Output Feedback Control of Food–Chain Systems

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1 Introduction

In this chapter, we consider the problem of output feedback control of a class of non–linear mass balance models that describe the behavior of certain food–chain systems. These models are of interest, amoung other fields, in environmental engineering.

The control approach we use to solve the stabilization problem builds upon some recent developments on passivity–based stabilization of port–controlled Hamiltonian systems reported in [7], [5]. Since the design procedure is applicable to a broad class of mass–balance systems of similar structure (such as comportamental systems and stirred tank reactors, see [1] and the references therein), we present it in a rather general form. In this technique the original Hamiltonian structure of the system is preserved in closed–loop, and only the energy function and the dissipation are modified via the control. Preservation of the Hamiltonian structure allows stabilization to be understood in terms of energy. These feature makes the method very appealing in applications, since the action of the control has a clear physical interpretation that simplifies its comissioning. This task is particularly difficult in mass–balance systems where the control (and the system state) should be positive.

One further advantage of the method, central for the developments in this paper, is that the restriction of disposing only of output–feedback (as opposed to full–state feedback) can be naturally incorporated into the controller design. In particular, we show here that to obtain an output–feedback control strategy, some of the natural damping of the mass–balance equations should be removed, leaving only the damping of the measurable coordinate, which is necessary to ensure asymptotic stability. To better explain this modification we present first a state–feedback solution for the simplest second order model. In this case we leave untouched the natural damping of the system and apply *verbatim* the method proposed in [7]. A careful observation of the energy–shaping plus damping injection conditions of [7] reveals that with a, rather unusual, injection of positive damping we can easily obtain an output–feedback solution. Furthermore, the new solution is a simple linear controller, while the state–feedback controller is nonlinear, and rather involved. It is interesting to note that, the injection of positive damping allows us to obtain a stabilizing controller for the n–th order model, while the solution without removal of damping cannot be extended beyond the second order case.

Some simulation results are presented to illustrate the properties of the controller, and we conclude the chapter with some open questions and final remarks.

2 Controller Design Procedure

In this section we review the basic material of [7] presented in a form suitable for the problem considered in this chapter. Even though we deal with mass–balances instead of energy–balances, to keep up with the standard notation we will use throughout the word "energy".

We consider, so–called port–controlled Hamiltonian models of the form [6], [11], [10]

$$\Sigma \;:\; \dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u, \tag{6.1}$$

where $x \in \Re^n_+ \subset \Re^n$, $u \in \Re^m_+ \subset \Re^m$, are the mass variables, and the control, respectively. The set $\Re^n_+$ is the n–dimensional positive orthant. The smooth function $H(x) : \Re^n \to \Re$, which typically represents the total stored energy, will denote for our mass–balance systems the *total mass*, and it will be non–negative. The matrices

$$J(x) = -J^\top(x), \;\; R(x) = R^\top(x) \geq 0, \;\; \forall\, x \in \Re^n_+,$$

capture the internal interconnections and the natural damping of the system, respectively, while $g(x)$ defines the interconnection of the system with its environment. We assume measurable the q–dimensional output vector function $y = h(x)$. This output should not be confused with the natural outputs associated to the port–controlled Hamiltonian system Σ defined as $g^\top(x)\frac{\partial H}{\partial x}(x)$ [7].

The control objective is to stabilize, via output–feedback, an equilibrium $\bar{x} \in \Re^n_+$ preserving in closed–loop the Hamiltonian structure. The latter property allows us to provide an *energy* interpretation of the control action.

We will consider only static controllers, but as shown in [7] the procedure can be easily modified to incorporate controller dynamics.

Following the principles of passivity–based control [8], [10], we will achieve the stabilization objective by the standard energy–shaping plus damping injection stages. That is:

1. Assigning to the closed–loop an energy function $H_d(x)$, which should have a strict local minimum at $\bar{x}$. (That is, there exists an open neighbourhood $\mathcal{B}$ of $\bar{x}$ such that $H_d(x) > H_d(\bar{x})$ for all $x \in \mathcal{B}$.) We will define

$$H_d(x) \stackrel{\triangle}{=} H(x) + H_a(x) \tag{6.2}$$

where $H_a(x)$ is a function to be defined.

2. Injecting some additional damping $R_a(x)$ to get

$$R_d(x) \stackrel{\triangle}{=} R(x) + R_a(x) \geq 0, \quad \forall\, x \in \Re^n_+ \tag{6.3}$$

That is, we look for an output–feedback control $u(h(x))$ such that

$$[J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u(h(x)) = [J(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x)$$

holds $\forall\, x \in \Re^n_+$, with $H_d(x)$, $R_d(x)$ defined by (6.2) and (6.3), respectively. In this way, the closed-loop dynamics will be defined as

$$\dot{x} = [J(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x), \tag{6.4}$$

and along the trajectories of (6.4) we will have

$$\frac{d}{dt}H_d = -\left[\frac{\partial H_d}{\partial x}(x)\right]^\top R_d(x)\frac{\partial H_d}{\partial x}(x) \leq 0,\ \forall x \in \Re^n_+ \tag{6.5}$$

Thus, $\bar{x}$ will be a *stable* equilibrium.

For ease of presentation we will assume throughout the following:

Assumption A $[J(x) - R_d(x)]$ is invertible for every $x \in \Re^n_+$.

□□□

It is important to remark that this does not imply that the closed–loop system is fully damped. That is, we do not require $R_d(x) > 0$, $\forall x \in \Re^n_+$. Actually, it is shown in [7] that Assumption A is not needed for the proof of the proposition below.

We have the following basic result.

Proposition 6.1 *[7] Given $J(x), R(x), H(x), g(x)$. Assume we can find and output-feedback control $u(h(x))$ and a matrix $R_a(x)$ such that $R(x) + R_a(x) \geq 0$, Assumption A hold, and the vector function $K(x)$, defined as,*

$$K(x) \triangleq [J(x) - (R(x) + R_a(x))]^{-1}[R_a(x)\frac{\partial H}{\partial x}(x) + g(x)u(h(x))] \tag{6.6}$$

satisfies

- (Integrability) *$K(x)$ is the gradient of a scalar function. That is,*

$$\frac{\partial K}{\partial x}(x) = \left[\frac{\partial K}{\partial x}(x)\right]^{\top} \tag{6.7}$$

- (Equilibrium assignment) *$K(x)$, at $\bar{x}$, verifies*

$$K(\bar{x}) = -\frac{\partial H}{\partial x}(\bar{x}) \tag{6.8}$$

- (Lyapunov stability) *The Jacobian of $K(x)$, at $\bar{x}$, satisfies the bound*

$$\frac{\partial K}{\partial x}(\bar{x}) > -\frac{\partial^2 H}{\partial x^2}(\bar{x}) \tag{6.9}$$

Then, $\bar{x}$ will be a locally stable *equilibrium of the closed–loop. It will be* asymptotically *stable if, furthermore, the largest invariant set under the closed–loop dynamics contained in*

$$\left\{ x \in \Re^n_+ \cap \mathcal{B} \mid \left[\frac{\partial H_d}{\partial x}(x)\right]^T R_d(x)\frac{\partial H_d}{\partial x}(x) = 0 \right\} \tag{6.10}$$

equals $\{\bar{x}\}$, where $H_d(x)$ is given by (6.2). The latter condition will be automatically satisfied if we can achieve full damping, that is, if $R_d(x) > 0$ for every $x \in \Re^n_+$.

Proof

First, notice that, using (6.2), (6.3) and Assumption A, the identity (6.4) may be equivalently written as

$$\frac{\partial H_a}{\partial x}(x) = [J(x) - R_d(x)]^{-1}[R_a(x)\frac{\partial H}{\partial x}(x) + g(x)u(h(x))] \tag{6.11}$$

For every given $u(h(x)), R_a(x)$, this is a linear PDE. A necessary and sufficient condition for the solvability of this PDE (on every contractible neighbourhood of $\Re^n_+$) is that the gradient of the right hand side of (6.11) is a symmetric matrix. From (6.3), (6.6) and (6.11) we see that

$$K(x) = \frac{\partial H_a}{\partial x}(x) \tag{6.12}$$

Henceforth, the matrix mentioned above will be symmetric iff the integrability condition (6.7) of the proposition is satisfied.

The stability proof is concluded invoking standard Lyapunov stability arguments [4]. Namely, from (6.5), we conclude that, under the standing assumptions, $H_d(x)$ qualifies as a Lyapunov function. Asymptotic stability follows from a direct application of La Salle's invariance principle and (6.10).

□□□

Remark 6.1 *Notice that the construction above* does not require *the explicit derivation of the Lyapunov function* $H_d(x)$*. This can be obtained, though, as a by-product integrating* $K(x) = \frac{\partial H_a}{\partial x}(x)$.

Remark 6.2 *Port-controlled Hamiltonian models (6.1) encompass a very large class of physical nonlinear systems, strictly containing the class of Euler–Lagrange models considered, for instance, in [8]. They result from the network modeling of energy–conserving lumped–parameter physical systems with independent storage elements, and have been advocated in a series of recent papers [6], [11] as an alternative to more classical Euler–Lagrange (or standard Hamiltonian) models.*

3 State–Feedback Control of a Simple Prey–Predator System

As pointed out in the introduction, to motivate our output–fedback control (which is given in the next section) we present first a state–feedback stabilizer for a simple second order food–chain system. The controller is obtained from a *verbatim* application of the method described above. This is a systematic technique that can be efficiently combined with symbolic computation. See, for instance, the simple Maple code given in Appendix A.

System Model

We consider the normalized second order prey–predator system (see e.g. [3])

$$\begin{aligned} \dot{x}_1 &= f(x) - x_1 \\ \dot{x}_2 &= -f(x) - x_2 + u \end{aligned} \tag{6.13}$$

The state variables x_1, x_2 represent the amount of mass of the two species (preys and predators) involved in the system. The function $f(x)$ describes the predation mechanism, we consider here the classical Lotka–Volterra mechanism $f(x) = x_1 x_2$. The terms $-x_1, -x_2$ in (6.13) represent the natural mortality of the species, while the control action u is a feeding inflow rate of preys. For the output feedback case, we will consider that the variable available for measurement is the last one in the chain, in this case, x_2.

The evolution of the system is clearly restricted to the positive orthant with $u \geq 0$. That is,

$$x_i(0) \geq 0, \text{ and } u(t) \geq 0, \ \forall t \geq 0 \ \Rightarrow \ x_i(t) \geq 0, \ \forall t \geq 0$$

It is possible to show that any equilibrium of the open–loop system with a lit *constant* input $\bar{u} \geq 0$ is globally asymptotically stable. The *control objective* is, then, to asymptotically stabilize a *given* non–zero equilibrium $\bar{x} \in \Re^2_+$ with a positive control. The *achievable* equilibria are $\bar{x} = [\bar{x}_1, \bar{x}_2]^\top = [x_1^*, 1]^\top$, with $x_1^* > 0$ the reference for x_1.

If we define the total mass

$$H(x) = x_1 + x_2$$

the system (6.13) may be written in the form (6.1) with

$$J(x) = \begin{bmatrix} 0 & x_1 x_2 \\ -x_1 x_2 & 0 \end{bmatrix}, \ R(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \ g(x) = g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The skew–symmetry of $J(x)$ captures the mass–conservative feature of the system without inflows and outflows.

State–Feedback Stabilization

Since the system is already fully damped, i.e., $R(x) > 0, \ \forall x \neq 0, x \in \Re^n_+$, it seems reasonable (as our first try) to set $R_a(x) = 0$. That is, we will not inject additional damping, but rely instead on the natural damping of the system to ensure the attractivity. In this case, the vector function (6.6) reduces to

$$K(x) = \begin{bmatrix} k_1(x) \\ k_2(x) \end{bmatrix} = \frac{-u(x)}{1 + x_1 x_2} \begin{bmatrix} 1 \\ \frac{1}{x_2} \end{bmatrix}$$

From which we immediately conclude that

$$x_2 k_2(x) = k_1(x) \tag{6.14}$$

The integrability condition (6.7) in this two–dimensional case reduces to

$$\frac{\partial k_1}{\partial x_2}(x) = \frac{\partial k_2}{\partial x_1}(x),$$

which, combined with (6.14), yields the linear PDE

$$\frac{\partial k_1}{\partial x_1}(x) - x_2 \frac{\partial k_1}{\partial x_2}(x) = 0 \tag{6.15}$$

A family of solutions of this PDE is easily obtained as

$$\begin{aligned} k_1(x) &= \Phi(\zeta(x)) \\ \zeta(x) &= x_1 + \log x_2, \end{aligned}$$

for all differentiable functions $\Phi(\cdot)$. From (6.14) we also obtain

$$k_2(x) = \frac{1}{x_2}\Phi(\zeta(x))$$

The equilibrium condition (6.8) imposes

$$\begin{bmatrix} k_1(\bar{x}) \\ k_2(\bar{x}) \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6.16}$$

Hence, $\Phi(\cdot)$ must be such that $\Phi(\zeta(\bar{x})) = -1$, where $\zeta(\bar{x}) = \bar{x}_1 + \log \bar{x}_2 = x_1^*$. It is clear then that we cannot take $\Phi(\zeta) = \zeta$. We propose the function

$$\Phi(\zeta) = c_1 \exp^{c_2\zeta},$$

with c_1, c_2 constants to be defined. (Although this choice of function might seem a bit contrived, we should note that this is the function that results if we directly apply the method of undetermined coefficients to the PDE (6.15). See Appendix A). The equilibrium condition $\Phi(\zeta(\bar{x})) = -1$ fixes the first constant as

$$c_1 = -\exp^{-c_2 x_1^*}$$

We will now verify the Hessian condition (6.9). Some simple calculations yield

$$\frac{\partial K}{\partial x}(x) = c_1 c_2 \exp^{c_2\zeta} \begin{bmatrix} 1 & \frac{1}{x_2} \\ \frac{1}{x_2} & \frac{1}{x_2^2}(1 - \frac{1}{c_2}) \end{bmatrix} \left(= \frac{\partial^2 H_d}{\partial x^2}(x) \right),$$

which evaluated in the equilibrium point gives

$$\frac{\partial K}{\partial x}(\bar{x}) = -c_2 \begin{bmatrix} 1 & 1 \\ 1 & \frac{c_2 - 1}{c_2} \end{bmatrix}$$

The determinant of this matrix is 1, hence it is positive definite iff $c_2 < 0$.

We will investigate now the asymptotic stability properties. To this end, we see that the ω–limit set (6.10) is defined as

$$\left\{ x \in \Re_+^2 \cap \mathcal{B} \mid -x_1(1 + k_1(x))^2 - x_2(1 + k_2(x))^2 = 0 \right\},$$

which consists only of the points $x = 0$ and $x = \bar{x}$. But, it can be easily shown, that $x = 0$ is an unstable equilibrium of the closed loop dynamics.

We have established the following result.

Proposition 6.2 *Consider the system (6.13), with $f(x) = x_1x_2$, in closed–loop with the positive control*

$$u(x) = (1 + x_1x_2)x_2^c \exp^{c(x_1 - x_1^*)} \tag{6.17}$$

with $x_1^* > 0$ the reference for x_1, and $c < 0$. Then, all trajectories starting in $x(0) \in \Re_+^2$, will converge asymptotically to the desired equilibrium point $(x_1^*, 1)$.

□□□

Let us summarize the calculations carried out above:

1. Fix the added damping $R_a(x)$ – to 0 in this case, since the open–loop system is fully damped –;
2. Define the vector $K(x)$, (6.6), as a function of $u(x)$;
3. Use the integrability conditions (6.7) to eliminate the control and obtain a linear PDE (6.15) to be solved for $K(x)$;
4. Find a solution of this PDE that satisfies the equilibrium (6.8) and Lyapunov stability conditions (6.9);
5. Derive the control law (6.17) from the definition of $K(x)$.

Remark 6.3 *As pointed out in Remark 2 as a by–product of our analysis we can get a Lyapunov function, which in our case is*

$$H_d(x) = \underbrace{x_1 + x_2}_{H(x)} \underbrace{- \frac{1}{k} x_2^k \exp^{k(x_1 - x_1^*)}}_{H_a(x)} + \frac{1}{k} - (1 + x_1^*)$$

where the third and fourth right hand constant terms are added to enforce $H_d(\bar{x}) = 0$. It is worth noting that $H_d(x)$ above is the classical Lyapunov function for the stability analysis *of Lotka–Volterra ecologies (see e.g. [3] and [9] among many other references). The design procedure of this paper allows to* rediscover *this Lyapunov function in a very natural way.*

Remark 6.4 *There is an easier way to derive the structural constraint (6.14) that does not require the inversion of the matrix $J(x) - R_d(x)$. To this end, rewrite (6.6) as*

$$[J(x) - (R(x) + R_a(x))]K(x) = [R_a(x)\frac{\partial H}{\partial x}(x) + g(x)u(x)] \tag{6.18}$$

The first equation of (6.18) for this example yields

$$-x_1k_1(x) + x_1x_2k_2(x) = 0$$

which, upon division by x_1, is precisely (6.14). The second equation simply defines the control law, in terms of $K(x)$, as

$$u(x) = -x_2 k_2(x) - x_1 x_2 k_1(x) \tag{6.19}$$

It is precisely this observation that will motivate the modification, introduced in the next section, that yields an output–feedback stabilizer.

4 Output–Feedback Stabilization

There are two important drawbacks of the solution proposed in the previous section. First, it requires measurement of all the state. Second, it can not be extended to treat the general food–chain system model, which is of the form

$$\begin{aligned}
\dot{x}_1 &= x_1 x_2 - x_1 \\
\dot{x}_2 &= x_2 x_3 - x_1 x_2 - x_2 \\
\dot{x}_3 &= x_3 x_4 - x_2 x_3 - x_3 \\
\vdots & \qquad \vdots \\
\dot{x}_n &= -x_{(n-1)} x_n - x_n + u \\
y &= x_n
\end{aligned} \tag{6.20}$$

To prove the second statement, let us write the model in the form (6.1) with $H(x) = \Sigma_{i=1}^n x_i$ and

$$J(x) = \begin{bmatrix} 0 & x_1 x_2 & 0 & \cdots & 0 \\ -x_1 x_2 & 0 & x_2 x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = -J^\top(x)$$

$$R(x) = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} = R^\top(x) \geq 0, \; g(x) = g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then, notice that the distribution spanned by the vector fields defined by the column vectors obtained from the first $n-1$ rows of $J(x) - R(x)$ is not involutive. Consequently, the key PDE

$$[J(x) - R(x)] \frac{\partial H_a}{\partial x}(x) = g u(x)$$

can not be solved.

In this section we show how, for our second order example (6.13), these limitations can be overcome modifying the damping of the closed–loop. In the next section we extend this result to the general n–th order model (6.20).

Towards this end, let us remove the damping from the first coordinate. That is, define $R_a(x)$ like

$$R_a(x) = \begin{bmatrix} -x_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Notice the negative sign. With this choice, the vector function (6.6) becomes now

$$K(x) = \begin{bmatrix} k_1(x) \\ k_2(x) \end{bmatrix} = \begin{bmatrix} -\frac{1}{x_1 x_2}(u-1) \\ -\frac{1}{x_2} \end{bmatrix}$$

Choosing the control law as the simple output–feedback

$$u(x_2) = cx_2 + 1,$$

with c some constant to be defined, yields

$$K(x) = \begin{bmatrix} -\frac{c}{x_1} \\ -\frac{1}{x_2} \end{bmatrix} \tag{6.21}$$

which is clearly the gradient of a scalar function. Hence, the integrability condition (6.7) is satisfied. We will now verify if we can find a constant c such that the remaining stability conditions of Proposition 6.1 are also satisfied. The equilibrium condition (6.16) imposes $c = x_1^*$. For the Hessian condition (6.9) we first observe from (6.21) and $\frac{\partial^2 H}{\partial x^2}(x) = 0$, that

$$\frac{\partial^2 H_d}{\partial x^2}(x) = \frac{\partial K}{\partial x}(x) = \begin{bmatrix} \frac{x_1^*}{x_1^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{bmatrix}$$

Evaluated in the equilibrium point gives

$$\frac{\partial^2 H_d}{\partial x^2}(\bar{x}) = \frac{\partial K}{\partial x}(\bar{x}) = \begin{bmatrix} \frac{1}{x_1^*} & 0 \\ 0 & 1 \end{bmatrix},$$

which will be positive definite for any $x_1^* > 0$.

Finally, asymptotic stability is ensured because the ω–limit set (6.10) is now defined as

$$\left\{ x \in \Re_+^2 \cap \mathcal{B} \mid \frac{x_2 - 1}{x_2} = 0 \right\},$$

which consists only of the point $x = \bar{x}$.

The new Lyapunov function is

$$H_d(x) = \underbrace{x_1 + x_2}_{H(x)} - \underbrace{x_1^* \ln(x_1) - \ln(x_2)}_{H_a(x)} - (x_1^* + 1 - x_1^* \ln(x_1^*)),$$

where the third right hand constant term is, again, added to enforce $H_d(\bar{x}) = 0$.

We have established the following result.

Proposition 6.3 *Consider the system (6.13), with $f(x) = x_1 x_2$, in closed–loop with the positive output–feedback control*

$$u(x_2) = 1 + x_1^* x_2 \tag{6.22}$$

with $x_1^* > 0$ the reference for x_1. Then, all trajectories starting in $x(0) \in \mathcal{R}_+^2$, will converge asymptotically to the desired equilibrium point $(x_1^*, 1)$.

□□□

Remark 6.5 *To increase the speed of convergence it is possible to inject some additional damping on the actuated coordinate x_2. To this end, we choose*

$$R_a(x) = \begin{bmatrix} -x_1 & 0 \\ 0 & (r-1)x_2 \end{bmatrix},$$

with the desired damping a constant $1 < r < 1 + x_1^*$. Going through the calculations we get the control law

$$u(x_2) = r + (x_1^* - r + 1)x_2 \tag{6.23}$$

It can be shown that this control law is also globally asymptotically stabilizing. Notice that with $r = 1$ we recover the controller (6.22).

5 Main Result

In this section we present the generalization of the previous result to the n–th order case.

Theorem 6.1 *Consider the general food chain system (6.20) in closed-loop with the output–feedback positive control*

$$u(x_n) = mx_n + m + x_1^*, \quad m = \frac{n-1}{2}$$

for n odd, and

$$u(x_n) = (m + x_1^*)x_n + \frac{n}{2}, \quad m = \frac{n}{2} - 1$$

for n even, with $x_1^* > 0$ the reference for x_1. Then, all trajectories starting in $x(0) \in \Re^n_+$ will converge asymptotically to the desired equilibrium point $\bar{x} = [x_1^*, \bar{x}_2, ..., \bar{x}_n]$.

□□□

Proof

Motivated by the developments of the second order case above we propose to remove the damping from all non–actuated coordinates. That is, we choose

$$R_a(x) = \begin{bmatrix} -x_1 & 0 & \cdots & 0 \\ 0 & -x_2 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

We will now verify the three conditions of Proposition 6.1.

- *Integrability*

The key equation (6.11) becomes then

$$\begin{bmatrix} 0 & x_1x_2 & 0 & \cdots & 0 & 0 \\ -x_1x_2 & 0 & x_2x_3 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 0 & x_{n-1}x_n \\ 0 & 0 & 0 & \cdots & -x_{n-1}x_n & x_n \end{bmatrix} \begin{bmatrix} k_1(x) \\ k_2(x) \\ . \\ . \\ k_{n-1}(x) \\ k_n(x) \end{bmatrix}$$
$$= \begin{bmatrix} -x_1 \\ -x_2 \\ . \\ . \\ -x_{n-1} \\ u(x) \end{bmatrix},$$

which can be compactly written as $\bar{\mathcal{J}}(x)K(x) = \bar{g}(x)$. Now, $\bar{\mathcal{J}}(x)$ admits a factorization of the form

$$\bar{\mathcal{J}}(x) = \text{diag}\{x_i\} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & -\frac{1}{x_n} \end{bmatrix} \text{diag}\{x_i\}$$

This leads to

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & -\frac{1}{x_n} \end{bmatrix} \begin{bmatrix} x_1k_1(x) \\ x_2k_2(x) \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}k_{n-1}(x) \\ x_nk_n(x) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ \frac{u(x)}{x_n} \end{bmatrix}$$

From which we obtain a system of equations of the form

$$\begin{aligned} x_2k_2(x) &= -1 \\ -x_1k_1(x) + x_3k_3(x) &= -1 \\ -x_2k_2(x) + x_4k_4(x) &= -1 \\ \vdots \qquad & \qquad \vdots \\ -x_{n-2}k_{n-2}(x) + x_nk_n(x) &= -1 \\ -x_{n-1}k_{n-1}(x) - k_n(x) &= \frac{u(x)}{x_n} \end{aligned} \tag{6.24}$$

Notice that from the first equation of (6.24) we have

$$k_2(x) = -\frac{1}{x_2}$$

Subsequently, the functions $k_i(x)$, for i even, have a unique solution, which is furthermore of the form $k_i(x) = k_i(x_i)$. Now, choosing

$$k_1(x) = -\frac{c}{x_1}$$

we can also obtain a unique solution $k_i(x_i)$, for i odd. The vector function $K(x)$ is finally given by

$$K(x) = \begin{bmatrix} -\frac{1}{x_1} & -\frac{1}{x_2} & \cdots & -\frac{m}{x_{n-1}} & -\frac{m+c}{x_n} \end{bmatrix}^T, \quad m = \frac{n-1}{2},$$

for n odd, and

$$K(x) = \begin{bmatrix} -\frac{c}{x_1} & -\frac{1}{x_2} & \cdots & -\frac{m+c}{x_{n-1}} & -\frac{\frac{n}{2}}{x_n} \end{bmatrix}^T, \quad m = \frac{n}{2} - 1,$$

for n even. It is clear that, in both cases, the integrability conditions are satisfied.

Also, from the last equation of (6.24) we compute the control law

$$u(x) = -x_n[x_{n-1}k_{n-1}(x) + k_n(x)]$$

- *Equilibrium Assignment*

The equilibrium condition is

$$K(\bar{x}) = -\frac{\partial H}{\partial x}(\bar{x}) = -\begin{bmatrix} 1 & \dots & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} -\frac{c}{x_1^*} & \cdots & -\frac{m+c}{x_{n-1^*}} & -\frac{\frac{n}{2}}{x_n^*} \end{bmatrix}^T,$$

which is satisfied with $c = x_1^*$.

- *Lyapunov Stability*

We will now verify the Hessian condition. Some simple calculations yield

$$\frac{\partial K}{\partial x}(x) = \begin{bmatrix} \frac{x_1^*}{x_1^2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{x_2^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1+x_1^*}{x_3^2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{m+x_1^*}{x_n^2} \end{bmatrix} \tag{6.25}$$

This matrix will be positive definite for any $x \in \Re_+^n$ and any $x_1^* > 0$. Finally, the ω limit set for n odd is defined as $\{x \in \Re_+^n \cap \mathcal{B} \mid \frac{x_n-(m+x_1^*)}{x_n} = 0\}$ and $\{x \in \Re_+^n \cap \mathcal{B} \mid \frac{x_n-(\frac{n}{2})}{x_n} = 0\}$ for n even. In both cases the ω limit set consists only of the point $x_n = \bar{x}_n$. This, together with uniqueness of the equilibrium, completes the proof of asymptotic stability.

Remark 6.6 *The proposed control design can be easily applied to the more general class of Lotka–Volterra ecologies defined as follows:*

$$\begin{aligned} \dot{x}_i &= x_i(-k_i + \sum_{j\neq i} a_{ij}x_j) \quad i = 1, ..., n-1 \\ \dot{x}_n &= x_n(-k_n + \sum_{j\neq n} a_{nj}x_j) + u \end{aligned}$$

with $k_i > 0$ the natural mortality rates, $a_{ij} = -a_{ij}, \forall i \neq j$, the predation coefficients and u the feeding rate of species x_n, with $u(t) \geq 0 \ \forall t$.

The procedure yields the classical Lyapunov function for Lotka–Volterra ecologies

$$\sum_{i=1}^{n} x_i - \bar{x}_i \ln(x_i),$$

and we obtain the following output feedback control law

$$u(x_n) = \bar{u} + \lambda(x_n - \bar{x}_n)$$

with $\bar{u}$ the constant control that assigns the desired equilibrium, and $0 < \lambda < \frac{\bar{u}}{\bar{x}_n}$ an arbitrary design parameter.

6 Simulations

Numerical simulations of the second order model (6.13) were carried out in order to show the performance of the proposed controllers. The parameters used in the simulations were, $c = -0.2$ for the state feedback controller (6.17), and $r = 1, 2.1$, for the output feedback controller (6.23). The desired equilibrium of the system is $\bar{x} = [1.2, 1]^T$. The initial conditions in all the simulations are $x_1(0) = 2$ and $x_2(0) = 2$.

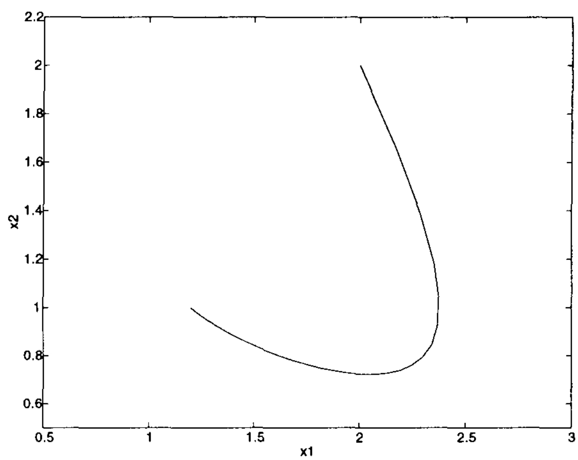

FIGURE 1. Open–loop trajectory

For the sake of comparison, in Fig. 1 we present the behaviour of the open loop trajectory in the state space with a constant input $\bar{u} = 2.2$, while Fig. 2 depicts the behaviour of the state and ouput feedback controllers. Finally, the control signals are shown in Fig. 3. As seem from the Figs. 2, 3 the addition of damping effectively increases the convergence rate with the additional advantage of reducing the control effort.

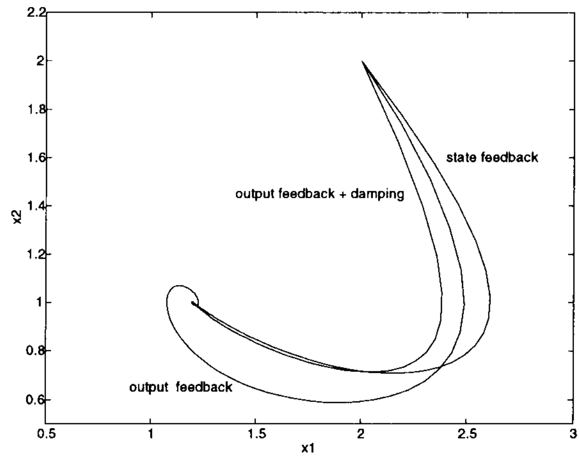

FIGURE 2. State space of the closed–loop trajectory

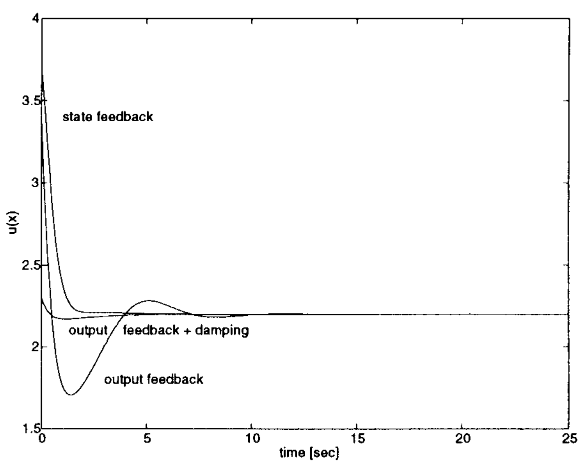

FIGURE 3. Control signals

7 Concluding Remarks

We have illustrated in this chapter how the application of the passivity–based controller design technique of [7] allows us to solve output–feedback stabilization problems for a class of mass–balance systems. The procedure is illustrated in detail with an n–th order food–chain model. It can, *mutatae–mutandi*, be applied also to other mass–balance models studied in [1], [3],

[9]. For instance, it can be shown that for the compartmental model of Section 4 in [1] the technique yields also asymptotically stabilizing controllers. However, we require in this case the knowledge of the full state.

We have not stressed here the advantages of taking a physically–based approach for controller design, see e.g. [8], [7], [10] for a detailed discussion. We should underscore, however, that the preservation of a physical interpretation to the control action (in terms of damping injection) was instrumental for our result. Finally, we bring to the readers attention the simplicity of the resulting control law. This important feature is a characteristic of passivity–based controllers.

As shown in this chapter the approach of [7] provides a flexible methodology to design controllers for physical systems. As discussed in that paper, we can also aim at modifying the internal interconnection structure $\mathcal{J}(x)$. In this way, we recover some of the results obtained with the technique of controlled Lagrangians, reported in [2]. Current research is under way to explore this interesting possibility for mass–balance systems.

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Appendix A: Maple Code

In this appendix we present a Maple code that guides us in the solution of the example of Section 2. The calculations proceed as follows

1. Definition of the system (with $JmR \triangleq J(x) - R(x)$ and $gu \triangleq gu(x)$) :

```
> with(linalg):
> JmR := matrix(2,2,[-x1,x1*x2,-x1*x2,-x2]);
                                       [ -x1       x1 x2]
                                JmR := [                 ]
                                       [-x1 x2     -x2 ]

> gu := vector([0,u(x1,x2)]);
                                  gu := [0, u(x1, x2)]
```

2. Computation of $K(x)$ and its Jacobian

```
> K := multiply(inverse(JmR),gu);
                           [  u(x1, x2)       u(x1, x2)    ]
                      K := [- ---------, - --------------]
                           [  1 + x1 x2    x2 (1 + x1 x2)]
```

```
> Jac := jacobian(K,[x1,x2]);

  Jac :=


  [                d                                   d              ]
  [                --- u(x1, x2)                       --- u(x1, x2)]
  [u(x1, x2) x2    dx1                u(x1, x2) x1     dx2            ]
  [------------ - -------------- ,  ------------ - ---------------]
  [           2     1 + x1 x2                    2      1 + x1 x2   ]
  [(1 + x1 x2)                       (1 + x1 x2)                      ]

  [                d
  [                --- u(x1, x2)
  [ u(x1, x2)      dx1
  [------------ - -------------- ,
  [           2    x2 (1 + x1 x2)
  [(1 + x1 x2)

                                              d              ]
                                              --- u(x1, x2) ]
     u(x1, x2)          u(x1, x2) x1          dx2            ]
  --------------- + --------------- - --------------]
   2                                2   x2 (1 + x1 x2)]
  x2  (1 + x1 x2)   x2 (1 + x1 x2)                     ]
```

3. Definition of the term

$$eq12 \triangleq \frac{\partial k_2}{\partial x_1}(x) - \frac{\partial k_1}{\partial x_2}(x)$$

```
> eq12 := Jac[2,1]-Jac[1,2];

                                 d
                                 --- u(x1, x2)
              u(x1, x2)          dx1                u(x1, x2) x1
  eq12 := ------------ - --------------- - ------------
                        2    x2 (1 + x1 x2)                   2
           (1 + x1 x2)                              (1 + x1 x2)

             d
             --- u(x1, x2)
             dx2
         + ------------
             1 + x1 x2
```

4. Determination of the control $u(x)$ which solves $eq12 = 0$, i.e., which ensures the integrability condition.

```
> u_star:=rhs(pdesolve(eq12=0,u(x1,x2)));
                                                     x2
                           u_star := _F1(--------) (1 + x1 x2)
                                                exp(-x1)
```

Notice that in the line above $\_F1(\cdot)$ is any differentiable function.

5. Evaluation of the Hessian for the given control expression.

```
> subs(u(x1,x2)=u_star,evalm(Jac));

[          x2                d                    x2                 d        ]
[_F1(--------) x2          --- %1          _F1(--------) x1         --- %1    ]
[      exp(-x1)            dx1                exp(-x1)              dx2       ]
[----------------- - ----------- ,  ----------------- - -----------]
[    1 + x1 x2          1 + x1 x2          1 + x1 x2           1 + x1 x2]

[          x2                d
[_F1(--------)             --- %1
[      exp(-x1)            dx1
[--------------- - ---------------- ,
[   1 + x1 x2       x2 (1 + x1 x2)
[

        x2                   x2                       d            ]
_F1(--------)        _F1(--------) x1               --- %1         ]
     exp(-x1)             exp(-x1)                  dx2            ]
--------------- + ----------------- - ----------------]
        2             x2 (1 + x1 x2)      x2 (1 + x1 x2)]
      x2                                                              ]

                  x2
%1 := _F1(--------) (1 + x1 x2)
            exp(-x1)
```

6. The design can be concluded selecting a function $\_F1(\cdot)$ that satisfies the equilibrium assignment and Lyapunov stability conditions of Proposition 6.1. In Section 2 we have chosen $\_F1(\zeta) = \zeta^k$.