust with respect to delay uncertainties. Our main contribution is to show that this lack of robustness is actually an artefact which stems from the assumption that the fluid viscosity is negligible when modelling the fluid motion. In the presence of a small unknown viscosity in the model, it appears that the non-robust feedback for the inviscid case is actually a perfectly robust stabilizer for the viscous system and that there is an intrinsic uniform margin of stability which is independent of the viscosity value even if it is asymptotically small.

Key words: Linear hyperbolic system; Boundary control; Feedback stabilization; Fluid flow system; Robustness; Viscosity; Diffusion.

1 Introduction

The lack of robustness of boundary feedback stabilization of hyperbolic systems when there is a transmission delay between the output measurement and the control input is a well known problem (see for instance [2], [3], [4], [14], [23], [33] and the references therein). When sensing and actuation are not co-located (see e.g. [1], [7], [8], [15], [22], [25], [28], [29], [32]), a similar lack of robustness may occur with respect to modelling uncertainties. The problem happens when the stabilization requires a dynamic feedback control law that includes delayed values of the output or relies, using a state observer, on the exact compensation of the characteristic velocities of the plant. In that case, the lack of robustness means that arbitrarily small modelling errors or unknown transmission delays may result in unstable solutions of a closed-loop system which is a priori theoretically exponentially stable.

In a previous paper [7], we have discussed this problem for a simple unstable transportation system with anti-located boundary actuation and sensing. For that system, we have shown that the presence of a small additional diffusion term in the model may be sufficient to guarantee the robustness of the control against delay uncertainties and to compute an upper bound to the exponential decay of the solutions. Related issues regarding the diffusion-robustness of feedback control were also recently addressed for an advection-convection process in [9] and for the viscous nonlinear Saint-Venant equations in [27]. Depending on the configuration of the boundary measurement and control devices, it could also be a relevant issue for the control of extended fluid flow systems represented, for instance, by Saint-Venant-Exner models as considered in [5, Chapter 5], [16] or [17].

In this article, using a simple model of fluid flow, we go further in the analysis of the problem by demonstrating that a strict margin of stability, guaranteed by the diffusion, holds even for an asymptotically small diffusion and

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that we are able to give the exact value of this margin in the left half complex plane. This means that, in the case of a liquid fluid flow, the apparent lack of robustness is actually an artefact which stems from the assumption that the fluid viscosity is negligible when modelling the fluid motion.

Our paper is organized as follows. The control problem is presented and motivated in Section 2. We consider a classical simple linear model of the motion of fluids when friction and viscosity are negligible. The control input is the flow rate at one boundary and the measured output is the density at the other boundary. It is first shown that this control system is open-loop unstable and cannot be stabilized by a simple proportional output feedback. Then, it is shown that the system can be stabilized by a dynamic controller that involves a delayed output feedback. However, this control turns out to be not robust with respect to delay uncertainties precisely because the delay requires a (utopian) exact knowledge of the characteristic velocity.

The main contribution of this paper is to show that the robustness of the control against delay uncertainties is recovered as soon as an arbitrary small diffusion is present in the system. For that purpose, in Section 3, it is assumed that the fluid is slightly viscous and the model is modified accordingly by introducing a viscosity coefficient η. The corresponding (unstable) input-output transfer function is computed. Then in Sections 4 and 5, we show that the dynamic output (non robust) feedback designed for the inviscid case stabilizes the viscous system. Furthermore, in this case, the control proves to be perfectly robust, even if the (unknown) viscosity is almost negligible. In addition, and this is a new result compared to our previous paper [7], using degree theory [10] we are able to determine the exact value of the stability margin which appears to be uniform with respect to the viscosity coefficient η (i.e. independent of the value of η when η is small).

Some final conclusions are given in Section 6.

2 Presentation and motivation of the control problem

Consider hyperbolic systems of two linear conservation laws over a finite interval in one spatial dimension with general form:

$$\partial_t H + \partial_x Q = 0, \tag{1a}$$

$$\partial_t Q + c_1 c_2 \partial_x H + (c_1 - c_2)\partial_x Q = 0, \tag{1b}$$

where $t \in [0, +\infty)$ is the time coordinate, $x \in [0, L]$ is the spatial coordinate, c_1 and c_2 are two real positive constants. In these equations $H(t, x)$ is the density and $Q(t, x)$ is the flow density of some extensive quantity of interest. Therefore, this system is called a "density-flow" system.

The model (1) can be used to represent many physical systems. In particular, it can be a valid approximate linearized model for applications in fluid mechanics where friction and diffusion are neglected. We can mention for example gas pipelines where H is the gas density and Q is the gas flow rate (see e.g. [18] and [21]), open channels where H is the water depth and Q is the water flow rate (see e.g. [5, Chapter 1], [6] and [26]) or the motion of liquid fluids in rigid pipes where H is the piezometric head and Q is the flow rate ([5, Chapter 1], [20], [30]).

In this paper, we are concerned with the solutions of the Cauchy problem for the system (1) under an initial condition:

$$H(0, x),\ Q(0, x), \qquad x \in [0, L], \tag{2}$$

and two boundary conditions of the form

$$Q(t, 0) = Q_0(t),\ Q(t, L) = Q^*,\ t \in [0, +\infty), \tag{3}$$

where $Q_0(t)$ is a time varying input flow which can be assigned by the operator while Q^* is a given constant outflow, arbitrarily imposed by the operating conditions.

Since any pair of constant states $H(t, x) = H^*, Q(t, x) = Q^*, \forall t$ and $\forall x \in [0, L]$, can be a steady-state, it is clear that the system (1)-(2)-(3) has a continuum of non-isolated equilibria which are not asymptotically stable. It is therefore relevant to study the feedback stabilization of this system.

In this paper, for the system (1)-(2)-(3), we address the *boundary control* issue where it is assumed that both actuation and sensing are located at the boundaries. The objective is to design an output feedback controller that regulates the density $H(t, x)$ at a desired set point H^* while keeping the system steady state (H^*, Q^*) exponentially stable.

A fairly common situation in practice occurs when the control input is the inflow rate $Q_0(t)$ and the measured output is the density $H(t, 0)$, that is a situation where actuation and sensing are *co-located* at the same boundary. In that case it is well known that a simple proportional output feedback control is sufficient to stabilize the system, see e.g. [5, Chapter 2].

In this paper, however, we address the more challenging situation where *actuation and sensing are anti-located*, with the control input $Q_0(t)$ at one boundary and the measured output $H(t, L)$ at the other boundary.

For simplicity, we consider the special case where the constant outflow $Q^* = 0$ and where the two characteristic velocities are identical, i.e. $c_1 = c_2 = c > 0$. In that case the model (1) becomes the following simple wave equation

$$\partial_t H + \partial_x Q = 0, \tag{4a}$$

$$\partial_t Q + c^2 \partial_x H = 0, \tag{4b}$$

where c is the wave celerity. For instance $c = \sqrt{gH^*}$ in the case of open channels or c is the speed of sound in the case of fluids in pipes.

Now, introducing the following notations for the deviations of the system states from the steady state

$$h(t,x) = H(t,x) - H^*, \qquad q(t,x) = Q(t,x) - Q^*, \tag{5}$$

the open-loop control system (2)-(3)-(4) may be written

$$\partial_t h(t,x) + \partial_x q(t,x) = 0, \tag{6a}$$

$$\partial_t q(t,x) + c^2 \partial_x h(t,x) = 0, \tag{6b}$$

$$q(t,0) = U(t), \tag{6c}$$

$$q(t,L) = 0, \tag{6d}$$

$$Y(t) = h(t,L), \tag{6e}$$

with control input $U(t)$ and measured output $Y(t)$.

In the frequency domain, the system (6) is written

$$s\mathrm{h}(s,x) + \partial_x \mathrm{q}(s,x) = 0, \tag{7a}$$

$$s\mathrm{q}(s,x) + c^2 \partial_x \mathrm{h}(s,x) = 0, \tag{7b}$$

$$\mathrm{q}(s,0) = \mathrm{U}(s), \tag{7c}$$

$$\mathrm{q}(s,L) = 0, \tag{7d}$$

$$\mathrm{Y}(s) = \mathrm{h}(s,L), \tag{7e}$$

with $s \in \mathbb{C}$ being the Laplace complex variable. In these equations $\mathrm{h}(s,x)$, $\mathrm{q}(s,x)$, $\mathrm{Y}(s)$ and $\mathrm{U}(s)$ denote the Laplace transforms of $h(t,x)$, $q(t,x)$, $Y(t)$ and $U(t)$ respectively.

By differentiating (7a) with respect to x, we have

$$s\partial_x \mathrm{h}(s,x) = -\partial^2_{xx}\mathrm{q}(s,x). \tag{8}$$

Using this relation in (7b), we get

$$s^2 \mathrm{q}(s,x) - c^2 \partial^2_{xx}\mathrm{q}(s,x) = 0. \tag{9}$$

Then, for any value of $s \neq 0$, the solution of this differential equation (9) is of the form

$$\mathrm{q}(s,x) = A(s)e^{sx/c} + B(s)e^{-sx/c}. \tag{10}$$

Using this expression in the boundary conditions (7c), (7d), (7e), we have

$$\mathrm{q}(s,0) = A(s) + B(s) = U(s), \tag{11}$$

$$\mathrm{q}(s,L) = A(s)e^{s\tau} + B(s)e^{-s\tau} = 0, \quad \text{with} \quad \tau = \frac{L}{c}, \tag{12}$$

$$\mathrm{Y}(s) = \mathrm{h}(s,L) = -\frac{1}{s}\partial_x \mathrm{q}(s,L) = \frac{1}{c}\left(-A(s)e^{s\tau} + B(s)e^{-s\tau}\right). \tag{13}$$

Eliminating $A(s)$ and $B(s)$ between these three equations, we obtain the transfer function of the open-loop control system (6):

$$G_o(s) = \frac{\mathrm{Y}(s)}{\mathrm{U}(s)} = \frac{2\,e^{-s\tau}}{c(1 - e^{-2s\tau})}. \tag{14}$$

The poles of the system are the roots of the characteristic equation

$$e^{2s\tau} - 1 = 0. \tag{15}$$

The open-loop system (6) is therefore clearly not asymptotically stable since all poles are located on the imaginary axis.

Despite its apparent simplicity, this unstable system cannot be stabilized with a simple proportional output feedback of the form

$$U(t) = -k_c Y(t) \tag{16}$$

where $k_c \neq 0$ is a control tuning parameter. Indeed, for the system (7) with the control law (16) the characteristic equation of the closed-loop system is:

$$e^{2s\tau} + 2k_c e^{s\tau} - 1 = 0. \tag{17}$$

Solving this equation for $e^{s\tau}$, we get

$$e^{s\tau} = -k_c \pm \sqrt{1 + k_c^2}. \tag{18}$$

Then for any $k_c \neq 0$ there is an infinity of system poles $\sigma + i\omega$ lying on two vertical lines with real parts:

$$\sigma = c \ln\left(\sqrt{1 + k_c^2} + |k_c|\right) > 0 \tag{19a}$$

$$\text{or} \quad \sigma = c \ln\left(\sqrt{1 + k_c^2} - |k_c|\right) < 0. \tag{19b}$$

It follows that the unstable system (6) cannot be stabilized with the static controller (16) since half of the poles of the closed loop have a strictly positive real part. We conclude that the feedback stabilization necessarily requires a dynamic controller that involves delayed output values (this includes full-state feedback control, given that the output measurement is located at a boundary).

From (14) it follows that the input-output dynamics of system (6) in the time domain can alternatively be represented by the delay-difference equation

$$Y(t) - Y(t-2\tau) = \frac{2}{c}\,U(t-\tau). \tag{20}$$

Hence the system is exponentially stabilized with a simple delayed output feedback of the form

$$U(t) = -\frac{c}{2}Y(t-\tau) \tag{21}$$

such that the closed-loop dynamics reduce to $Y(t) = 0$ $\forall t$ after the initial transient.

It is however well known that the boundary feedback stabilization of hyperbolic systems with delayed control should be considered with caution because it is sensitive to arbitrarily small delay modeling errors. For our case, this lack of robustness can be highlighted by rewriting the model in Riemann coordinates defined as

$$y_1(t,x) := \tfrac{1}{2}(q(t,x) + c\,h(t,x)), \tag{22a}$$

$$y_2(t,x) := \tfrac{1}{2}(q(t,x) - c\,h(t,x)). \tag{22b}$$

In these coordinates, the open-loop control system (6) is equivalent to

$$\partial_t y_1(t,x) + c\partial_x y_1(t,x) = 0, \tag{23a}$$

$$\partial_t y_2(t,x) - c\partial_x y_2(t,x) = 0, \tag{23b}$$

$$y_1(t,0) = -y_2(t,0) + U(t), \tag{23c}$$

$$y_2(t,L) = -y_1(t,L), \tag{23d}$$

$$Y(t) = \frac{2}{c}y_1(t,L). \tag{23e}$$

Moreover, a time domain representation of the dynamical control (21) may be defined as

$$\begin{aligned} &\partial_t y_3(t,x) + c\partial_x y_3(t,x) = 0, \\ &y_3(t,0) = Y(t), \\ &U(t) = -\frac{c}{2}y_3(t,L). \end{aligned} \tag{24}$$

Closing the system (23) with the controller (24), the boundary conditions of the closed-loop system can be expressed as

$$\begin{pmatrix} y_1(t,0) \\ y_2(t,L) \\ y_3(t,0) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & -c/2 \\ -1 & 0 & 0 \\ 2/c & 0 & 0 \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} y_1(t,L) \\ y_2(t,0) \\ y_3(t,L) \end{pmatrix}. \tag{25}$$

Now, for the matrix $\mathbf{K}$ defined in (25), it can be shown that

$$\bar{\rho}(\mathbf{K}) = \sqrt{2} \quad \text{for all } c > 0, \tag{26}$$

where the function $\bar{\rho}(\mathbf{K})$ is defined as follows:

$$\bar{\rho}(\mathbf{K}) := \max\left\{\rho\left(\operatorname{diag}\{e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\}\,\mathbf{K}\right); (\theta_1,\theta_2,\theta_3)^\mathsf{T} \in \mathbb{R}^3\right\}, \tag{27}$$

$\rho(M)$ denoting the spectral radius of the matrix M. For the computation of $\bar{\rho}(\mathbf{K})$ we refer the reader to [7, Appendix A].

This is an important point because it is well known (see [31], [24, Chapter 9, Theorem 6.1] or [5, Chapter 3, Theorem 3.8 and Corollary 3.10]) that $\bar{\rho}(\mathbf{K}) < 1$ is a necessary (and sufficient) condition to have a stability which is robust against uncertainties in the characteristic velocities. In the framework of this paper, this means that the control law (21) (or (24)) is not robust with respect to uncertainties on the value of the celerity parameter c. More precisely, if we assume that in the model (1) the characteristic velocities are $c_1 = c + \varepsilon_1$ and $c_2 = c + \varepsilon_2$ with ε_1 and ε_2 representing small modelling uncertainties, then equations (23a)-(23b) in Riemann coordinates are replaced by

$$\partial_t y_1(t,x) + (c+\varepsilon_1)\partial_x y_1(t,x) = 0, \tag{28a}$$

$$\partial_t y_2(t,x) - (c+\varepsilon_2)\partial_x y_2(t,x) = 0, \tag{28b}$$

and the closed-loop system may become unstable, with poles moving to the right half complex plane even for arbitrarily small ε_i perturbations. This will be illustrated in Figure 2 of Section 5.

In this paper, our contribution will be to show that this lack of robustness is actually an artefact which stems from the assumption that the viscosity can be neglected when modelling the fluid system (4). We shall show that the robustness of the output feedback stabilization is recovered as soon as an arbitrary small diffusion is present in the system even with the simple delay control law (21).

3 The open-loop control system with viscosity

Let us modify the control system (6) by assuming that the fluid is slightly viscous. For simplicity and without loss of generality, we assume a unit nominal length $L = 1$ and a unit nominal delay $\tau = L/c = 1$. The system dynamics in the time domain are therefore simplified as follows:

$$\partial_t h(t,x) + \partial_x q(t,x) = 0, \tag{29a}$$

$$\partial_t q(t,x) + \partial_x h(t,x) - \eta\partial^2_{xx} q(t,x) = 0, \tag{29b}$$

$$q(t,0) = U(t), \tag{29c}$$

$$q(t,1) = 0, \tag{29d}$$

$$Y(t) = h(t,1). \tag{29e}$$

An additional diffusion term $\eta \partial_{xx}^2 q$ is introduced in equation (29b) with the viscosity coefficient $\eta > 0$. The other equations remain unchanged.

Here also it is easily seen that any pair of constant states $h(t,x) = H^*, q(t,x) = 0$, $\forall t$ and $\forall x \in [0,1]$, is a steady-state corresponding to $U(t) = 0$. Thus the system (29) has a continuum of non-isolated equilibria which are therefore not asymptotically stable.

In the frequency domain, the system (29) is written

$$s\mathrm{h}(s,x) + \partial_x \mathrm{q}(s,x) = 0, \tag{30a}$$

$$s\mathrm{q}(s,x) + \partial_x \mathrm{h}(s,x) - \eta \partial_{xx}^2 \mathrm{q}(s,x) = 0, \tag{30b}$$

$$\mathrm{q}(s,0) = \mathrm{U}(s), \tag{30c}$$

$$\mathrm{q}(s,1) = 0, \tag{30d}$$

$$\mathrm{Y}(s) = \mathrm{h}(s,1). \tag{30e}$$

Using equation (8) which also holds for this system, we have from (30b)

$$s^2 \mathrm{q}(s,x) - (1+\eta s)\partial_{xx}^2 \mathrm{q}(s,x) = 0. \tag{31}$$

Then, for any value of $s \neq 0$, the solution of this differential equation (31) is of the form

$$\mathrm{q}(s,x) = A(s)e^{\lambda(s)x} + B(s)e^{-\lambda(s)x} \tag{32}$$

where $\lambda(s)$ and $-\lambda(s)$ are the roots of the polynomial

$$(1+\eta s)\lambda^2 - s^2 = 0. \tag{33}$$

Using the solution (32) in the boundary conditions (30c), (30d), (30e), we have

$$\mathrm{q}(s,0) = A(s) + B(s) = \mathrm{U}(s), \tag{34}$$

$$\mathrm{q}(s,1) = A(s)e^{\lambda(s)} + B(s)e^{-\lambda(s)} = 0, \tag{35}$$

$$\mathrm{Y}(s) = \mathrm{h}(s,1) = -\frac{1}{s}\partial_x \mathrm{q}(s,1) \tag{36}$$

$$= -\frac{\lambda(s)}{s}\left(A(s)e^{\lambda(s)} - B(s)e^{-\lambda(s)}\right). \tag{37}$$

Eliminating $A(s)$ and $B(s)$ between these three equations, the transfer function (see, for example, [12] and [13, Chapter 7]) of the open-loop control system (29) is the meromorphic function:

$$G(s) = \frac{\mathrm{Y}(s)}{\mathrm{U}(s)} = \frac{2\, e^{\lambda(s)}}{(\sqrt{1+\eta s})(e^{2\lambda(s)} - 1)}. \tag{38}$$

It can be checked that for all $\eta \neq 0$ the transfer function $G(s)$ has a pole at the origin. Therefore we recover that the open-loop system (29) is not asymptotically stable whatever the value of the viscosity η.

4 Stability of the closed-loop system with viscosity

Let us now assume that the system is closed with the control law (21) i.e. $U(t) = -\frac{1}{2}Y(t-1)$. In the frequency domain, this control law is

$$\mathrm{U}(s) = -\tfrac{1}{2}e^{-s}\mathrm{Y}(s). \tag{39}$$

It follows that the characteristic equation of the closed-loop system (38), (39) is:

$$\mathcal{G}(\eta,s) = e^s\sqrt{1+\eta s}\left(e^{s/\sqrt{1+\eta s}} - e^{-s/\sqrt{1+\eta s}}\right) + 1 = 0. \tag{40}$$

Our purpose is now to address the spectral stability of this closed-loop system. For a given value of the viscosity η, the spectrum $\mathcal{S}_\eta$ of the closed-loop system is the set of the poles which are the roots of the characteristic equation (40):

$$\mathcal{S}_\eta = \{s \in \mathbb{C} : \mathcal{G}(\eta,s) = 0\}. \tag{41}$$

Moreover, the maximal spectral abscissa is defined as the supremum of the real parts of the spectrum and denoted as follows:

$$\sigma_\eta := \sup\{\Re(s) : s \in \mathcal{S}_\eta\}. \tag{42}$$

Our goal is to know whether this maximal spectral abscissa is negative or, in other words, whether there are no unstable poles located in the right hand side of the complex plane. The spectrum is illustrated in Figure 1 for $\eta = 0.005, 0.01$ and 0.1. From this figure it can be seen that, at least for η sufficiently small, it appears that the maximal spectral abscissa $\sigma_\eta \simeq -\ln(2)$ is indeed negative and seems to be independent of η. This intuitive observation is in accordance with the following theorem which is the main contribution of this paper.

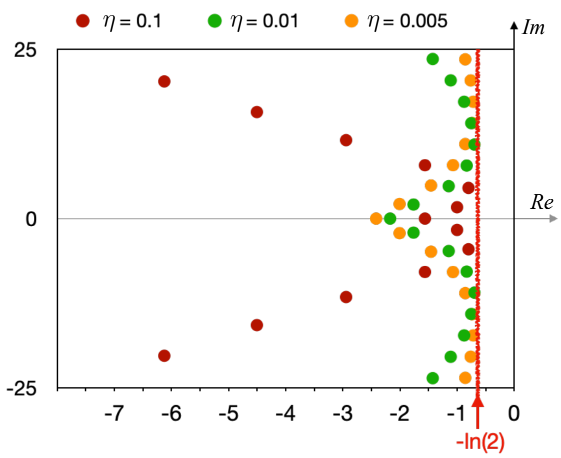

Fig. 1. The spectrum of the closed-loop system for $\eta = 0.005, 0.01$ and 0.1.

Theorem 1 *For every $\delta \in (0, \ln(2))$, there exists $\eta^* > 0$ such that*

$$-\ln(2) - \delta < \sigma_\eta < -\ln(2) + \delta \tag{43}$$

for all $\eta \in (0, \eta^*]$.

With a view to proving Theorem 1, let us now consider a sequence

$$(\eta_n)_{n\in\mathbb{N}} \quad \text{with} \quad 0 < \eta_n \in \mathbb{R},\ \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to+\infty} \eta_n = 0^+, \tag{44}$$

and an associated sequence of system poles

$$(s_n)_{n\in\mathbb{N}} \quad \text{such that} \quad s_n \in \mathcal{S}_{\eta_n}, \forall n \in \mathbb{N}. \tag{45}$$

Obviously it follows from (40) that

$$\mathcal{G}(\eta_n, s_n) = e^{s_n}\vartheta_n\left(e^{s_n/\vartheta_n} - e^{-s_n/\vartheta_n}\right) + 1 = 0,\ \forall n \in \mathbb{N} \tag{46}$$

with

$$\vartheta_n = \sqrt{1 + \eta_n s_n}. \tag{47}$$

In order to prove Theorem 1, we will look to the adherent points of the sequences $(s_n)_{n\in\mathbb{N}}$ when $n \to +\infty$ (i.e. when $\eta_n \to 0^+$). By definition, we know that $\bar{s}$ is an adherent point of a sequence $(s_n)_{n\in\mathbb{N}}$ if and only if there exists a subsequence which converges to $\bar{s}$. With a slight abuse of notation, we will often write

$$s_n \longrightarrow \bar{s} \quad \text{or} \quad \lim_{n\to+\infty} s_n = \bar{s} \tag{48}$$

to signify that $\bar{s}$ is an adherent point of a sequence $(s_n)_{n\in\mathbb{N}}$ but it is implied that the convergence in fact only relies on the adequate subsequence. This holds also for all other sequences that are introduced later in this article.

The proof of Theorem 1 is built from the two following lemmas.

Lemma 1 *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (44) and $(s_n)_{n\in\mathbb{N}}$ be an associated sequence of the form (45). Denoting $\sigma_n = \Re(s_n)$ and $\omega_n = \Im(s_n)$, let us consider induced associated sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$. Then*

(i) if $\bar{\sigma}$ is an adherence point of the sequence $(\sigma_n)_{n\in\mathbb{N}}$, then $\bar{\sigma} \in [-\infty, +\infty)$;
(ii) if $\bar{\sigma} \in (-\infty, +\infty)$ and 2θ is an adherence point of the sequence $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$, then $2\theta \in [0, +\infty)$.

The proof of this lemma is given in Appendix A.

Lemma 2 *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of the form (44) and $(s_n)_{n\in\mathbb{N}}$ be any associated sequence of the form (45) with induced associated sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\eta_n\omega_n^2)_{n\in\mathbb{N}}$. Let $\bar{\sigma}$ be an adherence point of the sequence $(\sigma_n)_{n\in\mathbb{N}}$. Then*

$$\bar{\sigma} \leqslant -\ln(2). \tag{49}$$

Proof. From Lemma 1, we know that $\bar{\sigma} \in [-\infty, +\infty)$. If $\bar{\sigma} = -\infty$ the lemma is obviously satisfied. Hence, according to Lemma 1, we assume from now on that

$$\lim_{n\to+\infty} \sigma_n = \bar{\sigma} \in (-\infty, +\infty) \tag{50a}$$

$$\text{and} \quad \lim_{n\to+\infty} \eta_n\omega_n^2 = 2\theta \in [0, +\infty). \tag{50b}$$

From (50), as $n \to +\infty$, we have

$$\vartheta_n = \sqrt{1 + \eta_n s_n} = 1 + o(1), \tag{51}$$

$$\frac{s_n}{\vartheta_n} = s_n\left(1 - \frac{\eta_n s_n}{2}\right) + o(1) = \bar{\sigma} + \theta + i\omega_n + o(1), \tag{52}$$

$$e^{s_n} = e^{\bar{\sigma}}e^{i\omega_n} + o(1). \tag{53}$$

Then from (46), (51), (52) and (53), we get

$$e^{\bar{\sigma}}e^{i\omega_n}\left(e^{\bar{\sigma}+\theta+i\omega_n} - e^{-\bar{\sigma}-\theta-i\omega_n}\right) + 1 = o(1), \tag{54}$$

which implies that

$$\left(e^{2\bar{\sigma}+\theta+2i\omega_n} - e^{-\theta}\right) + 1 = o(1). \tag{55}$$

Looking at the imaginary part of the left and right hand side of (55), we obtain that either

$$e^{2i\omega_n} = 1 + o(1) \text{ as } n \to +\infty, \tag{56}$$

or

$$e^{2i\omega_n} = -1 + o(1) \text{ as } n \to +\infty. \tag{57}$$

Let us first assume that (56) holds. From (55) and (56), we get

$$\left(e^{2\bar{\sigma}+\theta} - e^{-\theta}\right) + 1 = 0, \tag{58}$$

which implies that

$$e^{2\bar{\sigma}} = e^{-2\theta} - e^{-\theta}. \tag{59}$$

Since $\theta \in [0, +\infty)$, (59) leads to $e^{2\bar{\sigma}} \leq 0$ which is impossible. Hence (56) cannot hold and therefore we must have (57). As above we now get

$$e^{2\bar{\sigma}} = e^{-\theta} - e^{-2\theta}. \tag{60}$$

Note that

$$\max\{e^{-\theta} - e^{-2\theta};\ \theta \in [0, +\infty)\} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}. \tag{61}$$

Moreover the maximum is achieved for $\theta = \ln(2)$. Then, from (60) and (61), we finally get that

$$\bar{\sigma} \leqslant \frac{1}{2}\ln\left(\frac{1}{4}\right) = -\ln(2). \tag{62}$$

This completes the proof of Lemma 2.

Proof of Theorem 1.

Now, denoting $\sigma = \Re(s)$ and $\omega = \Im(s)$ and introducing the following change of variables

$$\gamma^2 = \eta, \qquad \varphi = \omega\gamma, \qquad \gamma \in [0, +\infty), \tag{63}$$

we have from (40) and (63):

$$\begin{aligned}\mathcal{G}(\eta, s) &= e^{\sigma + i\varphi/\gamma}\tilde{\vartheta}\left(e^{(\sigma+i\varphi/\gamma)/\tilde{\vartheta}} - e^{-(\sigma+i\varphi/\gamma)/\tilde{\vartheta}}\right) + 1 \\ &= \widetilde{\mathcal{G}}(\gamma, \sigma, \varphi)\end{aligned} \tag{64}$$

with

$$\tilde{\vartheta} := \sqrt{1 + \gamma^2\sigma + i\gamma\varphi}. \tag{65}$$

Let $B \in C^\infty(\mathbb{R}; \mathbb{C})$ be defined by

$$B(\varphi) := -e^{-\varphi^2/2} + 1 \tag{66}$$

and $A : (0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be defined by

$$A(\gamma, \sigma, \varphi) := (\widetilde{\mathcal{G}}(\gamma, \sigma, \varphi) - B(\varphi))e^{-2i\varphi/\gamma}. \tag{67}$$

From (64), (66) and (67), we can show that there exists $\gamma_0 > 0$ such that:

A can be extended as a continuous function on $[0, \gamma_0) \times (-2\ln(2), +\infty) \times \mathbb{R}$.

Indeed, there exists $\gamma_0 > 0$ such that for any $(\gamma, \sigma, \varphi) \in [0, \gamma_0) \times (-2\ln(2), +\infty) \times \mathbb{R}$,

$$\Re(\tilde{\vartheta}^2) = \Re(1 + \gamma^2\sigma + i\gamma\varphi) > 0, \tag{68}$$

and therefore the map $(\gamma, \sigma, \varphi) \to 1/\sqrt{1 + \gamma^2\sigma + i\gamma\varphi}$ is continuous (since the square root is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$). From (67) this means that A is continuous on $[0, \gamma_0) \times (-2\ln(2), +\infty) \times \mathbb{R}$. Therefore, in order to get (4), it only remains to look at the behavior of the map A as $\gamma \to 0^+$. From (64), (66) and (67), we have

$$\begin{aligned}A(\gamma, \sigma, \varphi) &= e^{\sigma(1+1/\tilde{\vartheta})}e^{-i(\varphi/\gamma)(1-1/\tilde{\vartheta})}\tilde{\vartheta} \\ &\quad - e^{\sigma(1-1/\tilde{\vartheta})}e^{-i(\varphi/\gamma)(1+1/\tilde{\vartheta})}\tilde{\vartheta} + e^{-\varphi^2/2}e^{-2i\varphi/\gamma},\end{aligned} \tag{69}$$

The right hand side of the first line of (69) tends to $e^{2\sigma+\varphi^2/2}$ when $\gamma \to 0^+$. For the terms in the second line of (69) we have, as $\gamma \to 0^+$,

$$\begin{aligned}&- e^{\sigma(1-1/\tilde{\vartheta})}e^{-i(\varphi/\gamma)(1+1/\tilde{\vartheta})}\tilde{\vartheta} + e^{-\varphi^2/2}e^{-2i\varphi/\gamma} \\ &= e^{-\varphi^2/2}e^{-2i\varphi/\gamma} \\ &\qquad - (1 + o(1))e^{-i(\varphi/\gamma)(2-(\gamma^2\sigma+i\gamma\varphi)/2+o((\gamma^2\sigma+i\gamma\varphi)))} \\ &= e^{-\varphi^2/2}e^{-2i\varphi/\gamma} - (1 + o(1))e^{-2i(\varphi/\gamma)-\varphi^2/2+o(1)} \\ &= (e^{-\varphi^2/2}e^{-2i\varphi/\gamma})o(1)\end{aligned} \tag{70}$$

which clearly tends to 0 when $\gamma \to 0^+$, irrespective of σ and φ. Still denoting by A this continuous extension, we then have

$$A(0, \sigma, \varphi) = e^{2\sigma+\varphi^2/2},\ \forall\sigma \in (-2\ln(2), +\infty),\ \forall\varphi \in \mathbb{R}. \tag{71}$$

Note that, by (66) and (71), we have

$$A(0, -\ln(2), \sqrt{2\ln(2)}) = \frac{1}{2}, \quad B(\sqrt{2\ln(2)}) = \frac{1}{2} \tag{72}$$

and consequently

$$-A(0, -\ln(2), \sqrt{2\ln(2)}) + B(\sqrt{2\ln(2)}) = 0. \tag{73}$$

In order to prove Theorem 1, the idea is then to look for a solution $(\tilde{\sigma}, \tilde{\varphi})$ of the characteristic equation (see (67))

$$\widetilde{\mathcal{G}}(\gamma, \tilde{\sigma}, \tilde{\varphi}) = e^{2i\tilde{\varphi}/\gamma}A(\gamma, \tilde{\sigma}, \tilde{\varphi}) + B(\tilde{\varphi}) = 0, \tag{74}$$

such that $\tilde{\sigma}$ is close to $-\ln(2)$, $\tilde{\varphi}$ is close to $\sqrt{2\ln(2)}$ and $e^{2i\tilde{\varphi}/\gamma}$ is close to -1 if γ is sufficiently small. We can look for this solution by using the degree theory (see [10, Appendix B]). Actually we shall see that the condition $e^{2i\tilde{\varphi}/\gamma} \simeq -1$ must not be imposed a priori but is a consequence of the requirements that $\tilde{\sigma} \simeq -\ln(2)$, $\tilde{\varphi} \simeq \sqrt{2\ln(2)}$ and (74) are satisfied.

In order to use the degree theory, we consider an open rectangular domain $\Omega^\gamma \subset \mathbb{R}^2$ defined by

$$\Omega^\gamma = \big(-\ln(2) - \delta, -\ln(2) + \delta\,\big) \times \big(\,k(\gamma)\pi\gamma, (k(\gamma)+1)\pi\gamma\,\big) \tag{75}$$

and the function

$$\phi^\gamma : (\sigma, \varphi) \to e^{2i\varphi/\gamma}A(\gamma, \sigma, \varphi) + B(\varphi) \in \mathbb{C} \equiv \mathbb{R}^2 \tag{76}$$

defined on the closure $\overline{\Omega^\gamma}$ of the domain Ω^γ, with the function $k(\gamma)$ defined as

$$k(\gamma) := \left\lfloor \frac{\sqrt{2\ln(2)}}{\pi\gamma} \right\rfloor. \tag{77}$$

We then have the following lemma.

Lemma 3 *There exists $\gamma_1 > 0$ such that, for every $\gamma \in (0, \gamma_1)$,*

$$\textit{degree}\big(\phi^\gamma, \Omega^\gamma, \mathbf{0}\big) = 1. \tag{78}$$

Proof. According to the degree theory (see [10, Appendix B]), equality (78) just means that, if (σ, φ) follows the boundary of the rectangle Ω^γ clockwise, then the function ϕ^γ describes a curve in $\mathbb{R}^2$ that does not pass through the origin $\mathbf{0}$ but encircles the origin exactly once and in the clockwise direction. This is shown in Appendix B.

It follows from Lemma 3 (see, for example, [10, Proposition B.10]) that, for any $\delta \in (0, \ln(2))$ and any $\gamma \in (0, \gamma_1]$, there exists

$$(\tilde{\sigma}, \tilde{\varphi}) \in (-\ln(2)-\delta, -\ln(2)+\delta) \times (k(\gamma)\pi\gamma, (k(\gamma)+1)\pi\gamma) \tag{79}$$

such that (74) holds. Let

$$\tilde{s} = \tilde{\sigma} + i\frac{\tilde{\varphi}}{\gamma}. \tag{80}$$

It follows from (64), (74) and (80) that for any $\gamma \in (0, \gamma_1]$

$$\mathcal{G}(\gamma^2, \tilde{s}) = 0. \tag{81}$$

Hence, we have shown that, for any $\eta \in (0, \gamma_1^2]$, there exists $\tilde{s} \in \mathcal{S}_\eta$ with $\Re(\tilde{s}) \in (-\ln(2) - \delta, -\ln(2) + \delta)$ and consequently that

$$-\ln(2) - \delta < \sigma_\eta. \tag{82}$$

Moreover (see the proof of Theorem 1 in [7]) Lemma 2 implies that there exists $\eta_1 > 0$ such that

$$\sigma_\eta < -\ln(2) + \delta \quad \forall \eta \in (0, \eta_1]. \tag{83}$$

We conclude that Theorem 1 is satisfied with $\eta^* = \min\{\gamma_1^2, \eta_1\}$.

Remark 1 *Since, for every $\eta > 0$, there exists $\delta_\eta > 0$ such that the function $s \to \mathcal{G}(\eta, s)$ is holomorphic on $\{z \in \mathbb{C} \mid \Re(z) > \delta_\eta\}$, then its degree gives exactly the number of zeroes of $\mathcal{G}$ (see, for example, [19, pages 45 and 46]). In particular, if $\eta = \gamma^2 > 0$ is sufficiently small, there is one and only one $\tilde{s}$ satisfying* (81) *such that*

$$\Re(\tilde{s}) \in (-\ln(2) - \delta_\eta, -\ln(2) + \delta_\eta), \tag{84}$$

$$\tilde{\omega} := \Im(\tilde{s}) \in (k(\gamma)\pi, (k(\gamma)+1)\pi). \tag{85}$$

Note that (77) *and* (85) *imply that*

$$\lim_{\eta \to 0^+} \eta \tilde{\omega}^2 = 2\ln(2). \tag{86}$$

Remark 2 *The degree theory was also used in [11] to study the spectrum of a closed-loop partial differential equation control system to deal with high frequency issues, but in a simpler situation.*

Remark 3 *Our proof of the existence of $(\tilde{\sigma}, \tilde{\varphi})$ as above uses very mild assumptions on the maps A and B, namely that they are simply two continuous functions. The proof can be extended to more general continuous functions $A : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ and $B : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$. Assume that there exists $\sigma_* \in \mathbb{R}$, $\delta_* \in (0, +\infty)$ and $\varphi_* \in (0, +\infty)$ such that*

$$A(0, \sigma_*, \varphi_*) = B(0, \sigma_*, \varphi_*) \neq 0, \tag{87}$$

and the map

$$\sigma \to |A(0, \sigma, \varphi_*)| - |B(0, \sigma, \varphi_*)| \tag{88}$$

is strictly monotone on $[\sigma_* - \delta_*, \sigma_* + \delta_*]$.

Then the above proof can be adapted to get that, for every $\delta_0 \in (0, \delta_*)$, there exists γ_0 such that, for every $\gamma \in (0, \gamma_0)$, there exists $\tilde{\sigma} \in \mathbb{R}$ and $\tilde{\varphi} \in \mathbb{R}$ such that

$$\widetilde{\mathcal{G}}(\gamma, \tilde{\sigma}, \tilde{\varphi}) = e^{2i\tilde{\varphi}/\gamma} A(\gamma, \tilde{\sigma}, \tilde{\varphi}) + B(\gamma, \tilde{\sigma}, \tilde{\varphi}) = 0, \tag{89}$$

$$(\tilde{\sigma}, \tilde{\varphi}) \in (\sigma_* - \delta_0, \sigma_* + \delta_0) \times (k(\gamma)\pi\gamma, (k(\gamma)+1)\pi\gamma), \tag{90}$$

where the function k is defined by

$$k(\gamma) := \left\lfloor \frac{\varphi_*}{\pi\gamma} \right\rfloor. \tag{91}$$

This generalization can be used, for instance, to solve Conjecture 1 in our previous paper [7] where we addressed the output feedback stabilization of an unstable interconnection of transport systems with anti-located sensing and control.

Remark 4 *In this section, to simplify the calculations, we addressed the special case where the model is normalized with a unit celerity $c = 1$ and a unit length $L = 1$. In that case we found that, with a small viscosity η, there is a stability margin $SM = \ln(2)$ which is independent of the value of η. But obviously, by following the same approach, the stability margin can also be determined in the general case where c and L have arbitrary positive real values. Considering the control system (with the no-*

tations defined in Section 2)

$$\partial_t H + \partial_x Q = 0, \tag{92a}$$

$$\partial_t Q + c^2 \partial_x H - \eta \partial_{xx} Q = 0, \tag{92b}$$

$$Q(t,0) = U(t), \tag{92c}$$

$$Q(t,L) = Q^*, \tag{92d}$$

$$Y(t) = H(t,L), \tag{92e}$$

under an output feedback control law

$$U(t) = \frac{c}{2}\big(H^* - H(t-\tau, L)\big) \quad \textit{with} \quad \tau = \frac{L}{c}, \tag{93}$$

it can be shown that the stability margin is

$$SM = \frac{c}{L}\ln(2) \tag{94}$$

for the closed-loop system (92)*,* (93)*. As one might intuitively expect, we see that the stability margin increases with the celerity c and decreases with the length L.*

5 Robustness analysis.

In order to analyze the robustness of the control, we now assume that there is some parameter uncertainty in the fluid flow model. We therefore introduce an additional perturbation ε such that the normalized open-loop system (29) in the time domain is now written with a celerity term $c^2 = 1 + \varepsilon$ instead of $c^2 = 1$:

$$\partial_t h(t,x) + \partial_x q(t,x) = 0, \tag{95a}$$

$$\partial_t q(t,x) + (1+\varepsilon)\partial_x h(t,x) - \eta \partial_{xx}^2 q(t,x) = 0, \tag{95b}$$

$$q(t,0) = U(t), \tag{95c}$$

$$q(t,1) = 0, \tag{95d}$$

$$Y(t) = h(t,1). \tag{95e}$$

We suppose, as before, that the system is closed with the control law (21):

$$U(t) = -\tfrac{1}{2} Y(t-1). \tag{96}$$

Remark that this control law depends on the theoretical delay ($\tau = 1$), ignoring the uncertainty represented by ε corresponding to a delay $\tau = 1/\sqrt{1+\varepsilon}$.

The characteristic equation of the closed-loop system in the frequency domain is then modified as follows:

$$\mathcal{G}_\varepsilon(\eta, s) = e^s \sqrt{1+\varepsilon+\eta s}\left(e^{s/\sqrt{1+\varepsilon+\eta s}} - e^{-s/\sqrt{1+\varepsilon+\eta s}}\right) + 1 = 0. \tag{97}$$

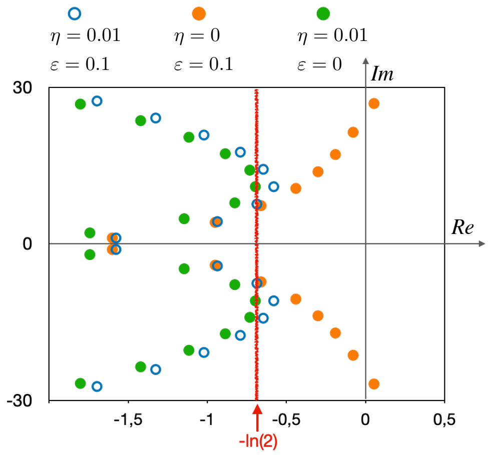

Fig. 2. Spectrum of the closed-loop system: influence of viscosity on stability in case of parameter uncertainty.

As in the previous section, we introduce the spectrum $\mathcal{S}_{\eta,\varepsilon}$ and the maximal abscissa $\sigma_{\eta,\varepsilon}$ defined by

$$\mathcal{S}_{\eta,\varepsilon} := \{s \in \mathbb{C} : s \text{ is solution of (97)}\}, \tag{98}$$

$$\sigma_{\eta,\varepsilon} := \sup\{\Re(s) : s \in \mathcal{S}_{\eta,\varepsilon}\}. \tag{99}$$

We remark that, by definition, we have

$$\mathcal{S}_{\eta,0} = \mathcal{S}_\eta \quad \text{and} \quad \sigma_{\eta,0} = \sigma_\eta. \tag{100}$$

We then have the following robustness theorem.

Theorem 2 *Let $\delta > 0$ and $\eta > 0$ be such that Theorem 1 holds, i.e.*

$$\sigma_{\eta,0} \leqslant -ln(2) + \delta. \tag{101}$$

Then there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ the maximal spectral abscissa $\sigma_{\eta,\varepsilon}$ satisfies

$$\sigma_{\eta,\varepsilon} \leqslant -ln(2) + 2\delta. \tag{102}$$

The proof of this theorem is omitted because it is quite similar to the proof of Theorem 2 in [7].

The theorem is illustrated in Figure 2. In this figure, we can see what happens in the situation where there is no viscosity ($\eta = 0$) but a slight parameter uncertainty ($\varepsilon = 0.1$) such that the actual delay is $\tau = 0.95$ instead of $\tau = 1$. Although the ideal system (without modelling uncertainty) should be exponentially stable, it appears that it becomes unstable with poles (represented by orange dots in Figure 2) moving to the right-half complex plane, showing clearly the lack of robustness.

In contrast, when there is some viscosity ($\eta = 0.01$) and no uncertainty ($\varepsilon = 0$), we know from Theorem 1 that the closed-loop system must be exponentially stable as it can be seen with the spectrum of green dots (actually reprinted from Figure 1) which is entirely strictly located in the left half plane.

Finally, illustrating Theorem 2, the robustness to modelling uncertainty ($\varepsilon = 0.1$) in the presence of a small viscosity ($\eta = 0.01$) is clearly evidenced by the spectrum of blue dots which, resulting from a small shift of the green spectrum, remains entirely in the left half plane.

6 Conclusion

We have discussed the output feedback stabilization of an unstable fluid system with anti-located boundary sensing and actuation. We have shown that the system can be stabilized by a dynamic controller that involves a delayed output feedback which is non-robust with respect to delay uncertainties. Then, we have shown that the designed control law can, however, stabilize the system in a robust way when there is a small unknown viscosity. Furthermore, there is an intrinsic uniform margin of stability which is independent of the viscosity value even if it is asymptotically small.

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A Proof of Lemma 1

For convenience, let us first recall the characteristic equation (46):

$$\mathcal{G}(\eta_n, s_n) = e^{s_n}\sqrt{1+\eta_n s_n}\Big(e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\Big) + 1 = 0,\ \forall n \in \mathbb{N}. \quad \text{(A.1)}$$

Proof of (i)

Let us assume by contradiction that $\bar{\sigma} = +\infty$ and therefore that $\sigma_n > 0$ if n is sufficiently large, which is always assumed from now on. This implies that

$$|e^{s_n}| \to +\infty \quad \text{and} \quad |\sqrt{1+\eta_n s_n}| \geqslant 1. \quad \text{(A.2)}$$

Then it follows from (A.1) that

$$\lim_{n\to+\infty}\Big(e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\Big) = 0. \quad \text{(A.3)}$$

This can be written as $\lim_{n\to+\infty}\left(X_n - X_n^{-1}\right) = 0$ with $X_n = e^{s_n/\sqrt{1+\eta_n s_n}}$ and implies that both X_n and X_n^{-1} are bounded. Therefore, since $||X_n| - |X_n|^{-1}| \leq |X_n - X_n^{-1}|$, (A.3) implies that 1 is the only possible adherence value for $|X_n|$ and therefore

$$|e^{s_n/\sqrt{1+\eta_n s_n}}| \to 1. \quad \text{(A.4)}$$

We denote $z_n = 1/\sqrt{1+\eta_n s_n}$. One has

$$z_n = \frac{\sqrt{1+\eta_n\sigma_n - i\eta_n\omega_n}}{\sqrt{(1+\eta_n\sigma_n)^2 + (\eta_n\omega_n)^2}}, \quad \text{(A.5)}$$

which implies that, since $1+\eta_n\sigma_n > 0$,

$$0 \leq |\Im(z_n)| \leq \frac{1}{\sqrt{2}}\frac{1}{((1+\eta_n\sigma_n)^2 + (\eta_n\omega_n)^2)^{1/4}} = \frac{1}{\sqrt{2}}|z_n|$$

and therefore that

$$0 \leq |\Im(z_n)| \leq \frac{1}{\sqrt{2}}|z_n| \leqslant \Re(z_n) \leqslant |z_n|. \quad \text{(A.6)}$$

With these notations (A.4) implies

$$|e^{\sigma_n\Re(z_n) - \omega_n\Im(z_n) + i(\sigma_n\Im(z_n) + \omega_n\Re(z_n))}| \to 1, \quad \text{(A.7)}$$

thus

$$\sigma_n\Re(z_n) - \omega_n\Im(z_n) \to 0. \quad \text{(A.8)}$$

We restrict to a subsequence such that ω_n converges in $[-\infty, +\infty]$. The case where $\omega_n \to 0$ can be discarded because, in this case, from (A.6), $\omega_n\Im(z_n) \to 0$ and $\sigma_n\Re(z_n) \to +\infty$ which is in contradiction with (A.8). Thus we can assume that n is large enough such that $\omega_n > 0$ (resp. $\omega_n < 0$). From (A.5), one can see that if $\omega_n > 0$ (resp. $\omega_n < 0$) then $\Re(z_n) > 0$ and $\Im(z_n) < 0$ (resp. $\Im(z_n) > 0$). Hence since $\sigma_n > 0$ we deduce from (A.8) that

$$|\sigma_n\Re(z_n)| + |\omega_n\Im(z_n)| \to 0. \quad \text{(A.9)}$$

Using (A.9) together with (A.6) gives

$$|\sigma_n||z_n| + |\omega_n\Im(z_n)| \to 0. \quad \text{(A.10)}$$

If there exists $C > 0$ such that for n sufficiently large $|\eta_n\omega_n| \leq C(1 + |\eta_n\sigma_n|)$ then

$$\begin{aligned} |\sigma_n||z_n| &= \frac{\sigma_n}{((1+\eta_n\sigma_n)^2 + (\eta_n\omega_n)^2)^{1/4}} \\ &\geq \frac{\sigma_n}{(1+C^2)^{1/4}(1+\eta_n\sigma_n)^{1/2}} \to +\infty, \end{aligned} \quad \text{(A.11)}$$

and this is in contradiction with (A.10). Thus, for n sufficiently large we can assume that $|\eta_n\omega_n| > 1 + |\eta_n\sigma_n|$ (at least up to a subsequence) and therefore, using (A.5), there exists a $c > 0$ independent of n such that $|\Im(z_n)| \geq c|z_n|$. This, combined with (A.10), gives

$$(|\sigma_n| + |\omega_n|)|z_n| \to 0. \quad \text{(A.12)}$$

From the definition of z_n, we have $s_n/\sqrt{1+\eta_n s_n} \to 0$. Taking the square, this leads to

$$\frac{|s_n|}{\left|\frac{1}{s_n} + \eta_n\right|} \to 0. \quad \text{(A.13)}$$

Since $|s_n| \geq \sigma_n \to +\infty$ and η_n is bounded, we have a contradiction. This concludes the proof of (i).

Proof of (ii)

Since $(\eta_n\omega_n^2)$ is non-negative for all n it follows directly that $2\theta \geqslant 0$. Let us assume by contradiction that $2\theta =$

$+\infty$. We have

$$\frac{s_n}{\sqrt{1+\eta_n s_n}} = \frac{(\sigma_n + i\omega_n)\sqrt{1+\sigma_n\eta_n - i\omega_n\eta_n}}{\sqrt{(1+\eta_n\sigma_n)^2 + (\omega_n\eta_n)^2}}. \quad \text{(A.14)}$$

Let α denote an adherence point of the sequence $(\eta_n\omega_n)$. We consider successively the case where $\alpha \in [-\infty, +\infty] \setminus \{0\}$ and the case where $\alpha = 0$.

• If $\alpha \in [-\infty, +\infty] \setminus \{0\}$, then the sequence (ω_n) is unbounded since $\eta_n \to 0$ and we can restrict to a subsequence (σ_n, ω_n) such that $\omega_n \to +\infty$ (resp. $\omega_n \to -\infty$) and, from (A.14),

$$\frac{s_n}{\sqrt{1+\eta_n s_n}} = \frac{\omega_n}{|\omega_n|}\frac{(o(1)+i)\sqrt{1+o(1) - i\omega_n\eta_n}}{\sqrt{o(1)+\eta_n^2}}, \quad \text{(A.15)}$$

where the $o(1)$ are real valued. Thus

$$\Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) = -\frac{\omega_n}{|\omega_n|}\left[\Im\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right) + o(1)\Re\left(\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right)\right]\sqrt{\frac{|\omega_n\eta_n|}{o(1)+\eta_n^2}}. \quad \text{(A.16)}$$

We then have, for $\alpha \in \mathbb{R} \setminus \{0\}$,

$$\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}} \to \sqrt{\frac{1-i\alpha}{|\alpha|}} \quad \text{(A.17)}$$

and for $\alpha = \pm\infty$

$$\sqrt{\frac{1+o(1)}{|\omega_n\eta_n|} - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}} \to \sqrt{\mp i} \quad \text{(A.18)}$$

such that, since $\alpha \neq 0$,

$$\lim_{n\to+\infty}\Re\left(\sqrt{\frac{1}{|\omega_n\eta_n|} + o(1) - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right) \in (0, +\infty),$$
$$\lim_{n\to+\infty}\Im\left(\sqrt{\frac{1}{|\omega_n\eta_n|} + o(1) - i\frac{\omega_n\eta_n}{|\omega_n\eta_n|}}\right) \in (-\infty, +\infty) \setminus \{0\}. \quad \text{(A.19)}$$

From (A.16) and (A.19) we deduce that

$$\Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) \to -\infty \quad \text{or} \quad \Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) \to +\infty, \quad \text{(A.20)}$$

which implies that

$$\lim_{n\to+\infty}\left|e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\right| = +\infty. \quad \text{(A.21)}$$

Since $\bar\sigma \in (-\infty, +\infty)$ by assumption, we deduce that $|e^{\sigma_n + i\omega_n}| \to e^{\bar\sigma} \in (0, +\infty)$ and $|\sqrt{1+\eta_n s_n}|^2 \geq 1/2$ for n sufficiently large. Hence, using (A.21),

$$\lim_{n\to+\infty}\left|e^{s_n}\sqrt{1+\eta_n s_n}\right|\left|e^{s_n/\sqrt{1+\eta_n s_n}} - e^{-s_n/\sqrt{1+\eta_n s_n}}\right| = +\infty, \quad \text{(A.22)}$$

which is in contradiction with (46).

• If $\alpha = 0$, then $\eta_n s_n \to 0$. Recall that $\omega_n^2\eta_n \to +\infty$ by assumption thus the sequence (ω_n) is again unbounded and we can select a subsequence such that $\omega_n \to +\infty$ (resp. $-\infty$). Then, we have

$$\begin{aligned}\frac{s_n}{\sqrt{1+\eta_n s_n}} &= s_n\left(1 - \frac{\eta_n s_n}{2} + o(\eta_n s_n)\right) \\ &= (\sigma_n + i\omega_n)\left(1 - \frac{\eta_n(\sigma_n + i\omega_n)}{2} + o(\eta_n s_n)\right),\end{aligned} \quad \text{(A.23)}$$

where the function $o(\eta_n s_n)$ can be complex valued. Hence

$$\Re\left(\frac{s_n}{\sqrt{1+\eta_n s_n}}\right) = (1+o(1))\sigma_n + \frac{\omega_n^2\eta_n}{2}(1+o(1)) \to +\infty, \quad \text{(A.24)}$$

where we use that $\sigma_n/\omega_n \to 0$. This means that we have again (A.21)–(A.22) and a contradiction.

This completes the proof of Lemma 1.

B Proof of Lemma 3

Recall that $\delta \in (0, \ln(2))$ and $\gamma \in (0, \gamma_0]$. Since A and B are continuous on $[0, \gamma_0) \times (-\ln(2) - \delta, -\ln(2) + \delta) \times \mathbb{R}$, we have

$$\phi^\gamma(\sigma, k(\gamma)\pi\gamma) = 2e^{2\sigma} + \frac{1}{2} + o(1), \quad \text{(B.1)}$$
$$\phi^\gamma(-\ln(2) + \delta, \varphi) = \frac{1}{2}(e^{2i\varphi/\gamma}e^{2\delta} + 1) + o(1), \quad \text{(B.2)}$$
$$\phi^\gamma(\sigma, (k(\gamma)+1)\pi\gamma) = 2e^{2\sigma} + \frac{1}{2} + o(1), \quad \text{(B.3)}$$
$$\phi^\gamma(-\ln(2) - \delta, \varphi) = \frac{1}{2}(e^{2i\varphi/\gamma}e^{-2\delta} + 1) + o(1), \quad \text{(B.4)}$$

where $o(1)$ refers to functions that tend to 0 in the C^0-norm when $\gamma \to 0^+$. These estimates are direct consequences of (66), (71) and (76). When (σ, φ) follows the boundary of the rectangle $(-\ln(2) - \delta, -\ln(2) + \delta) \times (k(\gamma)\pi\gamma, (k(\gamma)+1)\pi\gamma)$ one has:

• on the left and right boundaries, the function $2e^{2\sigma} + 1/2$ remains on the half real line $(0, +\infty)$;
• on the upper boundary $\sigma = -\ln(2) + \delta$, as φ/γ increases from $k(\gamma)\pi$ to $(k(\gamma)+1)\pi\gamma$, the function

$(1/2)(e^{2i\varphi/\gamma}e^{2\delta}+1)$ never meets $\mathbf{0}$ but describes a closed curve independent of $\gamma > 0$ which encircles $\mathbf{0}$ exactly once and in the clockwise direction since $2\varphi/\gamma - 2k(\gamma)\pi$ increases from 0 to 2π and $e^{2\delta} \in (1, +\infty)$;
• on the lower boundary $\sigma = -\ln(2) - \delta$, the function $(1/2)(e^{2i\varphi/\gamma}e^{-2\delta} + 1)$ does not pass through $\mathbf{0}$ but describes a curve independent of $\gamma > 0$ which remains in the strict right-hand side of the complex plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$ since $|e^{2i\varphi/\gamma}e^{-2\delta}| < 1$.

Hence, using also (B.1) to (B.4), there exists $\gamma_1 \in (0, \gamma_0)$ such that, for every $\gamma \in (0, \gamma_1)$, when (σ, φ) follows the boundary of the rectangle Ω^γ in the clockwise direction, the whole curve described by ϕ^γ does not pass through the origin $\mathbf{0}$ but encircles the origin exactly once and in the clockwise direction, which means that (78) holds. This completes the proof of Lemma 3.