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Boundary Feedback Control and Lyapunov Stability Analysis for Physical Networks of 2×2 Hyperbolic Balance Laws.

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***Abstract*— Sufficient dissipative boundary conditions are given for the exponential stability of equilibria in physical networks of 2×2 nonlinear hyperbolic balance laws under boundary feedback control. The analysis relies on the use of an explicit strict Lyapunov function.**

I. INTRODUCTION

Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g.[12]). Many physical networks having an engineering interest are described by systems of 2x2 hyperbolic balance laws. Among others, we may mention for instance Saint-Venant equations for hydraulic networks (e.g.[9],[5]), isothermal Euler equations for gas pipeline networks (e.g.[1]), or Aw-Rascle equations for road traffic networks (e.g.[7], [6]). In this paper, our concern is to analyse the stability (in the sense of Lyapunov) of the steady-states of such networks under boundary feedback control. The analysis relies on the use of an explicit strict Lyapunov function.

II. 2×2 HYPERBOLIC BALANCE LAWS

Definition

We consider 2×2 hyperbolic balance laws in one space dimension over a finite interval taking the following general form:

$$\partial_t p + \partial_x q = 0 \tag{1a}$$

$$\partial_t q + \alpha(p,q)\partial_x p + \beta(p,q)\partial_x q = \gamma(p,q) \tag{1b}$$

In these equations, the independent variables are the time t and a space coordinate x over a finite interval $(0, L)$. The dependent variables $p(t,x)$ and $q(t,x)$ are the states of the system. The first equation is a mass conservation law with p the density and q the flux. The second equation may be interpreted as a momentum balance law.

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Defining the vector $\mathbf{y} \triangleq (p, q)^T$, system (1) is written

$$\partial_t \mathbf{y} + F(\mathbf{y})\partial_x \mathbf{y} = G(\mathbf{y})$$

with

$$F(\mathbf{y}) \triangleq \begin{pmatrix} 0 & 1 \\ \alpha(p,q) & \beta(p,q) \end{pmatrix} \quad G(\mathbf{y}) \triangleq \begin{pmatrix} 0 \\ \gamma(p,q) \end{pmatrix}.$$

The system is supposed to be strictly hyperbolic, i.e. the matrix $F(\mathbf{y})$ has two real distinct eigenvalues called *characteristic velocities* $\lambda_1(\mathbf{y}) \neq \lambda_2(\mathbf{y})$.

A steady-state (or equilibrium state) for system (1) is a constant state $\bar{\mathbf{y}}$ which satisfies the condition $G(\bar{\mathbf{y}}) = 0$.

Riemann coordinates

It is a well known property (e.g. [8], [11]) that, for any system of the form (1), there exists a change of coordinates $\mathbf{u} = \Phi(\mathbf{y})$ such that the system can be rewritten in characteristic form

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1(\mathbf{u}) & 0 \\ 0 & c_2(\mathbf{u}) \end{pmatrix} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = H(\mathbf{u})$$

with $\mathbf{u} \triangleq (u_1, u_2)^T$, $c_i(\mathbf{u}) \triangleq \lambda_i(\Phi^{-1}(\mathbf{u}))$ and $H(\mathbf{u}) \triangleq (\partial\Phi/\partial\mathbf{y}(\Phi^{-1}(\mathbf{u}))G(\Phi^{-1}(\mathbf{u}))$.

Examples

- Open channels - Saint Venant equations, $\mathbf{y} = (h, q)^T$,

$$\alpha = gh - (q^2/h^2),\ \beta = 2q/h,\ \gamma = gSh - C(q^2/h^2),$$

with : h = water depth, q = water flow rate, g= gravity constant, S = canal slope, C = friction coefficient.
Characteristic velocities : $\lambda_{1,2} = (q/h) \pm \sqrt{gh}$.
Steady-state : $gS\bar{h}^3 = C\bar{q}^2$.
Riemann coordinates : $(q/h) \pm 2\sqrt{gh}$.

- Road traffic - Aw-Rascle equations, $\mathbf{y} = (\rho, q)^T$,

$$\alpha = qp'(\rho) - (q^2/p^2),\ \beta = 2q/\rho - \rho p'(\rho),\ \gamma = \sigma(\rho V(\rho) - q)$$

with ρ = traffic density, q = traffic flow rate, $p(\rho)$ = traffic pressure function such that $p'(\rho) > 0$, $V(\rho)$ = preferential velocity function, σ = constant.
Characteristic velocities : $\lambda_1 = q/\rho$, $\lambda_2 = q/\rho - \rho p'(\rho)$.
Steady-state : $\bar{\rho}V(\bar{\rho}) = \bar{q}$.
Riemann coordinates : q/ρ and $q/\rho + p(\rho)$.

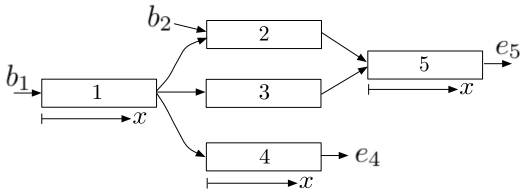

Fig. 1. Physical network

III. NETWORKS

Definition

We consider physical networks (e.g. irrigation or road networks) as illustrated in Fig.1. The structure of the network is reminiscent to the structure of so-called compartmental systems. The nodes of the network (called *compartments*) represent the physical links (i.e the canals or the roads) having dynamics expressed by hyperbolic balance laws

$$\partial_t p_j + \partial_x q_j = 0 \tag{2a}$$

$$\partial_t q_j + \alpha(p_j, q_j)\partial_x p_j + \beta(p_j, q_j)\partial_x q_j = g(p_j, q_j) \tag{2b}$$

$$t \geqslant 0,\ x \in (0, L),\ j = 1, \ldots, n,$$

or in matrix form

$$\partial_t \mathbf{y}_j + F(\mathbf{y}_j)\partial_x \mathbf{y}_j = G(\mathbf{y}_j),\ j = 1, \ldots, n \tag{3}$$

The directed arcs $i \to j$ of the network represent instantaneous mass transfers between the compartments (i.e. transfer of water between the pools in irrigation networks or transfer of vehicles at the road junctions in traffic networks). The transfer rate or *flow* from the output of a compartment i to the input of a compartment j is denoted $\varphi_{ij}(t)$. Additional input and output arcs represent interactions with the surroundings: either inflows $b_j(t)$ injected from the outside into some compartments or outflows $e_j(t)$ from some compartments to the outside. The set of $2n$ PDEs (2) is therefore subject to $2n$ boundary flow balance conditions of the form:

$$q_j(t, 0) = \sum_{i \neq j} \varphi_{ij}(t) + b_j(t), \quad j = 1, \ldots, n,$$

$$q_j(t, L) = \sum_{k \neq j} \varphi_{jk}(t) + e_j(t), \quad j = 1, \ldots, n.$$

A standard assumption, corresponding to many pratical engineering applications, is that the flows $e_j(t)$ and $\varphi_{ij}(t)$ are adequately modelled by static functions of the states $\mathbf{y}_i(t, L) = (p_i(t, L), q_i(t, L))^T$ at the output of the upstream compartment and the states $\mathbf{y}_j(t, 0) = (p_j(t, 0), q_j(t, 0))^T$ at the input of the downstream compartment. Moreover, we assume also that the network inflows $b_j(t)$ and (some) of the partial transfer flows $\varphi_{ij}(t)$ can be modulated by using appropriate actuators (like e.g. valves and pumps in irrigation channels or traffic lights in road networks). Therefore the boundary conditions are written as

$$q_j(t, 0) = \sum_{i \neq j} \varphi_{ij}(\mathbf{y}_i(t, L), \mathbf{y}_j(t, 0), w_{ij}(t)) + b_j(w_{oj}(t)),$$

$$q_j(t, L) = \sum_{k \neq j} \varphi_{jk}(\mathbf{y}_j(t, L), \mathbf{y}_k(t, 0), w_{jk}(t)) + e_j(\mathbf{y}_j(t, L)),$$

$$j = 1, \ldots, n. \tag{4}$$

with the notations w_{ij} and w_{oj} for the controls.

In equations (4), only the terms corresponding to actual links of the network are explicitly written. Otherwise stated, all the b_j, e_j and φ_{ij} for non existing links do not appear in the equations.

Finally, the control system (2)-(4) may be written in a compact form

$$\partial_t \mathbf{y} + \mathbf{F}(\mathbf{y})\partial_x \mathbf{y} = \mathbf{G}(\mathbf{y}), \tag{5a}$$

$$\mathbf{N_b}(\mathbf{y}(t, 0), \mathbf{y}(t, L), \mathbf{w}(t)) = 0, \tag{5b}$$

with obvious definitions of the notations for $\mathbf{y}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{N_o}$, $\mathbf{w}$.

Boundary control

Steady-state : For constant control actions $\mathbf{w}(t) = \bar{\mathbf{w}}$ a steady-state solution is a constant solution $\mathbf{y}(t, x) = \bar{\mathbf{y}}\ \forall t \in [0, +\infty),\ \forall x \in [0, L]$ which satisfies the condition $\mathbf{G}(\bar{\mathbf{y}}) = 0$ and the boundary conditions $\mathbf{N_b}(\bar{\mathbf{y}}, \bar{\mathbf{y}}, \bar{\mathbf{w}}) = 0$. Depending on the form of these boundary conditions, the steady-state solution may be stable or unstable.

We are concerned in analysing the stability of the steady-state $\bar{\mathbf{y}}$ when the system (5) is under boundary feedback control actions

$$\mathbf{w}(t) = \mathbf{w}(\mathbf{y}(t, 0), \mathbf{y}(t, L)). \tag{6}$$

With the control law (6), the closed-loop system is written

$$\partial_t \mathbf{y} + \mathbf{F}(\mathbf{y})\partial_x \mathbf{y} = \mathbf{G}(\mathbf{y}), \tag{7a}$$

$$\mathbf{N_c}(\mathbf{y}(t, 0), \mathbf{y}(t, L)) = 0, \tag{7b}$$

Riemann coordinates

In order to analyse the closed loop stability by a Lyapunov method, it is convenient to consider the system (2) expressed in Riemann coordinates:

$$\partial_t \mathbf{u}_i + \begin{pmatrix} c_i(\mathbf{u}_i) & 0 \\ 0 & c_{n+i}(\mathbf{u}_i) \end{pmatrix} \partial_x \mathbf{u}_i = H(\mathbf{u}_i) \tag{8}$$

$$i = 1, \ldots, n$$

with $\mathbf{u}_i = (u_i, u_{n+i})^T = \Phi(\mathbf{y}_i)$.

The change of coordinates $\mathbf{u}_i = \Phi(\mathbf{y}_i)$ is clearly defined up to a constant. It can therefore always be selected in such a way that $\Phi(\bar{\mathbf{y}}_i) = 0$ and the control problem can be stated as the problem of determining the control actions in such a way that the characteristic solutions $\mathbf{u}_i(t)$ converge towards the origin.

IV. Lyapunov stability of the linearised system

We consider the linear approximation of the system (8) around the origin

$$\partial_t \mathbf{u} + \mathbf{\Lambda}\partial_x u = \mathbf{B}\mathbf{u} \tag{9}$$

with $\mathbf{u} \triangleq (u_1, \ldots, u_{2n})$, $\mathbf{\Lambda} = \text{diag}\{c_1, \ldots, c_{2n}\}$ and an obvious definition of the matrix $\mathbf{B}$.

Moreover, using these notations, the linearisation of the boundary condition (7b) is written in the Riemann coordinates

$$\mathbf{N}_0\mathbf{u}(t,0) + \mathbf{N}_1\mathbf{u}(t,L)) = 0. \tag{10}$$

Our concern is to analyse the exponential stability of the solutions $\mathbf{u}(t,x)$ of the system (9)-(10) according to the following definition.

Definition 1. The linear hyperbolic system (9)-(10) is exponentially stable (in L^2-norm) if there exist $\nu > 0$ and $C > 0$ such that, for every initial condition

$$\mathbf{u}(0,x) = \mathbf{u}^0(x) \in L^2((0,L);\mathbb{R}^{2n}) \tag{11}$$

the solution to the Cauchy problem (9)-(10)-(11) satisfies

$$\|\mathbf{u}(t,\cdot)\|_{L^2((0,L);\mathbb{R}^{2n})} \leqslant Ce^{-\nu t}\|\mathbf{u}^0\|_{L^2((0,1);\mathbb{R}^{2n})}.$$

■

The following candidate Lyapunov function is defined:

$$V = \int_0^L \mathbf{u}^T\mathbf{P}(x)\mathbf{u}dx \tag{12}$$

where the weighting matrix $\mathbf{P}(x)$ is defined as follows: $\mathbf{P}(x) \triangleq \text{diag}\{p_i e^{-\sigma_i \mu x}, i = 1, \ldots, 2n\}$, with $\mu > 0$, $p_i > 0$ positive real numbers and $\sigma_i = \text{sign}(c_i)$.

The time derivative of V along the solutions of (9) is

$$\begin{aligned}
\dot{V} &= \int_0^L \left(\partial_t \mathbf{u}^T\mathbf{P}(x)\mathbf{u} + \mathbf{u}^T\mathbf{P}(x)\partial_t\mathbf{u}\right)dx \\
&= -\int_0^L (\partial_x \mathbf{u}^T\mathbf{\Lambda}\mathbf{P}(x)\mathbf{u} + \mathbf{u}^T\mathbf{P}(x)\mathbf{\Lambda}\partial_x\mathbf{u} \\
&\qquad - \mathbf{u}^T\mathbf{B}^T\mathbf{P}(x)\mathbf{u} - \mathbf{u}^T\mathbf{P}(x)\mathbf{B}\mathbf{u})dx \\
&= -\int_0^L \partial_x(\mathbf{u}^T\mathbf{R}(x)\mathbf{u})dx \\
&\quad + \int_0^L \mathbf{u}^T\Big(\mathbf{B}^T\mathbf{P}(x) + \mathbf{P}(x)\mathbf{B}\Big)\mathbf{u}\,dx
\end{aligned}$$

with the positive diagonal matrix

$$\mathbf{R}(x) \triangleq \text{diag}\{p_i|c_i|e^{-\sigma_i\mu x}, i = 1, \ldots, 2n\}.$$

Integrating by parts, we obtain:

$$\begin{aligned}
\dot{V} &= -\int_0^L \partial_x\left[\mathbf{u}^T\mathbf{R}(x)\mathbf{u}\right]dx \\
&\quad - \int_0^L \mathbf{u}^T\Big(\mu\mathbf{R}(x) - \mathbf{B}^T\mathbf{P}(x) - \mathbf{P}(x)\mathbf{B}\Big)\mathbf{u}\,dx \\
&= -\left[\mathbf{u}^T\mathbf{R}(x)\mathbf{u}\right]_0^L \\
&\quad - \int_0^L \mathbf{u}^T\Big(\mu\mathbf{R}(x) - \mathbf{B}^T\mathbf{P}(x) - \mathbf{P}(x)\mathbf{B}\Big)\mathbf{u}\,dx \\
&= -\left[\mathbf{u}^T(t,L)\mathbf{R}(L)\mathbf{u}(t,L) - \mathbf{u}^T(t,0)\mathbf{R}(0)\mathbf{u}(t,0)\right] \\
&\quad - \int_0^L \mathbf{u}^T\Big(\mu\mathbf{M}(x) - \mathbf{B}^T\mathbf{P}(x) - \mathbf{P}(x)\mathbf{B}\Big)\mathbf{u}\,dx.
\end{aligned}$$

The system (9)-(10) is exponentially stable if this function $\dot{V}$ is negative definite. We have thus shown the following result.

Theorem 1. The system (9)-(10) is exponentially stable if there exist $\mu > 0$ and $p_i > 0$ $i = 1, \ldots, 2n$ such that

C1. The boundary quadratic form $\mathbf{u}^T(t,0)\mathbf{R}(0)\mathbf{u}(t,0) - \mathbf{u}^T(t,L)\mathbf{R}(L)\mathbf{u}(t,L)$ is positive definite under the constraint of the linear boundary condition $\mathbf{N}_0\mathbf{u}(t,0) + \mathbf{N}_1\mathbf{u}(t,L) = \mathbf{0}$;

C2. The matrix $\mu\mathbf{M}(x) - \mathbf{B}^T\mathbf{P}(x) - \mathbf{P}(x)\mathbf{B}$ is positive definite $\forall x \in (0,L)$. ■

Boundary conditions that satisfy condition C1 are called *Dissipative Boundary Conditions*. Condition C1 is satisfied if and only if the leading principal minors of order $> 4n$ of the matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 \\ -\mathbf{N}_0^T & \mathbf{R}(0) & \mathbf{0} \\ -\mathbf{N}_1^T & \mathbf{0} & -\mathbf{R}(L) \end{pmatrix}$$

are strictly positive (see [14]).

V. Dissipative boundary conditions

In this section, we will present a variant of Theorem 1 with an explicit characterisation of a sufficient dissipative boundary condition which guarantees the system exponential stability in the case where $\|\mathbf{B}\|$ is sufficiently small or, in more intuitive terms, when the considered balance laws are viewed as perturbations of conservation laws. Again, we consider the linear approximation of the system (8) around the origin

$$\partial_t\mathbf{u}_i + \begin{pmatrix} c_i & 0 \\ 0 & c_{n+i} \end{pmatrix}\partial_x\mathbf{u}_i = M\mathbf{u}_i \qquad i = 1, \ldots, n \tag{13}$$

with $M \triangleq H'(0)$ and we assume that the characteristic velocities have opposite signs $c_{n+i} < 0 < c_i$. Then the system (13) is written

$$\begin{pmatrix} \partial_t\mathbf{u}^+ + \mathbf{\Lambda}^+\partial_x\mathbf{u}^+ \\ \partial_t\mathbf{u}^- - \mathbf{\Lambda}^-\partial_x\mathbf{u}^- \end{pmatrix} = \mathbf{M}\mathbf{u} \tag{14}$$

with $\mathbf{u}^+ \triangleq (u_1, \ldots, u_n)$, $\mathbf{u}^- \triangleq (u_{n+1}, \ldots, u_{2n})$, $\mathbf{u} \triangleq (\mathbf{u}^{+T}, \mathbf{u}^{-T})$, $\mathbf{\Lambda}^+ = \text{diag}\{c_1, \ldots, c_n\}$, $\mathbf{\Lambda}^- = \text{diag}$

$\{|c_{n+1}|,\ldots,|c_{2n}|\}$ and an obvious definition of the matrix $\mathbf{M}$.

Moreover, using these notations, the boundary condition (7b) is written in the Riemann coordinates

$$\mathbf{N_r}(\mathbf{u}^+(t,0),\mathbf{u}^+(t,L),\mathbf{u}^-(t,0),\mathbf{u}^-(t,L))=0 \qquad (15)$$

Then assuming that the map $\mathbf{N_r}$ is differentiable in a neighborhood of the origin and $\partial_{\mathbf{u}^+(0),\mathbf{u}^-(L)}\mathbf{N_r}(0,0,0,0)$ is nonsingular, by the implicit function theorem, the linearization of the boundary condition (15) about the origin is written

$$\begin{pmatrix}\mathbf{u}^+(t,0)\\ \mathbf{u}^-(t,L))\end{pmatrix}=\underbrace{\begin{pmatrix}K_{00} & K_{01}\\ K_{10} & K_{11}\end{pmatrix}}_{\mathbf{K}}\begin{pmatrix}\mathbf{u}^+(t,L)\\ \mathbf{u}^-(t,0)\end{pmatrix}. \qquad (16)$$

Let $\mathcal{D}_m$ denote the set of diagonal $m\times m$ real matrices with strictly positive diagonal entries. We introduce the following norm for the matrix $\mathbf{K}$:

$$\rho(\mathbf{K})\triangleq\inf\left\{\|\Delta\mathbf{K}\Delta^{-1}\|,\Delta\in\mathcal{D}_{2n}\right\}.$$

We then have the following stability Theorem.

Theorem 2. If $\rho(\mathbf{K})<1$, there exist $\varepsilon>0$ such that, if $\|\mathbf{M}\|<\varepsilon$, then the linear hyperbolic system (14)-(16) is exponentially stable.

Proof. The following candidate Lyapunov function is considered:

$$V=\int_0^L\left[(\mathbf{u}^{+T}P_0\mathbf{u}^+)e^{-\mu x}+(\mathbf{u}^{-T}P_1\mathbf{u}^-)e^{\mu x}\right]dx. \qquad (17)$$

with $P_0\in\mathcal{D}_n$, $P_1\in\mathcal{D}_n$ and $\mu>0$. The time derivative of V is

$$\begin{aligned}\dot V=&\int_0^L-\partial_x\left(\mathbf{u}^{+T}P_0\mathbf{\Lambda}^+\mathbf{u}^+\right)e^{-\mu x}\,dx\\ &+\int_0^L\partial_x\left(\mathbf{u}^{-T}P_1\mathbf{\Lambda}^-\mathbf{u}^-\right)e^{\mu x}\,dx\\ &+\int^L\mathbf{u}^T\left(\mathbf{M}^TP(x)+P(x)\mathbf{M}\right)\mathbf{u}\,dx.\end{aligned}$$

Using integration by parts we get

$$\dot V=\dot V_1+\dot V_2 \qquad (18)$$

with

$$\dot V_1\triangleq-\left[\mathbf{u}^{+T}P_0\mathbf{\Lambda}^+\mathbf{u}^+e^{-\mu x}\right]_0^L+\left[\mathbf{u}^{-T}P_1\mathbf{\Lambda}^-\mathbf{u}^-e^{\mu x}\right]_0^L$$

$$\dot V_2\triangleq\int_0^L\mathbf{u}^T\left(-\mu P(x)\mathbf{\Lambda}+\mathbf{M}^TP(x)+P(x)\mathbf{M}\right)\mathbf{u}\,dx$$

with

$$P(x)\triangleq\text{diag}\left\{P_0e^{-\mu x},P_1e^{\mu x}\right\}\text{ and }\mathbf{\Lambda}\triangleq\text{diag}\left\{\mathbf{\Lambda}^+,\mathbf{\Lambda}^-\right\}.$$

The two terms of (18) are analysed successively. For this analysis, we introduce the following notations:

$$\mathbf{u}_0^-(t)\triangleq\mathbf{u}^-(t,0)\quad \mathbf{u}_1^+(t)\triangleq\mathbf{u}^+(t,L).$$

Analysis of the first term.
Using the boundary condition (16), we have

$$\begin{aligned}\dot V_1=&-\left[\mathbf{u}^{+T}P_0\mathbf{\Lambda}^+\mathbf{u}^+e^{-\mu x}\right]_0^L+\left[\mathbf{u}^{-T}P_1\mathbf{\Lambda}^-\mathbf{u}^-e^{\mu x}\right]_0^L\\ =&-\left(\mathbf{u}_1^{+T}P_0\mathbf{\Lambda}^+\mathbf{u}_1^+\,e^{-\mu L}+\mathbf{u}_0^{-T}P_1\mathbf{\Lambda}^-\mathbf{u}_0^-\right)\\ &+\left(\mathbf{u}_1^{+T}K_{00}^T+\mathbf{u}_0^{-T}K_{01}^T\right)P_0\mathbf{\Lambda}^+\left(K_{00}\mathbf{u}_1^++K_{01}\mathbf{u}_0^-\right)\\ &+\left(\mathbf{u}_1^{+T}K_{10}^T+\mathbf{u}_0^{-T}K_{11}^T\right)P_1\mathbf{\Lambda}^-\left(K_{10}\mathbf{u}_1^++K_{11}\mathbf{u}_0^-\right)e^{\mu L}.\end{aligned}$$

Since $\rho(\mathbf{K})<1$ by assumption, there exist $D_0\in\mathcal{D}_n$, $D_1\in\mathcal{D}_n$ and $\Delta\triangleq\text{diag}\{D_0,D_1\}$ such that

$$\|\Delta\mathbf{K}\Delta^{-1}\|<1. \qquad (19)$$

The matrices P_0 and P_1 are selected such that $P_0\mathbf{\Lambda}^+=D_0^2$ and $P_1\mathbf{\Lambda}^-=D_1^2$. We define $\mathbf{z}_0\triangleq D_0\mathbf{u}_0^-$, $\mathbf{z}_1\triangleq D_1\mathbf{u}_1^+$ and $\mathbf{z}^T\triangleq(\mathbf{z}_0^T,\mathbf{z}_1^T)$. Then, using inequality (19), we have

$$\begin{aligned}&\left(\mathbf{u}_1^{+T}K_{00}^T+\mathbf{u}_0^{-T}K_{01}^T\right)P_0\mathbf{\Lambda}^+\left(K_{00}\mathbf{u}_1^++K_{01}\mathbf{u}_0^-\right)\\ &+\left(\mathbf{u}_1^{+T}K_{10}^T+\mathbf{u}_0^{-T}K_{11}^T\right)P_1\mathbf{\Lambda}^-\left(K_{10}\mathbf{u}_1^++K_{11}\mathbf{u}_0^-\right)\\ &=\|\Delta\mathbf{K}\Delta^{-1}\mathbf{z}\|^2<\|\mathbf{z}\|^2\\ &=\mathbf{u}_1^{+T}P_0\mathbf{\Lambda}^+\mathbf{u}_1^++\mathbf{u}_0^{-T}P_1\mathbf{\Lambda}^-\mathbf{u}_0^-.\end{aligned}$$

It follows that μ can be taken sufficiently small such that $\dot V_1$ is a negative definite quadratic form.

Analysis of the second term.

For any $\mu>0$, there exist clearly two positive constants $\varepsilon>0$ and $\alpha>0$ such that

$$\|\mathbf{M}\|<\varepsilon\;\Rightarrow\;\dot V_2\leqslant-\alpha V\;\Rightarrow\;\dot V=\dot V_1+\dot V_2\leqslant-\alpha V.$$

Consequently the solutions of the system (14)-(16) exponentially converge to the origin in L^2-norm. ∎

VI. Lyapunov stability of the nonlinear system

Our concern in this section is to briefly explain how the **linear** Lyapunov stability analysis of Section IV can be extended to the case of the **nonlinear** closed-loop control system (7). Assuming as above that the map $\mathbf{N_r}$ is differentiable and $\partial_{\mathbf{u}^+(0),\mathbf{u}^-(L)}\mathbf{N_r}(0,0,0,0)$ is nonsingular, the closed loop system in Riemann coordinates in a neighborhood of the origin is written

$$\begin{pmatrix}\partial_t\mathbf{u}^++\mathbf{\Lambda}^+(\mathbf{u})\partial_x\mathbf{u}^+\\ \partial_t\mathbf{u}^--\mathbf{\Lambda}^-(\mathbf{u})\partial_x\mathbf{u}^-\end{pmatrix}=\mathbf{H}(\mathbf{u}) \qquad (20a)$$

$$\begin{pmatrix}\mathbf{u}^+(t,0)\\ \mathbf{u}^-(t,L))\end{pmatrix}=\mathbf{K}\begin{pmatrix}\mathbf{u}^-(t,0)\\ \mathbf{u}^+(t,L))\end{pmatrix} \qquad (20b)$$

with appropriate nonlinear maps $\mathbf{H}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ and $\mathbf{K}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$.

With Theorem 2 we have proved the convergence to zero of the solutions of the linear system (14)-(16) in $L^2(0,L)$-norm. Unfortunately the same Lyapunov function cannot be directly used to analyse the local syability in the nonlinear case. As we have emphasized in detail in [4], in order to extend the Lyapunov stability analysis to the nonlinear case, it is needed to prove a convergence in $H^2(0,L)$-norm. We therefore adopt the following definition of the (local) exponential stability of the steady-state solution $\mathbf{u}(t,x) \equiv 0$

Definition 2. The equilibrium solution $\mathbf{u} \equiv 0$ of the nonlinear hyperbolic system (20) is exponentially stable (for the H^2-norm) if there exist $\delta > 0$, $\nu > 0$ and $C > 0$ such that, for every initial condition

$$\mathbf{u}(0,x) = \mathbf{u}^0(x) \in H^2((0,1),\mathbb{R}^n) \qquad (21)$$

satisfying

$$\|\mathbf{u}^0\|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta,$$

the classical solution u to the Cauchy problem (20)–(21) satisfies

$$\|\mathbf{u}(t,\cdot)\|_{H^2((0,1),\mathbb{R}^n)} \leqslant Ce^{-\nu t}\|\mathbf{u}^0\|_{H^2((0,1),\mathbb{R}^n)}, \qquad (22)$$
$$\forall t \in [0,+\infty).$$

■

The stability property may then be generalised as follows to the nonlinear case.

Theorem 3. If $\rho(\mathbf{K}'(0)) < 1$, there exist $\varepsilon > 0$ such that, if $\|\mathbf{H}'(0)\| < \varepsilon$, then the equilibrium $\mathbf{u} \equiv 0$ of the nonlinear hyperbolic system (20) is exponentially stable. ■

The proof of this theorem is much more complicated than its linear counterpart and can be established by using the approach followed in [3]. It makes use of an augmented Lyapunov function (see (17) for comparison) of the form

$$V = \int_0^L \left[(\mathbf{u}^{+T}P_0\mathbf{u}^+)e^{-\mu_1 x} + (\mathbf{u}^{-T}P_1\mathbf{u}^-)e^{\mu_1 x}\right]dx$$
$$+ \int_0^L \left[(\mathbf{v}^{+T}Q_0\mathbf{v}^+)e^{-\mu_2 x} + (\mathbf{v}^{-T}Q_0\mathbf{v}^-)e^{\mu_2 x}\right]dx$$
$$+ \int_0^L \left[(\mathbf{w}^{+T}R_0\mathbf{w}^+)e^{-\mu_3 x} + (\mathbf{w}^{-T}R_0\mathbf{w}^-)e^{\mu_3 x}\right]dx$$

with the weighting matrices

$$P_0 = D_0^2(\mathbf{\Lambda}^+)^{-1} \quad P_1 = D_1^2(\mathbf{\Lambda}^-)^{-1}$$
$$Q_0 = D_0^2(\mathbf{\Lambda}^+) \quad Q_1 = D_1^2(\mathbf{\Lambda}^-)$$
$$R_0 = D_0^2(\mathbf{\Lambda}^+)^3 \quad R_1 = D_1^2(\mathbf{\Lambda}^-)^3$$

and the additional state variables $\mathbf{v} \triangleq \partial_x \mathbf{u}$ and $\mathbf{w} \triangleq \partial_{xx}\mathbf{u}$.

VII. Concluding remarks

We have addressed the isue of stating sufficient boundary conditions for the exponential stability of the steady-states of physical networks described by interconnected systems of 2×2 nonlinear hyperbolic balance laws. In Theorem 1 we have first given a general implicit formulation of sufficient dissipative boundary conditions. Our analysis relies on the use of an explicit Lyapunov function. The weight $e^{\pm\mu x}$ is essential to get a strict Lyapunov function. It is similar to the one introduced in [2] to stabilize the Euler equation of incompressible fluids. More recently, it has also been used in [15] for linear symmetric hyperbolic systems.

Then in Theorems 2 and 3, we have shown that the explicit dissipativity condition $\rho(\mathbf{K}) < 1$ (or $\rho(\mathbf{K}'(0)) < 1$) gives a convergence in $L^2(0,L)$-norm for sytems of balance laws considered as perturbations of conservation laws. This new sufficient stability condition is weaker than the previous condition which was given in [13]. A variant of this property with convergence in $C^1(0,L)$-norm can also be found in the reference [10] where the analysis relies on the method of characteristics.

In this paper, for the sake of simplicity, we have considered the case where the steady-state is constant with respect to both t and x. The analysis can however be extended to the case where there is a non-uniform steady-state profile $\bar{\mathbf{y}}(x) \triangleq (\bar{q}, \bar{p}(x))$ with a constant flux $\bar{q}$ but a possibly space varying density $\bar{p}(x)$ which satisfies the steady-state scalar differential equation $\alpha(\bar{p}(x),\bar{q})\partial_x\bar{p}(x) = \gamma(\bar{p}(x),\bar{q})$.

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