CONTROLLABILITY AND STATE FEEDBACK STABILIZABILITY OF NON HOLONOMIC MECHANICAL SYSTEMS

G. Campion†, B. d'Andrea-Novel‡, G. Bastin†

† Laboratoire d'Automatique, Dynamique et Analyse des Systèmes
Université Catholique de Louvain
Place du Levant, 3-B1348 LOUVAIN-LA-NEUVE (Belgium)

‡ Centre d'Automatique,
Ecole Nationale Supérieure des Mines de Paris
Rue Saint Honoré, 35-F77305 FONTAINEBLEAU (France)

ABSRACT

The dynamics of non holonomic mechanical system are described by the classical Euler-Lagrange equations subjected to a set of non-integrable constraints. Non holonomic systems are strongly accessible whatever the structure of the constraints. They cannot be asymptotically stabilized by a smooth pure state feedback. However smooth state feedback control laws can be designed which guarantee the global marginal stability of non holonomic systems.

1. INTRODUCTION

A mechanical system, whose configuration is completely described by a set of generalized coordinates, can be subjected to kinematic constraints (such as the pure rolling condition of a wheel on a plane), which are expressed by relations between the coordinates and their time derivatives. If these constraints are holonomic (that is integrable) it is possible to characterize the system configuration by a smaller number of coordinates (i.e. to use the constraints in order to eliminate the redundant coordinates) in such a way that the constraints are automatically satisfied in the new coordinates. Unfortunately, in case of non holonomic constraints, this elimination is not possible and the constraints have to be taken into account explicitly in the derivation of the dynamical equations. The theory of mechanical systems with non holonomic constraints has been developped at the end of last century by many authors (e.g. Appell [1], Hamel [2]). The present paper deals with control design of such systems, for which, due to the nonholonomic constraints, the standard control laws developped for holonomic mechanical systems (for instance robotic manipulators) are not applicable .

Control of mechanical systems, with not integrable constraints which are linear in the generalized velocities, has been discussed in the literature through the special case of mobile wheeled robots (see e.g. [3, 4, 5]). In these papers however, the control is designed on the basis of a kinematic state-space model derived from the constraints, but not taking the internal dynamics of the system into account. The purpose of this paper is to derive a full dynamical description of such nonholonomic mechanical systems, including the constraints and the internal dynamics, and to show how a suitable change of coordinates allows to analyse globally the controllability and the state feedback stabilizability of the system. The feedback stabilizability of mechanical systems with constraints (holonomic or not) is also examined by Bloch and McClamroch [9]. However, they use another change of coordinates which is less efficient since it provides only local stability results and is not convenient for a controllability analysis.

The paper is organized as follows. The concept of non holonomic constraints for mechanical systems is introduced in Section 2 within the framework of the theory of nonlinear control systems. The dynamics of non holonomic systems can be partially described by a so-called kinematic state-space model. In Section 3.1, it is shown that this model is completely controllable. The existence of smooth stabilizing state feedback controls is then addressed in Section 3.2. It is shown that the origin of the generalized coordinates cannot be asymptotically stabilized by a smooth pure state feedback but can nevertheless be globally maginally stabilized. A general dynamical state-space model of non holonomic systems is then derived in Section 4, using the classical Euler-Lagrange formalism. By a suitable change of coordinates,this model can be partially linearized in such a way that the remaining nonlinearities only depend on the structure of the constraints. On this basis it is then shown, in Section 4.1, that non holonomic systems are strongly accessible whatever the structure of the constraints. Furthermore, as shown in Section 4.2, the stabilizability results of Section 3.2 can be extended to the general case: non holonomic systems cannot be asymptotically stabilized by a smooth pure feedback control but can be globally marginally stabilized. The design of the stabilizing control law is explicited.

2. NONHOLONOMIC CONSTRAINTS

We are concerned, in this paper, with mechanical systems whose configuration space is an n-dimensional simply connected manifold $\mathcal{M}$ and whose dynamics are described, in local coordinates, by the so-called Euler-Lagrange equations of motion. Usually, the local coordinates used for the description of these systems are termed "generalized coordinates" and denoted $q_1, q_2, ..., q_n$. Each configuration of the system is represented by the vector of these generalized coordinates and is denoted:

$$q \equiv [q_1, q_2, \dots, q_n]^T$$

The configuration manifold $\mathcal{M}$, which is the set of all possible configurations, is represented in local coordinates by an open set $\Omega \subseteq \mathbb{R}^n$. The position of each material point of the system is a function of the generalized coordinates. A motion of the system is represented in the q coordinates by a smooth time function q(t). The corresponding trajectory is a one-dimensional immersed submanifold of $\mathcal{M}$. The tangent vector at a point of the trajectory is then represented by the vector $\dot{q} \equiv [\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n]^T$ whose components $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ are termed generalized velocities.

In many instances, the motion of mechanical systems is subjected to various constraints which are permanently satisfied during the motion and which take the form of algebraic relationships between the positions and the velocities of particular material points of the system. Two kinds of constraints can be distinguished: geometric constraints and kinematic constraints.

Geometric constraints.

These constraints are represented by analytical relations between the generalized coordinates. When the system is subjected to m such constraints, there exists an m-dimensional vector function $\rho(q) : \Omega \to \mathbb{R}^m$ such that $\rho(q) = 0$ for all q in Ω. The m (<n) constraints are said *independent* when the jacobian matrix of $\rho(q)$ has full rank for all q. In that case m generalized coordinates can be eliminated and n - m generalized coordinates are sufficient to provide a full description of the configurations of the system.

Kinematic constraints.

These constraints are represented by analytical relations between the generalized coordinates and velocities. In most applications, these relations are linear with respect to the generalized velocities and written as:

$$a_j^T(q)\dot{q} = 0 \qquad (1)$$

where $a_1^T, a_2^T, \dots, a_m^T$ are smooth n-dimensional covector fields on $\mathcal{M}$. In matrix form, the constraints (1) are written :

$$A^T(q)\dot{q} = 0$$

where A(q) is the (n x m) matrix made up of the vector functions $a_j(q)$ as follows:

$$A(q) \equiv [a_1(q), a_2(q), \dots, a_m(q)]$$

The m (<n) constraints are said *independent* when this matrix has full rank for all q. Unlike geometric constraints, the kinematic constraints do not necessarily lead to the elimination of generalized coordinates from the system description. The elimination is possible only when the constraints are *holonomic* (that is: integrable). Our concern in this paper will precisely be to discuss the controllability and the feedback stabilization of mechanical systems with *nonholonomic* constraints.

Hence, without loss of generality, we can consider that all the redundant generalized coordinates associated to the geometric constraints have been eliminated and restrict our attention to mechanical systems subjected to m *independent* kinematic constraints only. These constraints are assumed to have the form (1).

We assume that the annihilator of the codistribution spanned by the covector fields $a_1^T, a_2^T, \dots, a_m^T$, is an (n-m)-dimensional *smooth* nonsingular distribution Δ on $\mathcal{M}$. This distribution Δ is spanned by a set of (n-m) smooth vector fields $s_1, s_2, \dots, s_{n-m}$:

$$\Delta \equiv \text{span}\{s_1, s_2, \dots, s_{n-m}\}$$

which satisfy, in local coordinates, the following relations:

$$a_j^T(q)s_i(q) = 0 \qquad \forall q \in \Omega \qquad j = 1,\dots,m \quad i = 1,\dots,n-m$$

Since Δ is nonsingular, any vector field τ of Δ can be expressed in the form:

$$\tau(q) = \sum_{i=1}^{n-m} c_i(q)s_i(q)$$

where $c_1(q), c_2(q), \dots, c_{n-m}(q)$ are smooth functions on Ω (see e.g. Isidori [6], Chapter 1, Section 1.3).

We introduce also the full rank matrix S(q) made up of the vector functions $s_i(q)$:

$$S(q) \equiv [s_1(q), s_2(q), \dots, s_{n-m}(q)]$$

It is then clear that the constraints (1) may be expressed as:

$$\dot{q} \in \Delta(q) \text{ or equivalently } \dot{q} \in \text{Im}[S(q)]$$

Consider now the involutive closure of Δ, denoted Δ^*, and defined as the smallest involutive distribution containing Δ. Assume that this distribution is regular (that is has constant dimension on $\mathcal{M}$). Clearly:

$$n - m \leq \dim(\Delta^*) \leq n$$

Let (n - m*) denote the dimension of Δ^*, with $m^* \leq m$. From Fröbenius Theorem, at each q in Ω, there exists a set of m* independent smooth functions denoted $\mu_1(q)$, $\mu_2(q)$, ..., $\mu_{m^*}(q)$, such that, for each vector field $\tau \in \Delta^*$ the following relations hold :

$$\mathcal{L}_\tau \mu_i(q) = 0 \qquad i = 1, ..., m^* \tag{2}$$

where $\mathcal{L}_\tau \mu_i$ denotes the Lie derivative of μ_i along τ.

Let us now define a change of coordinates $\xi = \Phi(q)$, with $\Phi(0) = 0$, with m* coordinates being the functions $\mu_1(q)$, $\mu_2(q)$, ..., $\mu_{m^*}(q)$, and the remaining n-m* coordinates being chosen to complete the diffeomorphism:

$$\xi = \Phi(q) = \begin{bmatrix} \varphi_1(q) \\ \varphi_2(q) \\ \cdot \\ \cdot \\ \varphi_{n-m^*}(q) \\ \mu_1(q) \\ \mu_2(q) \\ \cdot \\ \cdot \\ \mu_{m^*}(q) \end{bmatrix}$$

Hence the tangent vector to the trajectory at the point $\xi = \Phi(q)$ is represented in the ξ coordinates by $\dot{\xi}$, with :

$$\dot{\xi} \in \mathrm{Im}\left[\left(\frac{\partial \Phi}{\partial q}\right)_{\Phi^{-1}(\xi)} S(\Phi^{-1}(\xi))\right]$$

It then follows from (2) that, since $\dot{q} \in \Delta(q)$, the last m* components of $\dot{\xi}$ are identically zero:

$$\dot{\xi}_{n-m^*+1} = \dot{\xi}_{n-m^*+2} = ... = \dot{\xi}_n = 0$$

This means that the m* coordinates ξ_{n-m^*+1}, ξ_{n-m^*+2}, ..., ξ_n which are identical to the m* functions μ_i are constant along the motions of the system.

Then, depending on the dimension of Δ^*, several situations may arise:

a) If m* = m (that is if Δ is involutive) the system is said to be *holonomic* . The configuration space can be characterized with (n-m) coordinates only, namely ξ_1, ξ_2, ..., ξ_{n-m}. The configuration space is thus an (n-m)-dimensional manifold.

b) If $m^* = 0$ (that is if $\dim(\Delta^*) = n$) the constraints are completely nonintegrable and the system is said to be *nonholonomic*. The characterization of the configuration space requires n coordinates.

c) If $0 < m^* < m$ it is possible to eliminate m^* coordinates. The configuration space is a manifold of dimension $n - m^*$.

Without loss of generality, we can thus assume that all the geometric and all the integrable kinematic constraints have been eliminated from the system description and restrict our attention to the situation b) that is to nonholonomic mechanical systems evolving in an n-dimensional configuration manifold and subjected to m independent nonintegrable constraints.

3. THE KINEMATIC STATE-SPACE MODEL : CONTROLLABILITY AND FEEDBACK STABILIZATION

The dynamics of nonholonomic mechanical systems are partially described by a state-space model which is associated to the kinematic constraints and referred to as the *kinematic state-space model*. Our purpose, in this section is to examine the controllability properties of this model and to discuss its state feedback stabilization.

Along the motions of the system, the constraints (1) imply the existence of a vector time function $w(t) \in \mathbb{R}^{n-m}$ for all t, such that:

$$\dot{q} = S(q)w(t) \tag{3}$$

where $S(q)$ is the matrix defined above. Conversely, for any initial condition $q(0)$ and any time function $w(t)$, the solution $q(t)$ of (3) will satisfy the constraints (1) and be a possible motion of the system.

Hence the model (3) can be interpreted as an n-dimensional state space representation of the motion of a nonholonomic mechanical system with state q and control input w. Obviously, for a given choice of the generalized coordinates, this representation is not unique since it depends on the particular selection of the basis (i.e. the vector fields s_j) of the distribution Δ.

3.1. Controllability.

It follows immediately from the property of nonholonomy of the constraints that the *strong accessibility rank condition* (see [10]) is satisfied for all $q \in \Omega$ and, therefore, that the system (3) is strongly accessible from any configuration. Furthermore, since equation

(3) does not contain a drift vector field, strong accessibility implies controllability (see e.g. Nijmeijer and van der Schaft [7], Chapter 3, Section 3.1). We thus have the following result.

Lemma 1. The kinematic state space model of a nonholonomic system is controllable. •

In practice, this means that for any two configurations $q^{(1)}$ and $q^{(2)}$ in Ω, there exists a finite time T and an input function w(t) such that if $q(0) = q^{(1)}$ then $q(T) = q^{(2)}$. It is however worth noting that this does not mean that any velocity can be achieved since the generalized velocities are constrained to belong to the (n-m)-dimensional space spanned by the columns of S(q).

3.2. State feedback control.

In this section, we are concerned by the question of the existence of smooth pure state feedback stabilizing control laws for the kinematic state space model (3). More precisely, we would like to stabilize the system at a particular configuration which may be taken, without loss of generality, as the origin of the generalized coordinates (i.e. q = 0).

A smooth pure state feedback control law for the system (3) is defined as a smooth mapping:

$$w: \Omega \rightarrow \mathbb{R}^{n-m} : q \rightarrow w(q)$$

with the property that w(0) = 0. The application of this control law to the kinematic model (3) yields closed loop dynamics of the form:

$$\dot{q} = S(q)w(q) \tag{4}$$

which have the origin q = 0 as equilibrium point. Our concern is to find feedback controls w(q) that make this equilibrium point stable. Several definitions of the stability of equilibrium points are however in order here.

Definitions.

The equilibrium point q = 0 is Lagrange stable if, for any initial condition $q(0) = q_0$, there exist a bound $b(q_0)$ such that $\|q(t)\| \leq b(q_0)$ for all t.

The equilibrium point q = 0 is asymptotically stable (in the sense of Lyapunov) if there exists a positive constant ε such that if $\| q(0) \| \leq \varepsilon$, then q(t) is bounded and converges to zero as time tends to infinity.

The equilibrium state q = 0 is (globally) marginally stable if it is Lagrange stable but not asymptotically stable.

●

It follows from the controllability of the system (Lemma 1) that there exist control laws which ensure the convergence of q(t) to zero. However the controllability does not imply the existence of a *smooth feedback* control law which can make the origin asymptotically stable and which can be synthetized as a smooth function of the state q only. In fact, it is easily shown that such smooth feedback stabilizing controls do *not* exist for nonholonomic systems.

Lemma 2. The equilibrium point q = 0 of the closed loop system (4) cannot be made asymptotically stable by a smooth state feedback w(q).

Proof. From the smoothness of A(q) and the independence of the constraints, it results that there exists a neighbourhood of the origin in $\mathbb{R}^n$, say $\mathcal{U}_0$, such that a given set of m rows of A(q) are independent on $\mathcal{U}_0$. Without loss of generality, we assume that the first m rows of A(q) are independent on $\mathcal{U}_0$, and we partition A(q) as follows:

$$A(q) = \begin{pmatrix} A_1(q) \\ A_2(q) \end{pmatrix}$$

where $A_1(q)$ is a square matrix, non singular on $\mathcal{U}_0$.

Define a neighbourhood $\mathcal{U}_1$, in $\mathbb{R}^{n-m}$, containing the origin, and $\mathcal{U}$ as the cartesian product of $\mathcal{U}_0$ by $\mathcal{U}_1$. Consider the following mapping, inspired by Eq.(4):

$$(q,w) \rightarrow g(q,w) = S(q)w$$

and denote $\mathcal{V}$, the image of $\mathcal{U}$ by this mapping g.

Then, for any σ belonging to $\mathcal{V}$, there exists q such that σ belongs to Im(S(q)) and therefore that

$$A_1^T(q)\sigma_1 + A_2^T(q)\sigma_2 = 0$$

where σ is partitioned in a m-subvector σ_1 and a (n-m)-subvector σ_2. This implies that any σ, with σ_1 not equal to zero and σ_2 equal to zero, does not belong to $\mathcal{V}$, and therefore that $\mathcal{V}$, the image of the open set $\mathcal{U}$, is not an open neighbourhood of the origin. The result then follows from a necessary condition for the existence of smooth stabilizing feedback(see Brockett [8]). ●

Remarks:

-It must be noted that this proof is not based on the nonholonomy of the constraints and Lemma 2 holds therefore also for holonomic systems.

-Stabilization of non holonomic systems can however be achieved by open loop control, by non smooth state feedback control (see an example in Bloch and McClamroch[12]),or by using smooth state-feedback control depending explicitly on time, i.e. of the form w(q,t). Samson ([13]) proposes such a control ensuring the stability of the closed-loop, in the case of a mobile wheeled robot.

-However, as shown in the next theorem, there exists a smooth pure state-feedback control which can globally marginally stabilize the closed loop at the origin.

Theorem 1. With the smooth feedback control law:

$$w(q) = -S^T(q)q$$

the equilibrium point q = 0 of the closed loop system (4) is globally marginally stable. Precisely :

a) the state q(t) is bounded as follows for all t : $\| q(t) \| \leq \| q(0) \|$

b) the state q(t) converges to the invariant set U:

$$U \equiv \{ q \mid S^T(q)q = 0 \}$$

Proof. Straightforward by considering the Lyapunov function candidate $V(q) = q^Tq$ whose time derivative along the closed loop trajectories is:

$$\dot{V} = -2\, q^T S(q) S^T(q) q$$

●

Comment. We notice that:

$$\text{rank}\left[\frac{\partial}{\partial q}\{S^T(q)q\}\right]_{q=0} = n - m$$

This implies that, at least locally around the origin, the invariant set defined in the statement of Theorem 1 is an m - dimensional manifold.

4. THE DYNAMICAL STATE-SPACE MODEL / CONTROLLABILITY AND FEEDBACK STABILIZATION.

In Section 3, the kinematic state-space model has been advocated to analyse the controllability of nonholonomic systems. It is however worth noting that this model does not provide a full description of the dynamics of mechanical systems. The variables w considered as inputs in this model are actually internal states which are dynamically related to the physical inputs that is to the generalized forces and torques applied to the system by the actuators. Our purpose in this section is to examine the controllability properties of the dynamical state-space model of nonholonomic systems and to discuss its state feedback stabilization.

Using the Lagrange formalism, the dynamics of a mechanical system are described by the following differential equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = A(q)\lambda + B(q)u \tag{5}$$

with the following notations and definitions:

a) $L(q, \dot{q}) = T(q, \dot{q}) - W(q)$ is the Lagrangian of the system with $T(q, \dot{q})$ the kinetic energy and $W(q)$ the potential energy.

b) $B(q)u$ is the set of generalized forces applied to the system with $B(q)$ a (n x p) kinematic matrix and u the p-vector of external forces and torques applied to the system by the actuators.

c) $A(q)$ is the matrix associated to the constraints (see Section 2); λ is the m-vector of Lagrange multipliers.

The n-dimensional vector function $A(q)\lambda$ is the vector of the generalized forces acting on the system in order to satisfy the constraints. These forces are said "ideal" which means that their potential power is zero for any potential velocity field compatible with the constraints.

The kinetic energy $T(q, \dot{q})$ is defined as:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

where $M(q)$ is the (n x n) definite positive symmetric inertia matrix. We define also the matrix $C(q,\dot{q})$ and the vector $g(q)$ as follows:

$$C(q, \dot{q}) \equiv \frac{dM(q)}{dt} - \frac{1}{2}\frac{\partial}{\partial q}[\dot{q}^T M(q)]$$

$$g(q) \equiv \frac{\partial W(q)}{\partial q}$$

With these definitions, the model (5) is rewritten as follows:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = A^T(q)\lambda + B(q)u \tag{6}$$

This equation, together with the constraints (1) written in matrix form as:

$$A^T(q)\dot{q} = 0 \qquad (7)$$

provide a full description of the dynamics of the nonholonomic system.

We note that the following equality is a consequence of the definitions of section 1:

$$S^T(q)A(q) = 0 \qquad \forall q \in \Omega$$

Using this expression, we eliminate the Lagrange multipliers by premultiplying equation (6) by $S^T(q)$ to obtain:

$$S^T(q)[M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)] = S^T(q)B(q)u \qquad (8)$$

Moreover, the constraints (7) imply the existence of a vector time function $\eta(q, \dot{q})$ smooth in q and linear in $\dot{q}$ which satisfies the following equality along the trajectories of the system:

$$\dot{q} = S(q)\eta(q, \dot{q}) \qquad (9)$$

This is precisely the kinematic state-space model introduced in the previous section which appears now as a part of the system dynamics.

By differentiating (9), one obtains:

$$\ddot{q} = S(q)\dot{\eta} + R(q, \dot{q})\eta \qquad (10)$$

with:

$$R(q, \dot{q}) \equiv \frac{dS(q)}{dt} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}[S(q)]\dot{q}_i$$

Substituting (9) and (10) into (6) then leads to the following alternative state space description of the system:

$$\Sigma(q)\dot{\eta} = S^T(q)\{-[M(q)R(q,S(q)\eta)\eta + C(q,S(q)\eta)S(q)\eta + g(q)] + B(q)u\} \qquad (11.a)$$

$$\dot{q} = S(q)\eta \qquad (11.b)$$

where $\Sigma(q) = S^T(q)M(q)S(q)$ is a definite positive symmetric matrix. The state vector $\{\eta, q\}$ of this model, referred to as "the dynamical state-space model" of the system, has dimension (2n - m). It shows clearly that η is an internal state instead of being regarded as a fictitious input function w(t) in the kinematical model.

As a first step towards the analysis of the controllability of this system, we have the following property.

Lemma 3. If $p \geq n - m$ (recall that p is the number of inputs) and if $S^T(q)B(q)$ has full rank for all q in Ω, the dynamical state-space model (11) is partially feedback linearizable with a control law $u(\eta,q)$ chosen such that:

$$S^T(q)B(q)u = \Sigma(q)v + S^T(q)[M(q)R(q,S(q)\eta)\eta + C(q,S(q)\eta)S(q)\eta + g(q)] \tag{12}$$

where v denotes an (n-m) - dimensional external input. Indeed, with such a control law, the closed loop is written:

$$\dot{\eta} = v \tag{13.a}$$

$$\dot{q} = S(q)\eta \tag{13.b}$$

•

Thus it appears that the static state feedback (12) allows to reduce the system (11) to the simple form (13) whose structure only depends on the nonholonomic constraints. Our concern is now to discuss the controllability properties of this model and the design of a second state feedback loop $v(\eta,q)$ to stabilize the system around the origin.

4.1. Controllability.

Due to the presence of a drift vector field in the model, the controllability of the system (13) cannot be analyzed without an explicit knowledge of the matrix S(q). However, we know that a necessary controllability condition is that the strong accessibility rank of the system be equal to the state dimension (2n - m). As a matter of fact, this condition holds for nonholonomic systems whatever the structure of S(q) as is shown in the following theorem.

Theorem 2. The strong accessibility rank of a nonholonomic system evolving in an n-dimensional configuration manifold and subjected to m constraints is (2n - m).

Proof.

Results directly from the fact that, if a system is strongly accessible from an input, then it is also strongly accessible from the derivative of this input.

•

4.2. State feedback control.

In this section, we are concerned with the design of smooth state feedback stabilizing controls for the dynamical state space model (13). When a smooth state feedback control law $v(\eta, q)$, such that $v(0, 0) = 0$, is applied to the system (13), the closed loop dynamics :

$$\dot{\eta} = v(\eta, q) \tag{16.a}$$

$$\dot{q} = S(q)\eta \tag{16.b}$$

have the origin $(\eta, q) = (0, 0)$ as an equilibrium point.

We have properties quite similar to those that have been emphasized for the kinematic state-space model, namely that the equilibrium $(\eta, q) = (0, 0)$ of the closed loop cannot be made asymptotically stable by pure state feedback, but can be marginally stabilized.

Lemma 4. The equilibrium point $(\eta, q) = (0, 0)$ of the closed loop (16) cannot be made asymptotically stable by a smooth state feedback $v(\eta, q)$.

Proof. Similar to that of Lemma 2. •

Theorem 3. With the smooth state feedback control law:

$$v(\eta, q) = -S^T(q)S(q)\eta - D(\eta, q)q - \Lambda[S^T(q)q + \eta] - S^T(q)q \tag{17}$$

where:

$$D(\eta, q) \equiv \frac{d}{dt} S^T(q)$$

the equilibrium point $(\eta, q) = (0, 0)$ of the closed loop system (16)-(17) is Lagrange stable. Precisely:

a) the state $\eta(t)$, $q(t)$ is bounded for all t

b) the state $\eta(t)$, $q(t)$ converges to the invariant set U:

$$U = \{ (\eta, q) \mid \eta = 0 \text{ and } S^T(q)q = 0 \}$$

Proof. We define

$$\tilde{\eta} \equiv -S^T(q)q - \eta$$

The closed loop (16)-(17) is then easily shown to be equivalent to:

$$\dot{\tilde{\eta}} = -D(\eta, q)q - S^T(q)S(q)\eta - v(\eta, q) = -\Lambda\tilde{\eta} + S^T(q)q \tag{18.a}$$

$$\dot{q} = -S(q)S^T(q)q - S(q)\tilde{\eta} \tag{18.b}$$

The theorem follows by considering the following Lyapunov function candidate:

$$V(\tilde{\eta}, q) = \frac{1}{2} [\tilde{\eta}^T\tilde{\eta} + q^Tq]$$

whose time derivative along the solutions of (18) is given by:

$$\dot{V} = -\frac{1}{2} \tilde{\eta}^T(\Lambda + \Lambda^T)\tilde{\eta} - q^TS(q)S^T(q)q \leq 0$$

●

5. CONCLUSIONS.

Our main contribution in this paper has been to show that non holonomic systems: (i) are strongly accessible whatever the structure of the constraints; (ii) cannot be asymptotically stabilized by a smooth pure state feedback; (iii) can nevertheless be globally marginally stabilized by a smooth state feedback. Furthermore the design of these stabilizing controls has been explicited.

An application of the foregoing theory to mobile wheeled robots can be found in reference [11]. A brief sketch of this application is given in Appendix as a matter of illustration.

6. ACKNOWLEDGEMENTS.

We thank M. Fliess and C. Samson for their interesting suggestions and comments concerning the dynamics of non holonomic systems.

The results presented in this paper have been obtained within the framework of the Belgian Program on Concerted Research Actions and on Interuniversity Attraction Poles initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility rests with its authors.

7. REFERENCES.

[1] P. Appell , "Traité de mécanique rationnelle",Gauthier-Villars, Paris, 1903.

[2] G. Hamel, "Die Lagrange-Euler'schen Gleichungen der Mechanik", Teubner, Leipzig, 1903.

[3] J.P. Laumond, "Feasible trajectories for mobile robots with kinematic and environment constraints", Int. Conf. on Intelligent Autonomous Systems, Amsterdam, 1986, pp .346,354.

[4] J. Barraquand, J.C. Latombe, "On non holonomic mobile robots and optimal manoeuvring", Revue d'intelligence Artificielle, Vol 3-2, 1989, Ed. Hermes, pp. 77,103.

[5] C. Samson, K. Ait-Abderrahim, "Mobile robot control. Part 1: Feedback control of non holonomic wheeled cart in Cartesian space", INRIA Report, N°1288, 1990

[6] A. Isidori, "Nonlinear control systems", Springer Verlag, 1989.

[7] H. Nijmeijer and A.J. van der Schaft, "Nonlinear Dynamical Control Systems", Springer Verlag, 1990.

[8] R.W. Brockett, "Asymptotic stability and feedback stabilization", in Differential Geometric Control Theory (eds. Brockett, Millmann, Sussmann), Birkhauser, Boston, pp.181-191, 1983.

[9] Bloch and Mc Clamroch, "Control of mechanical systems with classical non holonomic constraints", Proc. 28th IEEE Conf.on Decision and Control, Tampa, 1989, pp.201-205.

[10] H.J. Sussmann and V. Jurdjevic, "Controllability of nonlinear systems", J. Diff. Eqns., 12, pp.95-116, 1972.

[11] B. d'Andrea, G.Bastin and G.Campion, "Modelling and Feedback Control of Nonholonomic Systems: The case of mobile robots", to be presented at the IEEE Conf. on Robotics and Automation, Sacramento, 1991.

[12] A. M. Bloch and N. H. McClamroch, "Controllability and stabilizability properties of a nonholonomic control system", Proc. 29th Conference on Decision and Control, Hawaii, pp.1312-1314, 1990.

[13] C. Samson,"Velocity and torque feedback control of a nonholonomic cart" Int. Workshop in Adaptive and Nonlinear Control: Issues in Robotics, Grenoble, 1990.

APPENDIX: Simplified model of a mobile wheeled robot.

We consider a mobile robot moving on an horizontal plane, constituted by a rigid trolley equipped with non deformable wheels. During the motion, the plane of each wheel remains vertical and the wheel rotates around its (horizontal) axis. The orientation of 2 wheels with respect to the trolley is fixed, while the orientation of the third wheel is varying (see Figure 1). The contact between the wheels and the ground satisfied the *pure rolling* and *non slipping* conditions. The motion of the robot is achieved by 2 motors which provide torques acting on the rotation of the 2 wheels whose orientation is fixed.

In order to characterize the position of the trolley , we define an inertial reference frame in the plane of motion $\{0, I_1, I_2\}$, a reference point Q on the trolley and a basis $\{x_1, x_2\}$ attached to the trolley. The position of the trolley in the plane is therefore characterized by 3 variables:

- x, y : the coordinates of the reference point Q in the inertial frame,

-θ : the orientation of the basis $\{x_1, x_2\}$ with respect to the inertial frame.

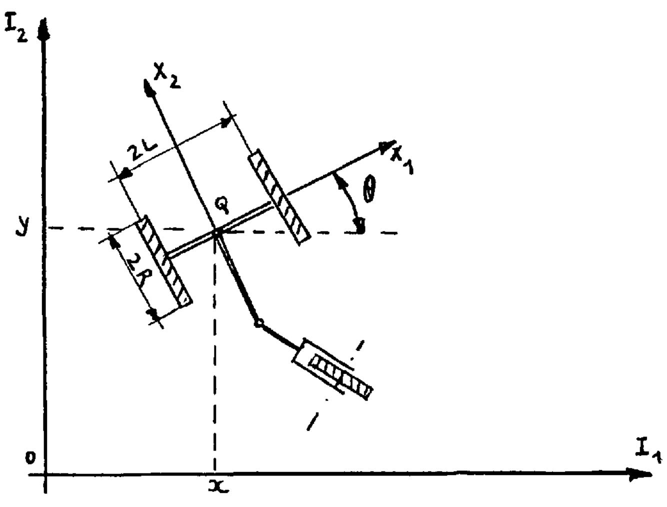

Figure 1: Mobile robot configuration

The configuration of the robot is described by 7 variables : (x, y, θ) for the position of the trolley, 3 angles characterizing the rotations of the 3 wheels, and 1 angle describing the orientation of the mobile wheel. A complete description can be found in [11]. In the present simplified illustrative analysis we restrict ourself in describing the motion of the robot in the plane and we define therefore the generalized coordinates vector q as:

$$q = (x \;\; y \;\; \theta)^T$$

There is one constraint involving the time dervative of q only , namely the non slipping condition of the axis of the 2 front wheels. This constraint is written as:

$$\dot{x}\cos\theta + \dot{y}\sin\theta = 0$$

The matix A(q) is therefore defined as follows:

$$A(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

A particular choice for S(q) is the following:

$$S(q) = (s_1(q) \;\; s_2(q)) = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix}$$

The Lie brackett of the vector fields associated with the colums of S(q) is computed as:

$$[s_1(q)\,,\, s_2(q)] = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

Since this new vector field does not belong to the distribution represented by S(q), we conclude that the system is nonholonomic.

A1. Kinematic model.

According to Eq.(4) the kinematic model is written as follows:

$$\dot{x} = -w_1 \sin\theta$$

$$\dot{y} = w_1 \cos\theta$$

$$\dot{\theta} = w_2$$

The 2 inputs w_1 and w_2 have a physical interpretation: they are respectively the velocity of the robot in the x_2 direction and its angular velocity.

The state feedback control of Theorem 1 is given by:

$$w_1 = x \sin\theta - y \cos\theta$$

$$w_2 = -\theta$$

and the invariant set U is described by:

$$-x \sin\theta + y \cos\theta = 0$$

$$\theta = 0$$

or, equivalently by:

$$y = \theta = 0$$

A2. Dynamical model

Neglecting the masses and inertias of the wheels, the kinetic energy reduces to:

$$T(q,\dot{q}) = \frac{1}{2} (\dot{x}\ \dot{y}\ \dot{\theta}) \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

where m is the mass of the robot, and I_0 is its inertia moment around the vertical axis at point Q.

We consider now as inputs $(u_1, u_2)$the torques provided by the 2 motors. The corresponding generalized forces are given by :

$$B(q)\, u = \frac{1}{R} \begin{pmatrix} -\sin\theta & -\sin\theta \\ \cos\theta & \cos\theta \\ L & -L \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where R is the radius of the wheels and 2L the length of the axis of the front wheels.

Equation (6) takes the following form:

$$m\ddot{x} = \lambda \cos\theta - \frac{1}{R}(u_1 + u_2)\sin\theta$$

$$m\ddot{y} = \lambda \sin\theta + \frac{1}{R}(u_1 + u_2)\cos\theta$$

$$I_0\ddot{\theta} = \frac{L}{R}(u_1 - u_2)$$

According to Eq.(9) we define η by:

$$\eta_1 = -\dot{x}\sin\theta + \dot{y}\cos\theta$$

$$\eta_2 = \dot{\theta}$$

After elimination of the Lagrange multiplier λ, we obtain the following dynamical model of the robot:

$$m\dot{\eta}_1 = \frac{1}{R}(u_1 + u_2)$$

$$I_0\dot{\eta}_2 = \frac{L}{R}(u_1 - u_2)$$

$$\dot{x} = -\eta_1 \sin\theta$$

$$\dot{y} = \eta_1 \cos\theta$$

$$\dot{\theta} = \eta_2$$

The following static state feedback allows to reduce these equations to the form of Eq.(13):

$$u_1 = \frac{R}{2}\left(m v_1 + \frac{I_0}{L} v_2\right)$$

$$u_2 = \frac{R}{2}\left(m v_1 - \frac{I_0}{L} v_2\right)$$

The stabilizing state feedback control (17) takes the following form:

$$v_1 = -(1 + k_1)(\eta_1 + x\sin\theta + y\cos\theta) + \eta_2(x\cos\theta + y\sin\theta)$$

$$v_2 = -(1 + k_2)(\theta + \eta_2)$$

where the gains k_1 and k_2 are non negative design parameters.

This choice of v, combined with the first state feedback, ensures the convergence of the closed-loop to the invariant set characterized by $\eta = 0$, $y = 0$ and $\theta = 0$.