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Modelling and Control of Non Holonomic Wheeled Mobile Robots

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Abstract: Using Lagrange formalism and differential geometry, a general dynamical model is derived for 3-wheels mobile robots with non holonomic constraints. It is shown that a static state feedback allows to reduce the dynamics of the system to a form for which stabilizing input-output linearizing control is possible.

1 Introduction

In this introduction, we give a brief account of the theory of mechanical systems with non holonomic constraints, which was developed by many authors at the end of the last century (see e.g. [2,5]). A more comprehensive treatment can be found in [1].
A mechanical system whose configuration is completely described by a n-vector

$$q = (q_1, \cdots, q_n)^T$$

of generalized coordinates, can be subjected to m kinematic independent constraints $(m < n)$ of the form :

$$a_j^T(q)\dot{q} = 0 \tag{1}$$

where $a_1, \cdots, a_m$ are smooth linearly independent vector fields on R^n and $\dot{q}$ denotes the time derivative of q as usual. We introduce a $(m \times n)$ matrix $A(q)$ made up of the vector fields $a_j(q)$ as follows :

$$A(q) = (a_1(q), \cdots, a_m(q))^T \tag{2}$$

The independence of the constraints implies that this matrix $A(q)$ has full rank m for all q in R^n. The number of degrees of freedom $(d.o.f)$ is defined as the difference between the number of generalized coordinates and the number of independent constraints :

$$d.o.f. = n - m \tag{3}$$

We define a smooth distribution Δ, associated with the constraints (1) :

$$\Delta(q) = Ker(A(q)) \tag{4}$$

The constraints are holonomic or completely integrable iff Δ is integrable, that is by Frobenius theorem, iff Δ is involutive. Our concern in this paper is to deal with mechanical systems which are non holonomic and whose associated distribution Δ is not involutive. We consider its involutive closure denoted $\bar{\Delta}$.
Let $(n - m^\star)$ denote the dimension of $\bar{\Delta}$, with $m^\star < m$. Since $\bar{\Delta}$ is involutive, it is completely integrable. Hence, the set of independent constraints can be partitioned in two parts : $m^\star$ holonomic constraints and $(n - m^\star)$ non holonomic constraints. This implies that $m^\star$ generalized coordinates can be eliminated from the dynamical description of the system as we shall see in the next sections.

2 Description of a mobile robot

Mobile robots constitute a typical example of non holonomic systems (see e.g. [3,6,4]). We consider here a robot moving on an horizontal plane, constituted by a rigid trolley equipped with non deformable wheels. During the motion, the plane of each wheel remains vertical and the wheels rotate around their (horizontal) axis whose orientation with respect to the trolley can be fixed or varying. The contact between the wheels and the ground satisfies the conditions of **pure rolling and non slipping**. The motion of the robot is achieved by actuators which provide torques acting on the rotation and/or the orientation of the axis of some of the wheels.
We now introduce the definition of the generalized coordinates and some additional notations which will allow us to describe the configuration and the dynamics of the robot.

2.1 Robot position

Consider an inertial reference frame $\{0, I_1, I_2\}$ in the plane of motion. Define a reference point Q on the trolley, and a basis $\{x_1\,,\,x_2\}$ attached to the trolley (see Fig.1). The position of the trolley in the plane is completely specified by the following 3 variables :

- $x\,,\,y$: the coordinates of the reference point Q in the inertial frame,
- θ : the orientation of the basis $\{x_1\,,\,x_2\}$ with respect to the inertial basis.

We define the vector ξ as :

$$\xi = (x\;\;y\;\;\theta)^T \tag{5}$$

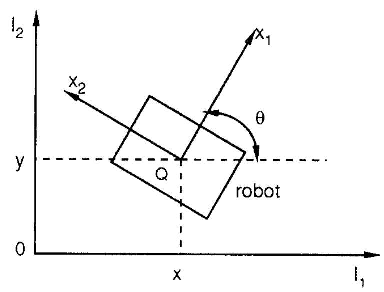

Figure 1: Position of the robot in the plane

2.2 Characterization of a wheel

We now characterize the position of a particular wheel (see Fig. 2). Consider the mobile frame $\{Q, x_1, x_2\}$ attached to the trolley. The center B of the wheel is connected to the trolley by a rigid rod AB (of constant length d), which can rotate around a fixed vertical axis at A. The position of this point A with respect to the trolley is specified by 2 constants : the length l and the angle α. The rotation angle of the rod AB with respect to QA is denoted β. The orientation of the plane of the wheel with respect to AB is given by the constant angle γ. The rotation angle of the wheel around its (horizontal) axis is denoted ϕ. The radius of each wheel is R. The position of the wheel is therefore characterized by a set of five constants : $\{R\,,\,l\,,\,d\,,\,\alpha\,,\,\gamma\}$, and its motion by 2 varying angles : $\beta(t)$ (orientation of the rod AB) and $\phi(t)$ (the rotation angle). Obviously if the rod AB is fixed the angle β becomes a constant.

With this description it becomes easy to compute the velocity of the point of the wheel in contact with the ground. The component of this velocity in the plane of the wheel is :

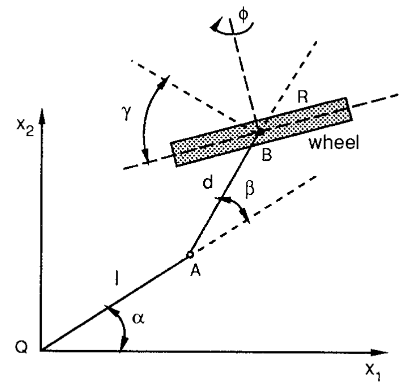

Figure 2: Characterization of a wheel

$$[-sin(\alpha+\beta+\gamma)\;;\;cos(\alpha+\beta+\gamma)\;; \tag{6}$$
$$dcos\gamma + lcos(\beta+\gamma)]R(\theta)\dot{\xi} + dcos\gamma\dot{\beta} + R\dot{\phi}$$

and the component orthogonal to the wheel :

$$[cos(\alpha+\beta+\gamma)\;;\;sin(\alpha+\beta+\gamma)\;; \tag{7}$$
$$dsin\gamma + lsin(\beta+\gamma)]R(\theta)\dot{\xi} + dsin\gamma\dot{\beta}$$

where $R(\theta)$ is a (3×3) orthogonal rotation matrix :

$$R(\theta) = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

These expressions will be used in the next sections to explicit the pure rolling and non slipping conditions.

2.3 Generalized coordinates

We use a lower index notation to identify the quantities relative to each wheel. Throughout the paper, we shall examine in more details the particular case of 3-wheels robots represented in Fig. 3 : the 2 front wheels (index 2 and 3) have a fixed orientation while the orientation of wheel 1 is varying. The theory is easily extended to robots with an arbitrary number of wheels (see [4]). According to our previous description, the geometry of the wheels is completely described by the following set

$$\{R_i\,,\,l_i\,,\,d_i\,,\,\alpha_i\,,\,\gamma_i\,,\,\beta_i\,,\,\phi_i\;;\;i=1\,,\,\cdots\,,\,3\}$$

The reference point Q is the center of the segment B_2B_3 (see Fig. 3). The basis vector x_1 is aligned with B_2B_3. The geometric characteristics are :

- Wheel 1 : $R_1 = R$, $l_1 = L$, $d_1 = d$, $\alpha_1 = \frac{3\pi}{2}$, $\beta_1 = \beta$, $\gamma_1 = \frac{\pi}{2}$.
- Wheel 2 : $R_2 = R$, $l_2 = L$, $d_2 = 0$, $\alpha_2 = 0$, $\beta_2 = 0$, $\gamma_2 = 0$.
- Wheel 3 : $R_3 = R$, $l_3 = L$, $d_3 = 0$, $\alpha_3 = \pi$, $\beta_3 = 0$, $\gamma_3 = 0$.

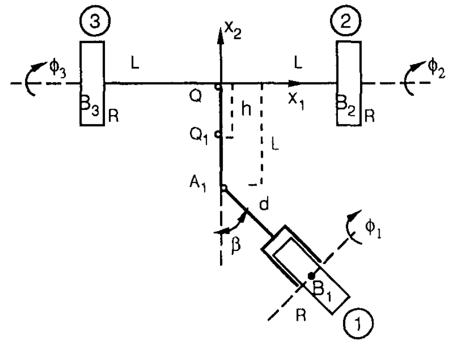

Figure 3: 3-wheels example

The robot motion is then completely described by the following vector of 7 generalized coordinates :

$$q(t) = (x \ y \ \theta \ \beta \ \phi_1 \ \phi_2 \ \phi_3)^T \tag{9}$$

2.4 Kinematical constraints

The **pure rolling** conditions, i.e. the fact that the component of the velocity of the contact point of the wheel with the ground in the plane of the wheel is zero, are deduced from (6) :

$$J_1(\beta)R(\theta)\dot{\xi} + J_2\dot{\phi} = 0 \tag{10}$$

with :

$$\begin{cases} J_1(\beta) = \begin{pmatrix} -sin\beta & cos\beta & -Lsin\beta \\ 0 & 1 & L \\ 0 & -1 & L \end{pmatrix} \\ J_2(i,i) = R \ , \ i = 1, \cdots, 3 \\ J_2(i,j) = 0 \ , \text{ if } i \neq j \end{cases}$$

The **non slipping** conditions, i.e. the fact that the component of the velocity of the contact point, orthogonal to the plane of the wheel is zero, are deduced from (7) :

$$C_1(\beta)R(\theta)\dot{\xi} + C_2\dot{\beta} = 0 \tag{11}$$

where C_1 and C_2 are partitioned in two blocks :

$$C_1(\beta) = \begin{pmatrix} C_{10}(\beta) \\ C_{11} \end{pmatrix} , \ C_2 = \begin{pmatrix} C_{20} \\ 0 \\ 0 \end{pmatrix}$$

with :

$$\begin{cases} C_{10}(\beta) = \begin{pmatrix} cos\beta & sin\beta & d + Lcos\beta \end{pmatrix} \\ C_{20} = d \ , \ C_{11} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \end{cases}$$

We note that these constraints (10)-(11) are in the general form of kinematical constraints (1).

2.5 Degrees of freedom

To obtain the number of degrees of freedom (3), we compute the rank of the following 6x7-matrix associated with the constraints (10) and (11) :

$$A(\beta,\theta) = \begin{pmatrix} J_1(\beta)R(\theta) & 0 & J_2 \\ C_{10}(\beta)R(\theta) & C_{20} & 0 \\ C_{11}R(\theta) & 0 & 0 \end{pmatrix} \tag{12}$$

Due to the block triangular structure of $A(\beta,\theta)$, it is easy to check that $A(\beta,\theta))$ has rank 5. Consequently, the robot has 2 degrees of freedom. Moreover, $rank(C_{11})$ being equal to 1, we deduce that the two last constraints are equivalent. Without loss of generality, we select the constraint corresponding to the second wheel. Defining the following row vector, $C_{11}^{\star} = (1 \ , \ 0 \ , \ 0)$, the corresponding constraint is written :

$$C_{11}^{\star}R(\theta)\dot{\xi} = 0 \tag{13}$$

For any $\dot{\xi}$ satisfying (13), there exist only one value of $\dot{\beta}$ and one value of $\dot{\phi}$ which satisfy the other constraints. These values are expressed as :

$$\dot{\beta} = -C_{20}^{-1}C_{10}(\beta)R(\theta)\dot{\xi} \stackrel{\text{def}}{=} D_1(\beta)R(\theta)\dot{\xi} \tag{14}$$

$$\dot{\phi} = -J_2^{-1}J_1(\beta)R(\theta)\dot{\xi} \stackrel{\text{def}}{=} D_2(\beta)R(\theta)\dot{\xi} \tag{15}$$

2.6 Determination of the non holonomic constraints

We select a basis $\{p_1 \ , \ p_2\}$ of the 2-dimensional space $Ker(C_{11}^{\star})$ as follows :

$$p_1 = (0 \ 1 \ 0)^T \quad ; \quad p_2 = (0 \ 0 \ 1)^T \tag{16}$$

Then, using (13)-(15), the 2-dimensional distribution Δ associated to the constraints is defined by :

$$Span \left\{ \begin{pmatrix} R^T(\theta)p_1 \\ D_1(\beta)p_1 \\ D_2(\beta)p_1 \end{pmatrix} , \begin{pmatrix} R^T(\theta)p_2 \\ D_1(\beta)p_2 \\ D_2(\beta)p_2 \end{pmatrix} \right\} \tag{17}$$

After a few computations we can check that :

$$dim\,\bar{\Delta} = 6 \tag{18}$$

This means that the constraints are not completely holonomic but that there exists one (i.e. $n - m^\star = 1$) function of q, constant along the trajectories satisfying the constraints. This function can be exhibited by summing the constraints of pure rolling of wheels 2 and 3. This sum is written :

$$L\dot{\theta} + R(\dot{\phi}_1 + \dot{\phi}_2) = 0 \tag{19}$$

which implies that $L\theta(t) + R(\phi_1(t) + \phi_2(t))$ is constant along all the trajectories satisfying the constraints.

3 Dynamical equations

In the case of mobile robots the Lagrangian is reduced to the kinetic energy $T(q, \dot{q})$. Using the Lagrangian formalism, the dynamical equations have the following form :

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) - \frac{\partial T}{\partial \xi} &= R^T(\theta)(J_1^T(\beta)\lambda + \\ &\quad C_{10}^T(\beta)\mu + {C_{11}^\star}^T\nu) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} &= C_{20}^T\mu + B_1 u_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} &= J_2^T\lambda + B_2 u_2 \end{aligned} \tag{20}$$

$\lambda = (\lambda_1\,,\,\lambda_2\,,\,\lambda_3)$, μ, ν being the 5 Lagrange multipliers associated with the 5 independent kinematical constraints, Bu is the set of generalized forces applied to the system with u the 2-vector of external forces and torques applied to the system by the actuators. These 7 equations (20) together with the 5 independent kinematical constraints describe the dynamics of the mobile robot.
We now specify the implementation of the actuators. We consider 2 possible cases :

- **Case 1**: The 2 motors provide the torques for the rotation of wheels 2 and 3. In this case B_1 does not exist and B_2 is given by :

$$B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{21}$$

- **Case 2** : The 2 motors are implemented on wheel 1, the first one for its orientation, the second one for its rotation, which gives :

$$B_1 = (1) \text{ and } B_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \tag{22}$$

3.1 Kinetic energy

The kinetic energy of the system is expressed as the following symmetric quadratic form :

$$T = \frac{1}{2}\dot{q}^T \begin{pmatrix} R^T(\theta)M(\beta)R(\theta) & R^T(\theta)V(\beta) & 0 \\ V^T(\beta)R(\theta) & I_\beta & 0 \\ 0 & 0 & I_\phi \end{pmatrix} \dot{q} \tag{23}$$

where $M(\beta)$ is a (3×3) symmetric matrix defined by :

$$\left\{ \begin{aligned} & M_{11}(\beta) = M_{22}(\beta) = M + \sum_{i=1}^{3} m_i\;,\; M_{12} = 0 \\ & M_{33}(\beta) = M(e_1^2 + e_2^2) + I_0 \\ &\qquad + \sum_{i=1}^{3} m_i(d_i^2 + l_i^2 + 2d_i l_i cos\beta_i) \\ & M_{13}(\beta) = -Me_2 - \sum_{i=1}^{3} m_i d_i sin(\alpha_i + \beta_i) \\ &\qquad - \sum_{i=1}^{3} m_i l_i sin\alpha_i \\ & M_{23}(\beta) = -Me_1 - \sum_{i=1}^{3} m_i d_i cos(\alpha_i + \beta_i) \\ &\qquad - \sum_{i=1}^{3} m_i l_i cos\alpha_i \end{aligned} \right. \tag{24}$$

$V(\beta)$ is a 3-vector defined as :

$$V(\beta) = \begin{pmatrix} -m_1 d_1 sin(\alpha_1 + \beta_1) \\ m_1 d_1 cos(\alpha_1 + \beta_1) \\ m_1 d_1^2 + m_1 d_1 l_1 cos(\beta_1) + I_{p1} \end{pmatrix} \tag{25}$$

I_β and I_ϕ have the following form :

$$\left\{ \begin{array}{l} I_\beta = m_1 d_1^2 + I_{p1} \\ I_\phi(i,i) = I_{ri}\;\;,\;\; i = 1, \cdots, 3 \\ I_\phi(i,j) = 0 \text{ if } i \neq j \end{array} \right. \tag{26}$$

In definitions (24)-(26), we have introduced various notations relying on the mass distribution of the robot :

- M : mass of the trolley,
- m_i : mass of wheel i, $i = 1, \cdots, 3$,
- e_1, e_2 : coordinates of the center of mass of the trolley in the frame attached to the trolley $\{Q, I_1, I_2\}$,
- I_0 : inertia moment of the trolley around the vertical axes passing through its center of mass,
- I_{pi} : inertia moment of wheel i, around the vertical axis passing through B_i,

- I_{ri} : inertia moment of wheel i, around its axis of rotation.

With this expression of the kinetic energy, the Lagrange equations (20) is rewritten as :

$$\left\{\begin{array}{lcl} M(\beta)R(\theta)\ddot{\xi} + f_1(\theta,\dot{\theta},\beta,\dot{\beta}) & = & J_1^T(\beta)\lambda + \\ & & C_{10}^T(\beta)\mu + {C_{11}^{\star}}^T\nu \\ V^T(\beta)R(\theta)\ddot{\xi} + f_2(\theta,\dot{\theta},\beta,\dot{\beta}) & = & C_{20}^T\mu + B_1u_1 \\ I_\phi\ddot{\phi} & = & J_2^T\lambda + B_2u_2 \end{array}\right. \tag{27}$$

where f_1 and f_2 are respectively a 3-vector and a 1-vector of functions of θ, $\dot{\theta}$, β, $\dot{\beta}$.

3.2 Elimination of variables

Consider the full rank (3×2) matrix P made up of the vectors p_1 and p_2 making a basis of $KerC_{11}^{\star}$ (see eq. (16)), that is :

$$C_{11}^{\star}P = 0 \tag{28}$$

We now eliminate the 5 Lagrange multipliers between the 7 equations (27). Premultiplying the first equality of (27) by P^T, this elimination leads to the following matrix equation :

$$\begin{array}{l} P^T(M(\beta) + D_1^T(\beta)V^T(\beta))R(\theta)\ddot{\xi} \\ +P^T(V(\beta) + D_1^T(\beta)I_\beta)\ddot{\beta} \\ +P^TD_2^T(\beta)I_\phi)\ddot{\phi} + P^Tf_1(\theta,\dot{\theta},\beta,\dot{\beta}) \\ +P^TD_1^T(\beta)f_2(\theta,\dot{\theta},\beta,\dot{\beta})) = P^TG(\beta)u \text{ with} \end{array} \tag{29}$$

$$G(\beta) = \left(\begin{array}{cc} D_1^T(\beta)B_1 & D_2^T(\beta)B_2 \end{array}\right) \tag{30}$$

The constraint (13) implies that there exists a 2-vector $\eta(t) = (\eta_1(t) \quad \eta_2(t))^T$ such that :

$$\left\{\begin{array}{l} R(\theta)\dot{\xi} = P\eta(t) \text{ i.e.} \\ \eta_1 = -\dot{x}sin\theta + \dot{y}cos\theta \\ \eta_2 = \dot{\theta} \end{array}\right. \tag{31}$$

The constraints (14) and (15) can then be rewritten as follows :

$$\left\{\begin{array}{l} \dot{\beta} = D_1(\beta)P\eta \\ \dot{\phi} = D_2(\beta)P\eta \end{array}\right. \tag{32}$$

Differentiating (31) and (32) with respect to time, replacing $\ddot{\xi}$, $\ddot{\beta}$, $\ddot{\phi}$ in (29) and noticing that :

$$\left\{\begin{array}{l} \dot{R}(\theta)\dot{\xi} = \dot{R}(\theta)R^{-1}(\theta)P\eta = \dot{\theta}EP\eta \text{ with} \\ E = \left(\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}\right. \tag{33}$$

equation (29) is rewritten as follows :

$$P^TM^{\star}(\beta)P\dot{\eta} + P^T\rho(\theta,\dot{\theta},\beta,\dot{\beta},\eta) = P^TG(\beta)u \tag{34}$$

where :

$$\begin{array}{l} M^{\star}(\beta) = M(\beta) + D_1^T(\beta)V^T(\beta) + V(\beta)D_1(\beta) \\ +D_1^T(\beta)I_\beta D_1(\beta) + D_2^T(\beta)I_\phi D_2(\beta) \end{array} \tag{35}$$

and

$$\begin{array}{l} \rho(\theta,\dot{\theta},\beta,\dot{\beta},\eta) = (V(\beta) + D_1^T(\beta)I_\beta)\dfrac{\partial D_1}{\partial\beta}(\beta)\dot{\beta} \\ -\dot{\theta}(M(\beta) + D_1^T(\beta)V^T(\beta))E \\ +D_2^T(\beta)I_\phi\dfrac{\partial D_2}{\partial\beta}(\beta)\dot{\beta} \\ +f_1(\theta,\dot{\theta},\beta,\dot{\beta}) + D_1^T(\beta)f_2(\theta,\dot{\theta},\beta,\dot{\beta}) \end{array} \tag{36}$$

From (31) and (32) we can express $\dot{\theta}$, $\dot{\beta}$ and consequently $\rho(\theta,\dot{\theta},\beta,\dot{\beta},\eta)$ as a function $f^{\star}$ depending only on θ, η, β which gives the following equivalent dynamical state space model of the mobile robot :

$$\left\{\begin{array}{l} P^TM^{\star}(\beta)P\dot{\eta} + P^Tf^{\star}(\theta,\beta,\eta) = P^TG(\beta)u \\ \dot{x} = -\eta_1 sin\theta \\ \dot{y} = \eta_1 cos\theta \\ \dot{\theta} = \eta_2 \\ \dot{\beta} = D_1(\beta)P\eta \\ \dot{\phi} = D_2(\beta)P\eta \end{array}\right. \tag{37}$$

4 Control design

4.1 System reduction

System (37) has a triangular structure : the variables (ξ,η) appear only in the first 5 equations. Since our purpose is to control the trajectory of the robot in the plane, i.e. only the variables ξ, we can restrict the analysis to these first 5 equations. No problem of internal stability can occur from this reduction, $\dot{\beta}$ and $\dot{\phi}$ being uniformly bounded provided η is bounded.

Moreover, it is easy to check that the input matrix $P^TG(\beta)$ has full rank for the two considered configurations, see eq. (21), (22), (30). Consequently, for any (v_1, v_2) there exists one and only one static state feedback $u(\theta,\dot{\theta},\beta)$ such that the system of equations (37) in the variables (ξ,η) reduces to :

$$\left\{\begin{array}{l} \dot{\eta}_1 = v_1 \quad , \quad \dot{\eta}_2 = \ddot{\theta} = v_2 \\ \dot{x} = -\eta_1 sin\theta \quad , \quad \dot{y} = \eta_1 cos\theta \\ \dot{\theta} = \eta_2 \end{array}\right. \tag{38}$$

4.2 Input-ouput feedback linearization

Consider for instance a point Q_1, whose coordinates in the frame attached to the robot are (0, - h) (see Fig. 3). Define as outputs the position of Q_1 in the plane i.e. :

$$z = \left(\begin{array}{c} \bar{x} \\ \bar{y} \end{array}\right) = \left(\begin{array}{c} x + hsin\theta \\ y - hcos\theta \end{array}\right) \tag{39}$$

It follows that :

$$\begin{array}{l} \ddot{z}_1 = -\eta_1\dot{\theta}cos\theta - h\dot{\theta}^2 sin\theta - v_1 sin\theta + v_2 hcos\theta \\ \ddot{z}_2 = -\eta_1\dot{\theta}sin\theta + h\dot{\theta}^2 cos\theta + v_1 cos\theta + v_2 hsin\theta \end{array} \quad (40)$$

Input-output linearization is therefore possible because the matrix of the coefficients of v_1 and v_2 has full rank, for any θ, provided h is not zero.
In order to analyse the internal stability we introduce the following change of coordinates $\zeta = \Phi(\xi, \eta)$:
(ξ denoting now the vector $(\bar{x}\ \bar{y}\ \theta)^T$)

$$\zeta_1 = \bar{x}\ ,\ \zeta_2 = \dot{\bar{x}}\ ,\ \zeta_3 = \bar{y}\ ,\ \zeta_4 = \dot{\bar{y}}\ ,\ \zeta_5 = \theta \quad (41)$$

in order to constitute a diffeomorphism, whose inverse is given by :

$$\begin{array}{l} \xi_1 = \zeta_1\ ,\ \xi_2 = \zeta_3\ ,\ \xi_3 = \zeta_5\ ,\ \eta = P_1^{-1}R_1(\zeta_5)\begin{pmatrix} \zeta_2 \\ \zeta_4 \end{pmatrix} \\ P_1 = \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix}\ ,\ R_1 = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \end{array} \quad (42)$$

In the new coordinates ζ, the system (38) can be rewritten as follows :

$$\left\{ \begin{array}{l} \dot{\zeta}_1 = \zeta_2\ ,\ \dot{\zeta}_2 = \alpha_1(\zeta) + S_1(\zeta)v \\ \dot{\zeta}_3 = \zeta_4\ ,\ \dot{\zeta}_4 = \alpha_2(\zeta) + S_2(\zeta)v \\ \dot{\zeta}_5 = \begin{pmatrix} 0 & 1 \end{pmatrix} P_1^{-1}R_1(\zeta_5)\begin{pmatrix} \zeta_2 \\ \zeta_4 \end{pmatrix} \end{array} \right. \quad (43)$$

where $\begin{pmatrix} S_1(\zeta) \\ S_2(\zeta) \end{pmatrix} = R_1^T(\zeta_5)P_1$ is a non singular (2×2) matrix. Equation (43) shows that provided ζ_2 and ζ_4 (i.e. $\dot{\bar{x}}$ and $\dot{\bar{y}}$) are bounded, $\dot{\zeta}_5$ (i.e. $\dot{\theta}$) remains bounded. This implies the boundedness successively of ξ, η and therefore of $\dot{\beta}$ and $\dot{\phi}$.

4.3 Trajectory tracking

Two different cases are considered.
a) Assume that the control purpose is to track a smooth reference trajectory $(\bar{x}_d(t), \bar{y}_d(t))$ in the plane. We choose (v_1, v_2) such that :

$$\begin{array}{l} -v_1 sin\theta + v_2 hcos\theta = \ddot{\bar{x}}_d + k_{11}(\dot{\bar{x}}_d - \dot{\bar{x}}) \\ +k_{12}(\bar{x}_d - \bar{x}) + \eta_1\dot{\theta}cos\theta + h\dot{\theta}^2 sin\theta \\ v_1 cos\theta + v_2 hsin\theta = \ddot{\bar{y}}_d + k_{21}(\dot{\bar{y}}_d - \dot{\bar{y}}) \\ +k_{22}(\bar{y}_d - \bar{y}) + \eta_1\dot{\theta}sin\theta - h\dot{\theta}^2 cos\theta \end{array} \quad (44)$$

Provided a suitable choice of the constant gains k_{ij}, the control law (44) ensures the exponential stability of the errors $\tilde{x} = \bar{x} - \bar{x}_d$ and $\tilde{y} = \bar{y} - \bar{y}_d$ as well as the internal stability.
b) Assume now that the control purpose is to track a smooth trajectory $(\bar{x}_d(t), \bar{y}_d(t), \theta_d(t))$ specified in the $(\bar{x}, \bar{y}, \theta)$ space, compatible with the constraints. After a simple computation of the tangent linearization of the dynamics of $\tilde{\theta} = \theta - \theta_d$, we can check that the control law (44) ensures stable trajectory tracking provided that the component along x_2 of the velocity of Q_1, η_{1d} is negative, with :

$$\eta_{1d} = -\dot{\bar{x}}_d sin\theta_d + \dot{\bar{y}}_d cos\theta_d - 2h\dot{\theta}_d sin\theta_d cos\theta_d \quad (45)$$

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