Systems & Control Letters 104 (2017) 66–71

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

A quadratic Lyapunov function for hyperbolic density–velocity systems with nonuniform steady states

Georges Bastin [a,*], Jean-Michel Coron [b]

[a] Department of Mathematical Engineering, ICTEAM, Université catholique de Louvain, 4, Avenue G. Lemaitre, 1348 Louvain-La-Neuve, Belgium
[b] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4, Place Jussieu, 75252 Paris Cedex, France

ARTICLE INFO

Article history:
Received 5 October 2016
Received in revised form 14 January 2017
Accepted 31 March 2017

Keywords:
Stability
Lyapunov
Hyperbolic
Euler equation
Partial differential equation

ABSTRACT

A new explicit Lyapunov function allows to study the exponential stability for a class of physical '2 by 2' hyperbolic systems with nonuniform steady states. In fluid dynamics, this class of systems involves isentropic Euler equations and Saint-Venant equations. The proposed quadratic Lyapunov function allows to analyze the local exponential stability of the system equilibria for suitable dissipative Dirichlet boundary conditions without additional conditions on the system parameters.

1. Introduction

Our purpose in this paper is to address the H^2-stability of the equilibria for a class of one-dimensional 2×2 hyperbolic systems. In fluid dynamics, this class of systems involves, for instance, isentropic Euler equations and Saint-Venant equations. These are systems of two balance laws with the density and the velocity of the considered fluid as state variables, and with source terms representing friction effects.

In [1–3] we have introduced a generic H^2-Lyapunov function, expressed in Riemann coordinates, to analyze the H^2-stability in the frictionless case when the system takes the form of conservation laws and has, therefore, a steady-state which is spatially uniform.

The problem which is addressed in this paper is more difficult because we consider the case where the friction is not negligible and the system is represented by balance laws with nonuniform steady-states. For that case, general implicit stability conditions are given in [4, Chapter 6] by using a weighted form of the H^2-Lyapunov function with a weighting matrix $Q(x)$ which is not a-priori given and has to be found for each particular application.

The issue addressed in this paper was previously considered in [5] (see also [6–8]) for the special case of isothermal Euler equations. In the paper [5], for a system with a non-uniform steady-state, the authors use a matrix $Q(x)$ which is related to the one proposed in [2] for conservation laws with uniform steady-state. This choice allows to obtain stability for suitable boundary conditions albeit with limitations depending on the values of some system parameters.

In this paper, we shall show how a more physical choice of the weighting matrix $Q(x)$ allows to remove the previous limitations. With this choice, we are able to prove the H^2 local exponential stability for suitable boundary conditions without any additional conditions on the system parameters.

2. Hyperbolic density–velocity systems

We are concerned with hyperbolic systems of two partial differential equations (PDEs) of the form

$$H_t + (HV)_x = 0, \tag{1a}$$

$$V_t + \frac{c^2(H)}{H} H_x + VV_x + f(H, V) = 0, \tag{1b}$$

where $t \in [0, +\infty)$, $x \in [0, L]$, $H : [0, +\infty) \times [0, L] \to (0, +\infty)$, $V : [0, +\infty) \times [0, L] \to (0, +\infty)$, $c \in C^2((0, +\infty); (0, +\infty))$, $f \in C^1((0, +\infty)^2; \mathbb{R})$.

In many applications, the model (1) represents the one-dimensional propagation of some physical quantity, with density H, propagation velocity V, flow density HV and friction $f(H, V)$. The first equation is a mass conservation law and the second equation is a momentum balance.

A very typical example is given by the isentropic Euler equations for the gas motion in pipelines where H is the gas density. In the

* Corresponding author.
E-mail addresses: georges.bastin@uclouvain.be (G. Bastin), coron@ann.jussieu.fr (J.-M. Coron).

http://dx.doi.org/10.1016/j.sysconle.2017.03.013

particular case of a polytropic gas, the gas pressure is the following function of the gas density:

$$p(H) = \kappa H^{\gamma} \tag{2}$$

with $\kappa > 0$ and $\gamma > 1$ real constants. In that case the function $c(H)$ is the sound velocity defined as:

$$c(H) = \sqrt{p'(H)} = \sqrt{\gamma\kappa H^{\gamma-1}}. \tag{3}$$

Another typical example is given by the Saint-Venant equations for the water motion in open channels where H is the water level and the function $c(H) = \sqrt{gH}$, with $g > 0$ the gravity acceleration constant.

In Eq. (1b), the coefficient $c^2(H)/H$ is the second order derivative of the internal energy $U(H)$ such that:

$$U''(H) = \frac{c^2(H)}{H} \tag{4}$$

with $U \in C^4((0, +\infty); \mathbb{R})$. Multiplying the first equation (1a) by $U'(H)$ and the second equation (1b) by HV, then summing up the two resulting equations, we obtain the following energy balance law which is satisfied along the smooth solutions of the system (1):

$$E(H, V)_t + F(H, V)_x + (HV)f(H, V) = 0, \tag{5}$$

with

$$E(H, V) = \frac{1}{2}HV^2 + U(H), \qquad F(H, V) = HV\left(U'(H) + \frac{1}{2}V^2\right). \tag{6}$$

The function $E(H, V)$ is the energy density: the first term $HV^2/2$ is the kinetic energy density and the second term $U(H)$ is the potential or internal energy density. Obviously, the total energy of the system (1) at the time t is given by the integral of E over the interval $[0, L]$:

$$\mathscr{E}(t) = \int_0^L E(H(t, x), V(t, x))dx. \tag{7}$$

In this paper, our purpose is to show how the energy $\mathscr{E}$ can be used to define a Lyapunov function for the exponential stability analysis of the steady-states of the system (1) when both the boundary conditions and the source term $f(H, V)$ are dissipative.

A steady state (or equilibrium) of the system (1) is a time-invariant solution $(H^*, V^*) : [0, L] \to (0, +\infty)^2$. It satisfies the system of ordinary differential equations

$$V^*_x = \frac{V^* f(H^*, V^*)}{c^2(H^*) - V^{*2}}, \qquad H^*_x = -\frac{H^* f(H^*, V^*)}{c^2(H^*) - V^{*2}}. \tag{8}$$

In this paper, we consider the standard physical case where

1. the steady-state density $H^*(x) > 0$ and velocity $V^*(x) > 0$ are *strictly positive* for all $x \in [0, L]$;
2. the steady state flow is *subcritical*, i.e. the denominators in (8) are *strictly positive*:

$$c^2(H^*) - V^{*2} > 0 \quad \forall x \in [0, L]. \tag{9}$$

In order to linearize the model, we define the deviations of the states $H(t, x)$ and $V(t, x)$ with respect to the steady states $H^*(x)$ and $V^*(x)$:

$$h(t, x) \triangleq H(t, x) - H^*(x), \qquad v(t, x) \triangleq V(t, x) - V^*(x). \tag{10}$$

The linearization of the system (1) about the steady state is then

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ \dfrac{c^2(H^*)}{H^*} & V^* \end{pmatrix}\begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V^*_x & H^*_x \\ \left(\dfrac{c^2(H^*)}{H^*}\right)_x + f^*_H & V^*_x + f^*_V \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix} = 0, \tag{11}$$

with the notations

$$f^*_H = \frac{\partial f}{\partial H}(H^*, V^*), \qquad f^*_V = \frac{\partial f}{\partial V}(H^*, V^*). \tag{12}$$

3. A physical quadratic Lyapunov function for the linearized system

In this section, we are concerned with the exponential stability of the L^2-solutions of the linearized system (11) under local linear boundary conditions of the form

$$v(t, 0) = -b_0 h(t, 0), \qquad v(t, L) = b_1 h(t, L), \tag{13}$$

with constants $b_0 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, and under an initial condition

$$h(0, x) = h^o(x), \quad v(0, x) = v^o(x), \tag{14}$$

such that

$$(h^o, v^o) \in L^2((0, L); \mathbb{R}^2). \tag{15}$$

The Cauchy problem (11), (13), (14) is well-posed (see [4, Appendix A]).

Definition 1. The system (11), (13) is exponentially stable (for the L^2-norm) if there exist $\nu > 0$ and $C > 0$ such that, for every initial condition $(h^o, v^o) \in L^2((0, L); \mathbb{R}^2)$, the solution to the Cauchy problem (11), (13), (14) satisfies

$$\|(h(t, \cdot), v(t, \cdot))\|_{L^2} \leqslant Ce^{-\nu t}\|(h^o, v^o)\|_{L^2}. \tag{16}$$

For this analysis, as a candidate Lyapunov function, we introduce the following quadratic functional:

$$\mathbf{V} = \int_0^L \left(\frac{c^2(H^*)}{H^*}h^2 + H^* v^2\right)dx = \int_0^L \begin{pmatrix} h & v \end{pmatrix}\begin{pmatrix} \dfrac{c^2(H^*)}{H^*} & 0 \\ 0 & H^* \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix} dx. \tag{17}$$

With the notations

$$Y = \begin{pmatrix} h \\ v \end{pmatrix}, \qquad A(x) = \begin{pmatrix} V^* & H^* \\ \dfrac{c^2(H^*)}{H^*} & V^* \end{pmatrix}, \tag{18}$$

$$B(x) = \begin{pmatrix} V^*_x & H^*_x \\ \left(\dfrac{c^2(H^*)}{H^*}\right)_x + f^*_H & V^*_x + f^*_V \end{pmatrix},$$

$$D(x) = \begin{pmatrix} \dfrac{c^2(H^*)}{H^*} & 0 \\ 0 & H^* \end{pmatrix}, \tag{19}$$

we compute the time derivative of $\mathbf{V}$ along the C^1-solutions of the system (11), (13):

$$\frac{d\mathbf{V}}{dt} = \int_0^L \left(Y^T D(x) Y_t + Y_t^T D(x) Y\right)dx = -\int_0^L \Big(Y^T D(x)\big(A(x)Y_x + B(x)Y\big) + \big(Y_x^T A^T(x) + Y^T B^T(x)\big)D(x)Y\Big)dx. \tag{20}$$

We observe that the matrix $M(x) = D(x)A(x)$ is symmetric:

$$M(x) = D(x)A(x) = A^T(x)D(x) = \begin{pmatrix} \left(\dfrac{c^2(H^*)}{H^*}\right)V^* & c^2(H^*) \\ c^2(H^*) & H^*V^* \end{pmatrix}. \tag{21}$$

Now, using (20), (21) and integration by parts, we have

$$\begin{aligned}\frac{d\mathbf{V}}{dt} &= -\int_0^L (Y^T M(x)Y)_x dx \\ &\quad - \int_0^L \left[Y^T\big(-M'(x)+B^T(x)D(x)+D(x)B(x)\big)Y\right]dx \\ &= -\left[Y^T M(x)Y\right]_0^L - \int_0^L \big(Y^T N(x)Y\big)dx,\end{aligned} \tag{22}$$

with, using (8),

$$N(x) = -M'(x) + B^T(x)D(x) + D(x)B(x) \tag{23}$$

$$= \begin{pmatrix} 2c(H^*)c'(H^*)V_x^* & H^* f_H^* \\ H^* f_H^* & 2H^*(V_x^* + f_V^*)\end{pmatrix}. \tag{24}$$

Hence, from Eq. (22), we see that $d\mathbf{V}/dt$ is a negative definite function along the solutions of the system (11), (13) if the two following conditions are satisfied:

(a) the boundary term is non negative (i.e. $\left[Y^T M(x)Y\right]_0^L \geqslant 0$) for every $Y(t,0) \in \mathbb{R}^2$ and every $Y(t,L) \in \mathbb{R}^2$ such that $v(t,0) = -b_0 h(t,0)$ and $v(t,L) = b_1 h(t,L)$;
(b) the matrix $N(x)$ is positive definite for all x in $[0,L]$.

Since, for all $x \in [0,L]$, the matrix $N(x)$ is positive definite, there exists a positive real number ν such that

$$Y^T N(x) Y \geqslant 2\nu Y^T D(x) Y, \quad \forall x \in [0,L], \quad \forall Y \in \mathbb{R}^2. \tag{25}$$

Then, it follows directly from the definition of $\mathbf{V}$ and from (22) that

$$\frac{d\mathbf{V}}{dt} \leqslant -2\nu\mathbf{V} \tag{26}$$

along the C^1-solutions of the system. However, since the C^1-solutions are dense in the set of L^2-solutions, inequality (26) is also satisfied in the sense of distributions for L^2-solutions (see [4, Section 2.1.3] for details). Consequently, $\mathbf{V}$ is an exponentially decaying Lyapunov function for the L^2-norm and there is a positive constant C such that the solutions satisfy inequality (16) of Definition 1.

In the next proposition, we now give conditions on the coefficients b_0 and b_1 such that the boundary condition (a) is fulfilled. For simplicity, we introduce the following notations:

$$h_0(t) = h(t,0), \quad H_0^* = H^*(0), \quad V_0^* = V^*(0), \tag{27}$$

$$h_1(t) = h(t,L), \quad H_1^* = H^*(L), \quad V_1^* = V^*(L). \tag{28}$$

Proposition. *Condition* (a) *is verified if and only if* b_0 *and* b_1 *satisfy the following inequalities:*

$$b_0 \in [b_0^-, b_0^+] \text{ with } b_0^{\pm} = \frac{c^2(H_0^*)}{H_0^*}\left(\frac{1}{V_0^*} \pm \sqrt{\frac{1}{V_0^{*2}} - \frac{1}{c^2(H_0^*)}}\right), \tag{29}$$

$$b_1 \in \mathbb{R} \setminus (b_1^-, b_1^+) \text{ with } b_1^{\pm} = \frac{c^2(H_1^*)}{H_1^*}\left(-\frac{1}{V_1^*} \pm \sqrt{\frac{1}{V_1^{*2}} - \frac{1}{c^2(H_1^*)}}\right). \tag{30}$$

Proof. We have

$$\left[Y^T M(x) Y\right]_0^L = \left[Y^T(t,L)M(L)Y(t,L)\right] - \left[Y^T(t,0)M(0)Y(t,0)\right]. \tag{31}$$

Under the boundary condition (13), we have

$$\left[Y^T(t,0)M(0)Y(t,0)\right] = \underbrace{\Big(H_0^* V_0^* b_0^2 - 2c^2(H_0^*)b_0 + \frac{c^2(H_0^*)}{H_0^*}V_0^*\Big)}_{\mathscr{P}_0} h_0^2(t). \tag{32}$$

$\mathscr{P}_0$ is a degree-2 polynomial in b_0 with two positive real roots b_0^-, b_0^+. Condition (29) holds for values of b_0 located in the closed interval $[b_0^-, b_0^+]$ such that $\mathscr{P}_0$ is nonpositive and, consequently, that $[Y^T(t,0)M(0)Y(t,0)] \leqslant 0$ for all t.

On the other hand, under the boundary condition (13), we have

$$\left[Y^T(t,L)M(L)Y(t,L)\right] = \underbrace{\Big(H_1^* V_1^* b_1^2 + 2c^2(H_1^*)b_1 + \frac{c^2(H_1^*)}{H_1^*}V_1^*\Big)}_{\mathscr{P}_1} h_1^2(t). \tag{33}$$

$\mathscr{P}_1$ is a degree-2 polynomial in b_1 with two negative real roots b_1^-, b_1^+. Condition (30) holds for values of b_1 located outside the open interval (b_1^-, b_1^+) such that $\mathscr{P}_1$ is non negative and, consequently, that $[Y^T(t,L)M(L)Y(t,L)] \geqslant 0$ for all t. This completes the proof of the proposition. $\square$

In the next two subsections we compute the matrix $N(x)$ and we show that condition (b) is satisfied for the examples of isentropic Euler equations and Saint-Venant equations.

Isentropic Euler equations

The isentropic Euler equations are typically used to describe the gas propagation in ducts, with a pressure $p(H) = \kappa H^\gamma$ (where $\gamma > 1$ is the adiabatic index) and a friction proportional to the square of the velocity:

$$f(H,V) = kV^2, \quad k > 0. \tag{34}$$

In this case, for all $x \in [0,L]$, we have:

$$\begin{aligned}&c^2(H^*) = \kappa\gamma H^{*(\gamma-1)} > 0,\\ &2c(H^*)c'(H^*) = \kappa\gamma(\gamma-1)H^{*(\gamma-2)} > 0,\end{aligned} \tag{35}$$

$$f(H^*,V^*) > 0, \qquad f_H^* = 0, \qquad f_V^* = 2kV^* > 0, \tag{36}$$

Under the subcritical flow condition (9), it follows directly that $V_x^* > 0$ and $N(x)$ is positive definite for all $x \in [0,L]$.

Saint-Venant equations

For a horizontal channel with a rectangular cross section and a unit width, the water propagation may be described by Saint-Venant equations that are a special case of the model (1) with

$$c^2(H) = gH \quad \text{and} \quad f(H,V) = k\frac{V^2}{H}, \quad k > 0. \tag{37}$$

In this case, for all $x \in [0,L]$, we have

$$2c(H^*)c'(H^*) = g, \qquad f_H^* = -\frac{kV^{*2}}{H^{*2}}, \qquad f_V^* = \frac{2kV^*}{H^*}. \tag{38}$$

Then, using (8), the matrix $N(x)$ is

$$N(x) = \begin{pmatrix} \dfrac{gkV^{*3}}{H^*(gH^* - V^{*2})} & -\dfrac{kV^{*2}}{H^*} \\ -\dfrac{kV^{*2}}{H^*} & \dfrac{2kV^{*3}}{(gH^* - V^{*2})} + 4kV^* \end{pmatrix}. \tag{39}$$

Here the subcritical flow condition (9) implies that $gH^* - V^{*2} > 0$. It follows that

$$\det[N(x)] = \left(\frac{kV^{*2}}{H^*}\right)^2 \left(\frac{2gH^*V^{*2}}{(gH^* - V^{*2})^2} + \frac{4gH^*}{(gH^* - V^{*2})} - 1\right) > 0, \tag{40}$$

and therefore that the matrix $N(x)$ is positive definite for all $x \in [0,L]$.

Remark 1 (*Comment on the Choice of the Lyapunov Function*)**.** Let us expand the energy function (7) in a neighborhood of the steady-state H^*, V^* for the Saint-Venant equations. We obtain:

$$\begin{aligned}\mathscr{E} &= \frac{1}{2}\int_0^L \Big((H^*+h)(V^*+v)^2 + g(H^*+h)^2\Big)dx \\ &= \mathscr{E}^* + \mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3, \end{aligned} \tag{41}$$

with

$$\mathscr{E}^* = \frac{1}{2}\int_0^L \left(H^*V^{*2} + gH^{*2}\right)dx, \tag{42}$$

$$\mathscr{E}_1 = \int_0^L \left[\left(\frac{V^{*2}}{2} + gH^*\right)h + (H^*V^*)v\right]dx, \tag{43}$$

$$\mathscr{E}_2 = \frac{1}{2}\int_0^L (gh^2 + 2V^*hv + H^*v^2)dx, \tag{44}$$

$$\mathscr{E}_3 = \frac{1}{2}\int_0^L (hv^2)dx. \tag{45}$$

We could believe that the quadratic term $\mathscr{E}_2$ should be a natural candidate for a Lyapunov function for this system. Let us therefore compute the time derivative of $\mathscr{E}_2$ along the solutions of (11):

$$\begin{aligned}\frac{d\mathscr{E}_2}{dt} &= \int_0^L [(gh+V^*v)\partial_t h + (V^*h + H^*v)\partial_t v]dx \\ &= -\int_0^L \Big[(gh+V^*v)\Big(V^*\partial_x h + H^*\partial_x v + V_x^* h + H_x^* v\Big) \\ &\quad + (V^*h + H^*v)\Big(g\partial_x h + V^*\partial_x v - \frac{kV^{*2}}{H^{*2}}h \\ &\quad + \Big(V_x^* + \frac{2kV^*}{H^*}\Big)v\Big)\Big]dx \\ &= -\int_0^L \Big[gV^*\partial_x(h^2) + (V^{*2} + gH^*)\partial_x(hv) + H^*V^*\partial_x(v^2) \\ &\quad + \Big(gV_x^* - \frac{kV^{*3}}{H^{*2}}\Big)h^2 + 2kV^*v^2 \\ &\quad + \Big(gH_x^* + 2V^*V_x^* + \frac{kV^{*2}}{H^*}\Big)hv\Big]dx. \end{aligned} \tag{46}$$

Integrating by parts and using the equilibrium conditions (8) we get:

$$\frac{d\mathscr{E}_2}{dt} = B + \int_0^L \left[\frac{kV^{*3}}{H^*}h\left(\frac{h}{H^*} - \frac{v}{V^*}\right) - 2kV^*v^2\right]dx \tag{47}$$

with

$$B = -\Big[gV^*h^2 + (V^{*2} + gH^*)hv + H^*V^*v^2\Big]_0^L. \tag{48}$$

In this case, we observe that the integral term can be positive (just take $h > 0$ and $v = 0$) and consequently that $\mathscr{E}_2$ may not be a Lyapunov function for our system. Here the interesting point is that, by ignoring the cross-term $2V^*hv$ in $\mathscr{E}_2$, we get a valid candidate for the Lyapunov function as we have shown above.

Remark 2 (*The Special Case of Isothermal Euler Equations*)**.** The special case of an *isothermal* gas flow is sometimes considered for modeling pipeline dynamics and control (e.g. [5–8]). In that case the Euler equations are said to be isothermal, the pressure is proportional to the density $p(H) = \kappa H$ and the sound velocity is constant $c = \sqrt{\kappa}$. It then follows that $c' = 0$ and therefore that the matrix $N(x)$ is *not* positive definite. In that case, the Lyapunov function introduced in this paper is not applicable and the use of Lyapunov functions expressed in Riemann coordinates with exponential weights as in [5–8] is not redundant.

4. Exponential stability of the steady-state of the nonlinear system

In this section, we shall now show that conditions (a) and (b) are also sufficient to guarantee the exponential stability in H^2 of the steady state $H^*(x)$, $V^*(x)$ of the nonlinear system (1). For this purpose it is first useful to transform the linearized system (11) into Riemann coordinates. Under the subcritical condition (9), the matrix $A(x)$ has two real distinct eigenvalues:

$$\lambda^+(x) = V^* + c(H^*) > 0, \qquad \lambda^-(x) = V^* - c(H^*) < 0. \tag{49}$$

Therefore, $A(x)$ is diagonalizable with the invertible matrix $S(x)$ defined as

$$S(x) = \frac{1}{2}\begin{pmatrix} \dfrac{H^*(x)}{c(H^*(x))} & -\dfrac{H^*(x)}{c(H^*(x))} \\ 1 & 1 \end{pmatrix} \tag{50}$$

such that

$$\Lambda(x) = S^{-1}(x)A(x)S(x) \quad \text{with} \quad \Lambda(x) = \begin{pmatrix}\lambda^+(x) & 0 \\ 0 & \lambda^-(x)\end{pmatrix}. \tag{51}$$

Then the vector of Riemann coordinates

$$R = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} \tag{52}$$

is defined as

$$R(t,x) = S^{-1}(x)Y(t,x) \tag{53}$$

so that

$$R^+ = v + h\frac{c(H^*)}{H^*}, \qquad R^- = v - h\frac{c(H^*)}{H^*}. \tag{54}$$

In these coordinates, the linearized system (11) is written

$$R_t + \Lambda(x)R_x + C(x)R = 0 \tag{55}$$

with

$$C(x) = S^{-1}(x)\big[A(x)S_x(x) + B(x)S(x)\big] = \begin{pmatrix}\gamma_1(x) & \delta_1(x) \\ \gamma_2(x) & \delta_2(x)\end{pmatrix}. \tag{56}$$

Moreover, the boundary conditions (13) are written as follows in the Riemann coordinates (54):

$$\begin{pmatrix} R^+(t,0) \\ R^-(t,L)\end{pmatrix} = \begin{pmatrix} 0 & k_0 \\ k_1 & 0 \end{pmatrix}\begin{pmatrix} R^+(t,L) \\ R^-(t,0)\end{pmatrix} \tag{57}$$

$$\text{with} \quad k_0 = \frac{b_0H_0^* - c(H_0^*)}{b_0H_0^* + c(H_0^*)}, \quad k_1 = \frac{b_1H_1^* - c(H_1^*)}{b_1H_1^* + c(H_1^*)}. \tag{58}$$

The linear system (55), (57) is the linearization, around a steady-state $H^*(x)$, $V^*(x)$ of our initial nonlinear system (1)

$$H_t + (HV)_x = 0, \tag{59a}$$

$$V_t + \frac{c^2(H)}{H}H_x + VV_x + f(H,V) = 0, \tag{59b}$$

with nonlinear boundary conditions

$$\begin{aligned} &V(t,0) = \mathscr{B}_0(H(t,0)), \quad V(t,L) = \mathscr{B}_1(H(t,L)) \\ &\quad \text{s.t.} \;\; b_0 = -\mathscr{B}_0'(H^*(0)), \quad b_1 = \mathscr{B}_1'(H^*(L)). \end{aligned} \tag{60}$$

Sufficient conditions for the exponential stability of the steady-state of the nonlinear system (59), (60) are then given in the following Theorem ($\mathcal{D}_n$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements).

Theorem 1. *The steady state $H^*(x)$, $V^*(x)$ of the system (59), (60) is exponentially stable for the H^2-norm if*

(i) there exists a map $Q \in C^1([0, L]; \mathcal{D}_n)$ *such that the matrix*

$$-\left(Q(x)\Lambda(x)\right)_x + Q(x)C(x) + C^T(x)Q(x)$$

is positive definite for all $x \in [0, L]$,

(ii) the following inequalities are satisfied:

$$k_0^2 \leqslant \frac{|\lambda^-(0)|}{\lambda^+(0)} \quad \text{and} \quad k_1^2 \leqslant \frac{\lambda^+(L)}{|\lambda^-(L)|}. \tag{61}$$

Proof. This theorem is a special case of Theorems 6.6 and 6.10 in [4]. $\square$

Now, in the next two lemmas, we shall show that conditions **(i)** and **(ii)** are fulfilled with the matrix

$$Q(x) = S^T(x)D(x)S(x) = \frac{1}{2}\begin{pmatrix} H^*(x) & 0 \\ 0 & H^*(x) \end{pmatrix} \tag{62}$$

if the matrix $N(x)$ is positive definite and if the linearized boundary conditions (60) satisfy inequalities (29), (30).

Lemma 1. *If the matrix $N(x)$ is positive definite for all $x \in [0, L]$, then the matrix*

$$-(Q(x)\Lambda(x))_x + Q(x)C(x) + C^T(x)Q(x)$$

is positive definite for all $x \in [0, L]$.

Proof. We omit the argument x for notational simplicity. Using the definition (62) of Q, the definition (56) of C, the equality (51) $S\Lambda = AS$, the equality (21) $M = DA = A^TD$ and the definition (23) of N, we have:

$$\begin{aligned}
-(Q\Lambda)_x + QC + C^TQ &= -(S^TDS\Lambda)_x + S^TD(AS_x + BS) \\
&\quad + (S^TB^T + S_x^TA^T)DS \\
&= -(S^TDAS)_x + S^TD(AS_x + BS) \\
&\quad + (S^TB^T + S_x^TA^T)DS \\
&= -S_x^TDAS - S^TD_xAS - S^TDA_xS - S^TDAS_x \\
&\quad + S^TDAS_x + S^TDBS + S^TB^TDS + S_x^TA^TDS \\
&= -S^TD_xAS - S^TDA_xS + S^TDBS + S^TB^TDS \\
&= S^T\left(-M'(x) + DB + B^TD\right)S \\
&= S^TNS.
\end{aligned}$$

Hence, since S is invertible, the positive definiteness of N implies the positive definiteness of $-(Q\Lambda)_x + QC + C^TQ$. $\square$

Moreover Lemma 2 states that conditions (29), (30) on b_0 and b_1 are equivalent to conditions (61) on k_0 and k_1.

Lemma 2. *The real numbers b_0 and b_1 satisfy inequalities (29), (30) if and only if the real numbers k_0 and k_1 satisfy inequalities (61).*

Proof. Let us denote $c_0 = c(H_0^*)$. Then we have

$$\begin{aligned}
k_0^2 \leqslant \frac{|\lambda^-(0)|}{\lambda^+(0)} &\\
\iff &(b_0H_0^* - c_0)^2(c_0 + V_0^*) \leqslant (b_0H_0^* + c_0)^2(c_0 - V_0^*) \\
&\text{by (49) and (58)} \\
\iff &((b_0H_0^*)^2 - 2b_0c_0H_0^* + c_0^2)(c_0 + V_0^*) \\
&\leqslant ((b_0H_0^*)^2 + 2b_0c_0H_0^* + c_0^2)(c_0 - V_0^*) \\
\iff &\underbrace{b_0^2H_0^{*2}V_0^* - 2b_0c_0^2H_0^* + c_0^2V_0^*}_{=H_0^*\mathscr{P}_0} \leqslant 0 \\
\iff &b_0 \in [b_0^-, b_0^+].
\end{aligned}$$

The proof is similar for b_1. $\square$

Remark 3. A numerical approach for characterizing necessary and sufficient conditions for the existence of a basic quadratic control Lyapunov function of the form

$$\mathbf{V} = \int_0^L \left(q_1(x)\left(R^+(t,x)\right)^2 + q_2(x)\left(R^-(t,x)\right)^2\right)dx \tag{63}$$

is given in the paper [9]. The quadratic Lyapunov function (17) of the present paper is the particular case where $q_1(x) = q_2(x) = H^*(x)$. In the paper [9] it is shown that the existence of such a basic quadratic control Lyapunov function is equivalent to the following property $(\mathcal{P})$.

$(\mathcal{P})$. The solution $\eta(x)$ of the ordinary differential equation

$$\eta'(x) = \left|\frac{\varphi(x)\delta_1(x)}{\lambda^+(x)} - \frac{\gamma_2(x)}{\varphi(x)\lambda^-(x)}\eta^2(x)\right|, \qquad \eta(0) = |k_0|, \tag{64}$$

with

$$\varphi(x) = \exp\left[\int_0^x \left(\frac{\gamma_1(s)}{\lambda^+(s)} - \frac{\delta_2(s)}{\lambda^-(s)}\right)ds\right], \tag{65}$$

exists for all x in the interval $[0, L]$ and satisfies

$$|k_L|\eta(L) < \varphi^2(L). \tag{66}$$

(Note that the functions $\gamma_i(x)$ and $\delta_i(x)$, $i = 1, 2$, in (64) and (65) are the entries of the matrix $C(x)$ defined by Eq. (56)).

In the book [4, Section 5.3], we give a numerical simulation for a practical example of Saint Venant equations, in the case where $k_0 = 0$ and we observe that, as predicted in the present paper, the solution η exists for the interval $[0, L]$ with L the maximal length of the pool for which the flow remains subcritical. Let us mention that there is a mistake in this numerical example in the paper [9, Section 4] which is corrected in the book [4, Section 5.3].

5. Conclusions

In this paper, our main contribution was to exhibit an explicit Lyapunov function which allows to study the exponential stability of a class of Euler-type 2×2 hyperbolic equations with nonuniform steady-states. Surprisingly enough, as seen in Remark 1, the quadratic term $\mathscr{E}_2$ in the expansion of the energy function is not a control Lyapunov function for this kind of physical system. However, it is rather remarkable that, by ignoring the cross-term $2V^*hv$ in $\mathscr{E}_2$, we get a valid control Lyapunov function. An interesting question is then to examine how this phenomenon can be extended to more general $n \times n$ physical hyperbolic systems.

Acknowledgment

The second author's work was supported by Agence Nationale de la Recherche (ANR) Projet Blanc Finite4SoS number ANR-15-CE23-0007.

References

[1] J.-M. Coron, B. d'Andréa-Novel, G. Bastin, A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws, IEEE Trans. Automat. Control 52 (1) (2007) 2–11.
[2] J.-M. Coron, G. Bastin, B. d'Andréa-Novel, Dissipative boundary conditions for one dimensional nonlinear hyperbolic systems, SIAM J. Control Optim. 47 (3) (2008) 1460–1498.
[3] J.-M. Coron, R. Vazquez, M. Krstic, G. Bastin, Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping, SIAM J. Control Optim. 51 (3) (2013) 2005–2035.
[4] G. Bastin, J.-M. Coron, Stability and Boundary Stabilisation of 1-D Hyperbolic Systems, in: Progress in Nonlinear Differential Equations and their Applications, vol. 88, Springer International, 2016.
[5] M. Gugat, G. Leugering, S. Tamasoiu, K. Wang, H^2-stabilization of the isothermal Euler equations: a Lyapunov function approach, Chinese Ann. Math. Ser. B 33 (4) (2012) 479–500.

[6] M. Gugat, M. Herty, Existence of classical solutions and feedback stabilisation for the flow in gas networks, ESAIM Control Optim. Calc. Var. 17 (1) (2011) 28–51.
[7] M. Dick, M. Gugat, G. Leugering, Classical solutions and feedback stabilisation for the gas flow in a sequence of pipes, Netw. Heterog. Media 5 (4) (2010) 691–709.
[8] M. Gugat, M. Dick, G. Leugering, Gas flow in fan-shaped networks: classical solutions and feedback stabilization, SIAM J. Control Optim. 49 (5) (2011) 2101–20117.
[9] G. Bastin, J.-M. Coron, On boundary feedback stabilization of non-uniform 2×2 hyperbolic systems over a bounded interval, Syst. Control Lett. 60 (11) (2011) 900–906.