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FEEDBACK STABILISATION WITH POSITIVE CONTROL OF A CLASS OF DISSIPATIVE MASS-BALANCE SYSTEMS

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Abstract. In many process control applications, the system under consideration is *positive*. This means that both the state variables and the control input are physically constrained to remain non-negative along the system trajectories. For such systems, the design of state feedback controllers makes sense only if the control function is guaranteed to provide a non-negative value at each time instant. The purpose of this paper is to present a positive control law for the feedback stabilisation of a class of positive mass-balance systems which are dissipative but can nevertheless be globally unstable. The approach is illustrated with an application to the control of an industrial grinding circuit.

Keywords : positive control systems, stabilisation, dissipative, mass balance.

1. INTRODUCTION

In many practical applications of control engineering, the dynamical system under consideration is *positive*. This means that both the state variables and the control input are physically constrained to remain non-negative along the system trajectories as stated in the following definition :

Definition. Positive System. A control system $\dot{x} = f(x,u)$ $x \in I\!R^n$, $u \in I\!R$ is positive if

$$\left.\begin{array}{l} x(0) \in I\!R^n_+ \\ u(t) \in I\!R_+ \;\; \forall t \geq 0 \end{array}\right\} \Rightarrow x(t) \in I\!R^n_+ \;\; \forall t \geq 0.$$

(**Notation.** The set of non-negative real numbers is denoted as usual $I\!R_+ = \{a \in I\!R, a \geq 0\}$. For any integer n, the set $I\!R^n_+$ is called the "non-negative orthant". Similarly the set of positive real numbers is denoted $I\!P = \{a \in I\!R, a > 0\}$ and $I\!P^n$ is called the "positive orthant".)
For such systems, it is an evidence that the design of state feedback controllers makes sense only if the control function is guaranteed to provide a non-negative value at each time instant.

The purpose of the present paper is to design a positive control law for the feedback stabilisation of a class of positive mass-balance systems which are described in Section 2. These systems are dissipative but can nevertheless be globally unstable. In Section 3, a positive control law is proposed in order to achieve global output stabilization with state boundedness in the positive orthant. The controlled output has a clear physical meaning : it is the total mass contained in the system. The approach is illustrated with an application to a compartmental model of an industrial grinding circuit in Section 4. Some final comments are given in Section 5.

2. POSITIVE MASS BALANCE SYSTEMS

We consider a class of single-input mass balance dynamical systems described by a state equation of the form :

$$\dot{x} = f(x) - Ax + bu \tag{1}$$

with state $x \in I\!R^n_+$ and control input $u \in I\!R_+$, under the following conditions:

C1. The function

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T : \mathbb{R}^n_+ \to \mathbb{R}^n$$

is continuous

C2. $f(0) = 0$

C3. $f_i(x) \geq 0 \;\; \forall x \in \mathbb{R}^n_+$ with $x_i = 0$

C4. $\sum_{i=1}^n f_i(x) = 0 \;\; \forall x \in \mathbb{R}^n_+$

C5. The matrix A is diagonal

$$A = \text{diag}(a_1, a_2, \ldots, a_n)$$

with $a_i \geq 0 \;\; \forall i$ and $a_i > 0$ for at least one i.

C6. The vector $b = (b_1, b_2, \ldots, b_n)^T$ has non negative entries and at least one of them is positive : $b_i \geq 0 \;\; \forall i$ and $b_i > 0$ for at least one i.

C7. The unforced system $\dot{x} = f(x) - Ax$ with output $y = \sum_{i=1}^n a_i x_i$ is zero state detectable in the sense of [5] (Definition 2.27).

System (1) with conditions (C1 - C7) is representative of a wide class of mass balance systems of interest in industrial and environmental applications. Typical examples are compartmental systems (see e.g. [2]), chemical or biological stirred tank reactors (see e.g. [1]), or Lotka-Volterra ecological systems (see e.g. [3]).

In these systems :

1. The state variables x_i represent the amount of mass of various species involved in the system. The state vector x is often called the *composition* because it represents the distribution of mass among the various species.

2. The functions $f_i(x)$ represent various mass transformation effects like :

   - exchange of material between compartments in compartmental systems
   - kinetic transformations in chemical and biological reactors
   - interactions (predation, competition for food, ...) between biological species in ecological systems.

3. The terms $a_i x_i$ represent an outflow of material leaving the system (withdrawal, excretion, mortality of living organisms, etc ...)

4. The terms $b_i u$ represent an inflow of material injected into the system from the outside (like feeding of reactants or nutrients for instance).

The total mass contained in the system is

$$M(x) = \sum_{i=1}^n x_i$$

In all cases, Condition C4 expresses that the system is *mass conservative* in the sense that the involved transformations preserve the mass balance inside the system. This is easily seen if we consider the special case of system (1) without inflows ($u = 0$) and without outflows ($a_i = 0, \forall i$). Then the system reduces to $\dot{x} = f(x)$ and $dM(x)/dt = 0$ which shows that the total mass is indeed conserved.

Under conditions C1-C7, the mass balance system (1) has the following properties.

Properties

1. The system is *positive*. Indeed, if $x_i = 0$, then $\dot{x}_i = f_i(x) + b_i u \geq 0$ from conditions C3 and C6.

2. If $u = 0$ (no inflow), the unforced system $\dot{x} = f(x) - Ax$ is *dissipative* in the sense that the total mass $M(x)$ decreases along the system trajectories, because from Condition C4 we have $dM(x)/dt = -\sum_{i=1}^n a_i x_i \leq 0$.

3. If $u = 0$ (no inflow), the origin $x = 0$ is a globally asymptotically stable equilibrium of the unforced system $\dot{x} = f(x) - Ax$. This follows readily from Condition C7 and Lasalle's Theorem.

3. CONTROL DESIGN FOR GLOBAL STABILISATION

Although the system is dissipative when the control input u is zero (no inflow), it can nevertheless be *globally unstable* when there is a non zero inflow $u(t) > 0$ which is the normal mode of operation in practical applications. The symptom of this instability is an unbounded accumulation of mass inside the system. An example will be given in the application section of the paper. This obviously makes the problem of feedback stabilisation of mass balance systems in the positive orthant relevant and sensible. One way of approaching the problem is to consider that the control objective is to globally stabilize the total mass $M(x)$ at a given positive set point $M^* > 0$ in order to prevent the unbounded mass accumulation. This control objective may be achieved with the following *positive* control law :

$$u(x) = \max(0, \tilde{u}(x)) \qquad (2)$$

$$\tilde{u}(x) = \left(\sum_{i=1}^n b_i\right)^{-1} \left(\sum_{i=1}^n a_i x_i + \lambda(M^* - M(x))\right) \qquad (3)$$

where λ is an arbitrary positive design parameter. The set $\Omega = \{x : M(x) = M^* \text{ and } x \in I\!R^n_+\}$ (which is a portion of hyperplane) is called an "iso-mass", because it is the set of all compositions x corresponding to the same total mass M^*.

The control law (2) is a saturated input-output feedback linearisation with the total mass $y = M(x)$ as regulated output. The stabilizing properties of this control law are given in the following theorem.

Theorem. For the closed loop system (1)-(2) with arbitrary initial conditions in the non-negative orthant $x(0) \in I\!R^n_+$:

(i) the iso-mass Ω is positively invariant

(ii) the state $x(t)$ is bounded for all $t > 0$ and converges to the iso-mass Ω.

Proof. Along the closed loop trajectories, we have :

$$\frac{dM(x)}{dt} = -\sum_{i=1}^{n} a_i x_i + \left(\sum_{i=1}^{n} b_i\right) u(x)$$

(i) if $x \in \Omega$, then $M(x) = M^*$ and

$$u(x) = \tilde{u}(x) = \left(\sum_{i=1}^{n} b_i\right)^{-1} \left(\sum_{i=1}^{n} a_i x_i\right)$$

hence $\dot{M}(x) = 0$ which proves that Ω is positively invariant.

(ii) if $x \neq \Omega$, then $u(x) = \begin{cases} 0 & \text{if } \tilde{u}(x) < 0 \\ \tilde{u}(x) & \text{if } \tilde{u}(x) \geq 0 \end{cases}$

Suppose that $\tilde{u}(x) < 0$ and $u(x) = 0$, then necessarily $M(x) > M^*$.

Consider the Lyapunov function candidate $V = \frac{1}{2}(M^* - M(x))^2$. We have :

$$\begin{aligned} \dot{V} &= -(M^* - M(x))\frac{dM(x)}{dt} \\ &= (M^* - M(x))\left(\sum_{i=1}^{n} a_i x_i\right) \leq 0 \end{aligned}$$

Suppose that $u(x) = \tilde{u}(x) \geq 0$, then

$$\dot{V} = -\lambda(M^* - M(x))^2 \leq 0$$

If $\dot{V} = 0$ then either $x \in \Omega$ which is a positively invariant set of the closed loop (see above)

$$\text{or } x \in \{x : \sum_{i=1}^{n} a_i x_i = 0 \text{ and } M(x) > M^*\}$$

which does not contain any invariant set from condition C7. The result then follows from Lasalle's theorem. ■

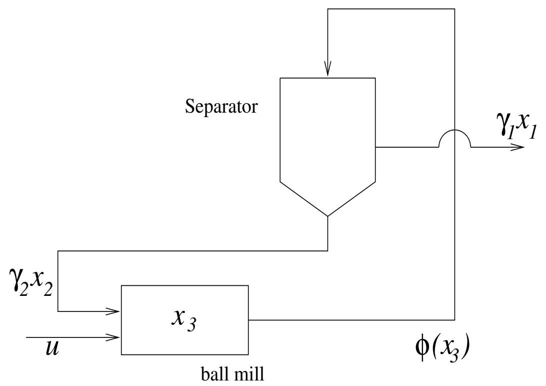

Figure 1: An industrial ball mill

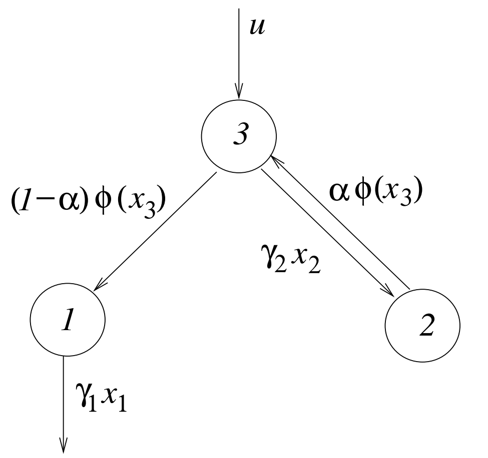

Figure 2: Compartmental structure of the system

4. APPLICATION TO A COMPARTMENTAL SYSTEM

A schematic lay-out of an industrial grinding circuit used in cement industries is depicted in Fig.1.

It is made up of the interconnection of a ball mill and a separator as shown in the figure. The ball mill is fed with raw material. After grinding, the milled material is introduced in a separator where the finished product is separated from the oversize particles which are recycled to the ball mill. A simple dynamical model has been proposed (see [4]) for this system under the form of a compartmental system with three compartments as represented in Fig.2.

The corresponding state space model is as follows :

$$\begin{aligned} \dot{x}_1 &= -\gamma_1 x_1 + (1-\alpha)\phi(x_3) \\ \dot{x}_2 &= -\gamma_2 x_2 + \alpha\phi(x_3) \\ \dot{x}_3 &= \gamma_2 x_2 - \phi(x_3) + u \end{aligned}$$

with the following notations and definitions :

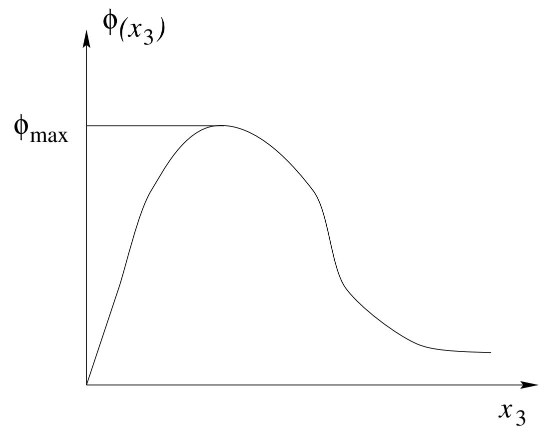

Figure 3: The grinding function

$$
\begin{array}{rcl}
x_1 & = & \text{amount of finished product in} \\
 & & \text{the separator} \\
x_2 & = & \text{amount of oversize particles in} \\
 & & \text{the separator} \\
x_3 & = & \text{amount of material in the ball mill} \\
u & = & \text{feeding rate} \\
\gamma_1 x_1 & = & \text{outflow rate of finished product} \\
\gamma_2 x_2 & = & \text{flowrate of recycled product} \\
\phi(x_3) & = & \text{grinding function}
\end{array}
$$

The parameter α is the separation constant of the separator ($0 < \alpha < 1$). The grinding function $\phi(x_3)$ is non monotonic as represented in Fig. 3. This model is readily seen to be a special case of the general mass-balance model (1) with the following definitions :

$$
\begin{array}{ll}
f_1(x) = (1-\alpha)\phi(x_3) & a_1 = \gamma_1 \\
f_2(x) = -\gamma_2 x_2 + \alpha\phi(x_3) & a_2 = 0 \\
f_3(x) = \gamma_2 x_2 - \phi(x_3) & a_3 = 0
\end{array}
$$

It is easy to check that Conditions C1 to C6 are satisfied. The following sequence of equalities for the unforced system (with $u = 0$) shows that condition C7 is also satisfied :

$$
\begin{array}{rcl}
x_1 = 0 & \Longrightarrow & \dot{x}_1 = 0 \Longrightarrow x_3 = 0 \\
 & \Longrightarrow & \dot{x}_3 = 0 \Longrightarrow x_2 = 0
\end{array}
$$

When the control input is constant $u = \bar{u}$ (constant) > 0, the global instability of the system appears if the state is initialised in the set D defined by the following inequalities (see Fig. 4) :

$$
D \left\{
\begin{array}{c}
(1-\alpha)\phi(x_3) < \gamma_1 x_1 < \bar{u} \\
\alpha\phi(x_3) < \gamma_2 x_2 \\
\partial\phi/\partial x_3 < 0
\end{array}
\right.
$$

Indeed, it can be shown that this set D is positively invariant and if $x(0) \in D$ then $x_1 \to 0 \;\; x_2 \to 0 \;\; x_3 \to \infty$. This means that there is an irreversible accumulation of material in the mill with a decrease of the production to zero. In the jargon of cement industries, this is called *mill plugging*. In practice, the state may be lead to the set D by intermittent disturbances like variations of hardness of the raw material.

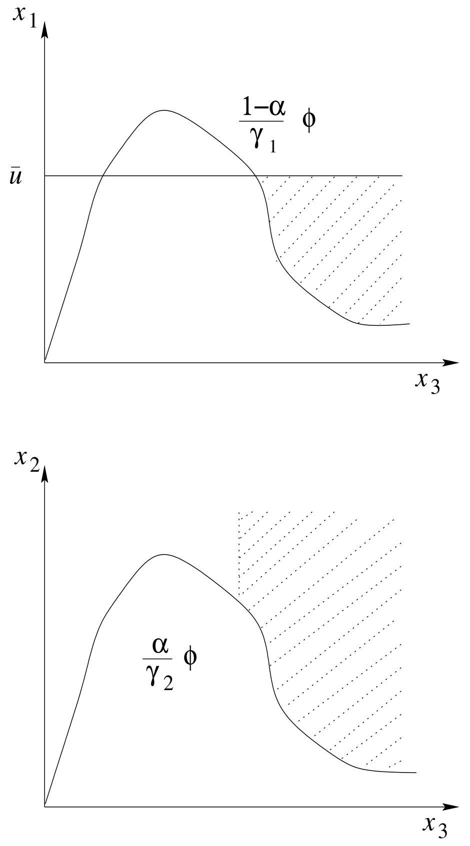

Figure 4: The stability invariant set D

The iso-mass is $M(x) = x_1 + x_2 + x_3$ and we have $\dot{M}(x) = -\gamma_1 x_1 + u$. The control law is written :

$$u(x) = \max(0, \tilde{u}(x))$$

with :

$$
\begin{array}{rcl}
\tilde{u}(x) & = & \gamma_1 x_1 + \lambda(M^* - M(x)) \\
 & = & \lambda M^* + (\gamma_1 - \lambda)x_1 - \lambda x_2 - \lambda x_3
\end{array}
$$

It is interesting to analyse the behaviour of the system in the invariant set Ω which is in fact the behaviour of the zero dynamics (computed for the constant regulated output $y = M(x) = x_1 + x_2 + x_3 = M^*$) :

$$
\left\{
\begin{array}{rl}
\dot{x}_2 = & -\gamma_2 x_2 + \alpha\phi(x_3) \\
\dot{x}_3 = & \gamma_1(M^* - x_2 - x_3) + \gamma_2 x_2 - \phi(x_3)
\end{array}
\right.
$$

The equilibria of the zero-dynamics must satisfy the following relation :

$$\underbrace{\left(\frac{(\gamma_1 - \gamma_2)}{\gamma_2}\alpha + 1\right)\phi(x_3) + \gamma_1 x_3}_{\psi(x_3)} = \gamma_1 M^*$$

The zero-dynamics have a unique equilibrium in Ω if the following inequality is satisfied :

$$\frac{\partial\psi}{\partial x_3} > 0 \Longrightarrow \frac{\partial\phi}{\partial x_3} > -\frac{\gamma_1\gamma_2}{\alpha\gamma_1 + (1-\alpha)\gamma_2}$$

If $\gamma_2 \geq \gamma_1$, this unique equilibrium is easily shown to be globally asymptotically stable in Ω by using the Bendixsson theorem (e.g. [6]).

In this case, it follows that the feedback controller is able not only to prevent the mill from plugging by regulating the total mass at an arbitrary set point, but also to stabilise the system at a unique equilibrium which is globally asymptotically stable in its domain of physical existence (the positive orthant).

5. FINAL COMMENTS

1. The controller (2) proposed in this paper has an interesting robustness property. Indeed it is fully independent from $f(x)$ which represents the internal transformations of the system. This means that the feedback stabilisation is robust against a full modelling uncertainty regarding $f(x)$ provided it satisfies conditions C1 to C4. This is quite important because in many practical applications, $f(x)$ is precisely the most uncertain term of the model.

2. The mass-conservative condition C4 is critical for our result. It is interesting to note that the positive models encountered in the literature are not always mass conservative, even when they describe mass transformations in chemical or biological systems. An archetype is the classical Lotka prey-predator model :

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 + k_1 x_1 \\ \dot{x}_2 &= x_1 x_2 - k_2 x_2 \end{aligned}$$

   If Condition C5 is satisfied, this model cannot verify Condition C4 and is therefore not mass conservative. The underlying reason is that the term $k_1 x_1$ represents a creation of preys "ex-nihilo", under the assumption that the resources for the development of preys are unlimited.

3. Condition C7 which guarantees the dissipativity of the unforced system is also critical for our result. It implies that, in absence of feeding ($u = 0$), there is a natural "wash-out" of the material contained in the system. Besides the fact that it is a common property in many practical applications, it must be emphasized that, without natural dissipativity, there is no hope to globally stabilise the total mass $M(x)$ at an *arbitrary* set point.

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