DISCRETE TIME ADAPTIVE CONTROL FOR A CLASS OF NONLINEAR CONTINUOUS SYSTEMS

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ABSTRACT.

It is shown that adaptive approximate state feedback linearization can be achieved for a class of sampled-data non linear models which arise from the sampling of state feedback linearizable and linearly parametrized continuous time systems.

1. INTRODUCTION.

Adaptive control of non linear *continuous time* systems is a subject of growing interest. Several direct and indirect adaptive schemes have been recently discussed and analysed for the class of non linear systems which are state feedback linearizable and linearly parametrized (see e.g. Taylor (1987), Taylor et al. (1988), Sastry and Isidori (1988), Pomet and Praly (1988), Bastin and Campion (1989)). In addition, specific applications have been reported in Biotechnology (Dochain and Bastin, 1984) and in Robotics (e.g. Craig et al., 1986; Middelton and Goodwin, 1988).
The interest of a continuous time design is however restricted by the fact that controllers are most often implemented digitally. The issue therefore arises of designing discrete time controllers based on the sampled-data models of continuous time systems. Our objective, in this paper, is to derive and analyse sampled-data counterparts of the adaptive continuous time linearizing controllers proposed in (Bastin and Campion, 1989). As we shall emphasize, this derivation is not immediate, mainly because both state feedback linearizability and linear parametrization can be destroyed by the sampling process. For simplicity, we shall limit ourselves to adaptive regulation of non linear systems which are full state linearizable without diffeomorphism. This restriction should be interpreted as a first attempt towards a more general theory of adaptive control of sampled-data non linear systems.
The paper is organized as follows. The class of non linear systems under consideration is described in section 2, characterized by three basic assumptions which guarantee discretizability, feedback linearizability and linear parametrization. The exact sampled-data model of these continuous systems is stated in section 3. The discrete time adaptive control problem we address in this paper is formulated in section 4, where the difficulties of the transposition from continuous time to discrete time are also emphasized. The control law is based on the "certainty equivalence principle" and is obtained combining an "approximate" linearizing control law and a suitable parameter estimator which is presented and analyzed in section 5. In section 6, we discuss non adaptive and adaptive regulation, in the special case where the system has as many inputs as there are states, and then generalize for the situation where the input dimension is lower than the state dimension. For the readability of the paper, the proofs are not given in the text but, within the bounds of the paper length, some of them are given in appendix.

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2. SYSTEM DESCRIPTION IN CONTINUOUS TIME.

We consider a class of non linear systems, with parametric uncertainty, which are state feedback linearizable, linearly parametrized and linear in the control input.

They are denoted as follows :

$$\dot{x} = f(x,\theta_0) + G(x,\theta_0)\, u \tag{2.1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $\theta_0 \in R^p$ is the parameter vector, $f(x,\theta_0) \in R^n$ is a vector field, $G(x,\theta_0)$ is a nxm matrix on R whose columns are the vector fields $g_i(x,\theta_0) \in R^n$ (i =1,m).

By "parametric uncertainty", we mean that the "true" value θ_0 of the parameter is unknown, but that an estimate θ is available.

The systems (2.1) are supposed to satisfy the following assumptions for every $(x,\theta) \in B_x \times B_\theta$, where B_x and B_θ denote *compact sets* containing x=0 and $\theta=\theta_0$ as interior points i.e. there exists 2 strictly positive constants k_1 and k_2 such that
$\| x \| \le k_1 \Rightarrow x \in B_x$ and $\|\theta - \theta_0\| \le k_2 \Rightarrow \theta \in B_\theta$

A1. (discretizability) : the vector fields $f(x,\theta)$ and $g_i(x,\theta)$ are analytic.

A2. (linearizability) : the matrix $G(x,\theta)$ has full rank and there exists a Hurwitz matrix Λ such that :

$$\Lambda x - f(x,\theta) \in \text{span}\,\{g_i(x,\theta),\ i = 1,m\}$$

A3. (linear parametrization) : the vector fields $f(x,\theta)$ are linearly parametrized as follows :

$$f(x,\theta) = F(x)\theta \quad \text{and} \quad g_i(x,\theta) = G_i(x)\theta$$

where $F(x)$ and $G_i(x)$ are nxp matrices independent of θ.

Comments :
1) Assumption **A1** is needed to guarantee the consistency of the sampled-data model of system (2.1) which will be introduced in the next section.

2) Assumption **A2** guarantees that stabilizing state feedback linearization of system (2.1) is achieved by the following state feedback control law :

$$u_0(x,\theta_0) = [G^T(x,\theta_0)\, G(x,\theta_0)]^{-1}\, G^T(x,\theta_0)\, [\Lambda x - f(x,\theta_0)] \tag{2.2}$$

Indeed, it is easily seen that, with control law (2.2), the closed loop is linear and stable as follows :

$$\dot{x} = f(x,\theta_0) + G(x,\theta_0)\, u_0(x,\theta_0) = \Lambda x \qquad (2.3)$$

In the special case where n=m (that is there are as many inputs as there are states), the matrix $G(x,\theta_0)$ is square full rank and the control law reduces to :

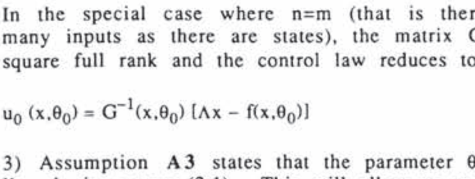
$$u_0\,(x,\theta_0) = G^{-1}(x,\theta_0)\,[\Lambda x - f(x,\theta_0)] \qquad (2.4)$$

3) Assumption **A3** states that the parameter θ appears linearly in system (2.1). This will allow us to write the model in regressor form and, hence, to use recursive linear regression to implement adaptive control laws.

Remark :
Assumptions **A2** and **A3** are not as restrictive as they might appear. There is a wide number of practical applications (from electromechanical systems to (bio)chemical kinetics) where linearly parametrized models are relevant . Moreover, assumption **A2** can be easily relaxed by considering feedback linearization through diffeomorphism. But the price to pay would be a considerable amount of technical complications without really gaining insight into the control problem we want to handle.

3.EXACT SAMPLING OF NON LINEAR CONTINUOUS SYSTEMS.

We are concerned with computer control of non linear continuous time systems of the form (2.1.), performed by sampling and zero-order-hold control action. The situation is depicted in fig.1.



The state x(t) and the control u(t) are supposed to be sampled at the same rate, with a sampling period δ.
The sampled state is defined at the sampling instants :

$$x(k) \equiv x(t = \delta k) \qquad (3.1)$$

The Z0H control is defined as follows :

$$u(t) = u(k\delta) \equiv u(k) \quad k\delta \le t < (k+1)\,\delta \qquad (3.2)$$

Notice that the argument "δ" is omitted in x(k) and u(k) without risk of confusion.

With definitions (3.1) and (3.2), the sampled-data version of (2.1) can be shown to be written as follows (e.g. Monaco and Normand Cyrot, 1985) :

$$Dx(k) = \left\{ \sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \left[L_f + \sum_{i=1}^{m} u_i(k)\, L_{g_i} \right]^j \right\} x(k) \qquad (3.3)$$

where L_f and L_{gi} denote Lie derivatives, and where Dx(k) is the finite difference operator :

$$Dx(k) = \frac{x\,(k+1) - x(k)}{\delta} \qquad (3.4)$$

The model (3.3) is called an "exact" sampled-data model because its state x(k) exactly coïncides with the state of the continuous system (2.1) at the sampling instants.

For convenience, this model will be occasionally written in the following compact form :

$$Dx(k) = h\,(x(k), u(k), \delta\,) \qquad (3.5)$$

4.STATEMENT OF THE ADAPTIVE CONTROL PROBLEM.

In this paper, our objective is to design adaptive linearizing controllers for the sampled-data model (3.3). A specific study is required because a direct transposition of the available continuous time results (e.g. Bastin and Campion, 1989) is not possible. This is due to the following features of the discrete model (3.3) :

F1. The RHS of (3.3) is an infinite series with respect to the sampling period δ. This means that the model may not be tractable for control design purpose and that truncation and approximation may be necessary.

F2. Although the system (2.1) is linear in the control input u(t), its discretized counterpart (3.3) is *not* linear in the control input. This implies that state feedback linearizability of (3.3) is not guaranteed a priori, despite assumption A2.

F3. Although the system (2.1) is linearly parametrized, its discretized counterpart (3.3) is *not* linearly parametrized. This will obviously introduce difficulties in the parameter adaptation design.

Hence, we shall address the problem of finding a parametrized discrete state feedback control law, denoted :

$$u\,(x(k), \bar{\beta}(\theta_0), \delta) \qquad (4.1)$$

which is able to realize *approximate* adaptive feedback linearization of the sampled-data model (3.3).

Let us explain the terminology and the notation.
By "approximate linearization" we mean that we attempt to approach the following "truncated" discrete reference model :

$$Dx(k) = \left(\sum_{j=1}^{r} \frac{\delta^{j-1}}{j!} \Lambda^j \right) x(k) \overset{\Delta}{=} \bar{\Lambda}_r(\delta)\; x(k) \qquad (4.2)$$

where Λ is the Hurwitz matrix defined in assumption A2. Notice that (4.2) tends to the continuous reference model (2.3) when $\delta \to 0$. For $r = \infty$, (4.2) is uniformly asymptotically stable for any δ, while for limited r, the following property hold :

Lemma 1 :

For any r, there exists an upper bound $\delta_r < \infty$ such that (4.2) is uniformly asymptotically stable for all $\delta \le \delta_r$.

We introduce the following rigourous definition of approximate linearization.

Definition :

The discretized model (3.3) is said to be r-th degree linearizable if there exists a control law $u(x(k), \delta, \bar{\beta}(\theta_0))$ such that the closed loop dynamics coïncides with the linear reference model (4.2) up to the degree r, that is it can be written :

$$h(x(k), u(x(k), \bar{\beta}(\theta_0), \delta), \delta) = \bar{\Lambda}_r(\delta)\; x(k) + R(\delta^{r+1}) \qquad (4.3)$$

"r" will be called the linearization degree.
Notation : throughout the paper we use $R(\delta^s)$ to denote a residual n-vector whose norm is of the order of δ^s i.e.
$\| R(\delta^s) \| \le O(\delta^s)$

Though the control law linearizes the discretized model only approximately,it will be designed in order to match exactly the continuous time linearizing control $u_0\,(x,\theta_0)$ given by (2.2) or (2.3) when $\delta \to 0$:

$$\lim_{\delta \to 0} u\,(x(k), \bar{\beta}(\theta_0), \delta) = u_0(x,\theta_0) \qquad (4.4)$$

Finally, the control law (4.1) is parametrized by a parameter β which is an (over)reparametrization of the "physical" parameter θ_0. Then, an adaptive control algorithm will be obtained by replacing $\bar{\beta}(\theta_0)$ by an on line estimate β (certainty equivalence principle) which will be computed with a recursive algorithm presented in the next section.

5. DISCRETE PARAMETER ESTIMATION.

Although the discrete model (3.3) is non linear with respect to the parameter θ_0, it can be seen, from assumption **A3**, that there exists an *infinite* sequence of linear reparametrizations, denoted $\bar{\beta}_j(\theta_0)$, $j = 1,\ldots,\infty$, such that (3.3) is rewritten as follows :

$$Dx(k) = \sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \varphi_j^T(k)\, \bar{\beta}_j(\theta_0) \tag{5.1}$$

where $\bar{\beta}_j(\theta_0)$ is the vector of the components of the j^{th} tensor product of θ_0 by itself and $\varphi_j(k)$ is an appropriate regressor (function of x(k) and u(k)).

Notice in particular that

$\bar{\beta}_1(\theta_0) = \theta_0$ and $\varphi_1^T(k)\, \bar{\beta}_1(\theta_0) = f(x(k),\theta_0) + G(x(k),\theta_0)\, u(k)$

Operating on both sides of (5.1) with the operator $(D+\omega)^{-1}$ where ω is any positive constant such that $\omega\delta<1$, the discrete model (5.1) is easily shown to be equivalent to :

$$x(k) = \varphi_{0F}(k) + \sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \varphi_{jF}^T(k)\, \bar{\beta}_j(\theta_0) \tag{5.2}$$

where $\varphi_{jF}(k)$ $(j=0,\ldots,\infty)$ can be computed on line by filtering x(k) and $\varphi_j(k)$ as follows :

$$\varphi_{0F}(k) = \frac{\omega}{D+\omega}\, x(k) \tag{5.3.a}$$

$$\varphi_{jF}(k) = \frac{1}{D+\omega}\varphi_j(k) \quad j = 1,\ldots,\infty \tag{5.3.b}$$

The model (5.3) is now rewritten as follows :

$$x(k) = \psi_0(k) + \psi^T(k)\, \bar{\beta}(\theta_0) + R_1(\delta^q) \tag{5.4}$$

where

$$\psi_0(k) = \varphi_{0F}(k) \tag{5.5.a}$$

$$\psi^T(k) = \left(\varphi_{1F}^T(k), \frac{\delta}{2!}\varphi_{2F}^T(k),\ldots,\frac{\delta^{q-1}}{q!}\varphi_{qF}^T(k)\right) \tag{5.5.b}$$

$$\bar{\beta}(\theta_0)^T = (\bar{\beta}_1(\theta_0)^T, \bar{\beta}_2(\theta_0)^T,\ldots, \bar{\beta}_q(\theta_0)^T) \tag{5.5.c}$$

If the term $R_1(\delta^q)$ in (5.4), is ignored, the model is clearly in a standard linear regressor form.
Bounds on $\psi(k)$ and $R_1(\delta^q)$ are given in the following lemma .

Lemma 2 :

For all $x \in B_x$, for all δ, if u(k) is uniformly bounded (i.e. $\exists U$: $\|u(k)\|<U$, $\forall k$), there exists positive uniform bounds $\rho(\delta)$ and $M_1(\delta)$ such that :

$$\|\psi(k)\| \le \rho(\delta) \quad \text{and} \quad \|R_1(\delta^q)\| \le \frac{\delta^q}{(q+1)!} M_1(\delta) \quad \forall k$$

Our approach is then to state a recursive prediction error algorithm based on the linear regression part of (5.5), and made robust against the "unmodelled" term $R_1(\delta^q)$ by introducing a dead zone.

With β denoting the estimate of $\bar{\beta}(\theta_0)$, the algorithm is as follows :

prediction : $\hat{x}(k) = \psi_0(k) + \psi^T(k)\,\beta(k)$ (5.6.a)

prediction error : $e(k) = x(k) - \hat{x}(k)$ (5.6.b)

parameter adaptation :

$$D\beta(k) = \alpha P(k)\psi(k)e(k) \quad \text{if } \|e\| > d(\delta)\frac{\delta^q}{(q+1)!} \tag{5.6.c}$$
$$= 0 \text{ otherwise}$$

Gain adaptation :

$$DP(k) = -\alpha P(k)\psi(k)\psi^T(k)P(k) \quad \text{if } \|e\| > d(\delta)\frac{\delta^q}{(q+1)!} \tag{5.6.d.}$$
$$= 0 \text{ otherwise}$$
$$\text{with } P(0) = \gamma I \quad (\gamma > 0)$$

"q" will be called the estimation degree.
The dead zone size $d(\delta)$ is defined as follows :

$$d(\delta) = \frac{M_1(\delta)}{\sqrt{1-\alpha\delta\gamma\rho^2(\delta)}} \tag{5.7.a}$$

where the step size α must satisfy the following inequality :

$$0 < \alpha < \frac{1}{\delta\gamma\rho^2(\delta)} \tag{5.7.b}$$

Notice that, if $\delta \to 0$, this algorithm exactly coïncides with the parameter adaptation scheme used for continuous time adaptive control in (Bastin and Campion, 1989).

The following theorem establishes the properties of the algorithm.

Theorem 1.:

For all $x \in B_x$, for all δ, if u(k) is uniformly bounded,

(i) the estimation error $\tilde{\beta}(k) = \bar{\beta}(\theta_0) - \beta(k)$ is bounded as follows : $\|\tilde{\beta}(k)\|^2 \le \|\tilde{\beta}(0)\|^2$ *(5.8)*

(ii) the prediction error is bounded as follows :

$$\|e(k)\| \le \rho(\delta)\, \|\tilde{\beta}(0)\| + M_1(\delta)\, \frac{\delta^q}{(q+1)!} \tag{5.9}$$

$$\text{(iii)} \quad \limsup_{k\to\infty} \|e(k)\| \le d(\delta)\, \frac{\delta^q}{(q+1)!} \tag{5.10}$$

6. ADAPTIVE r-LINEARIZING CONTROL.

In this section, we shall first focus on the particular class of systems which have as many inputs as there are states. $G(x,\theta)$ is then a square matrix and assumption **A2** reduces to :

A2bis For every (x,θ) in $B_x \times B_\theta$, the matrix $G(x,\theta)$ is regular. Λ is an arbitrary Hurwitz matrix.

The section will be organized is four parts. We first present two preliminary properties which are useful for the analysis of both non adaptive and adaptive situations (section 6.1). Then in section (6.2) we discuss the existence and the properties of non-adaptive (i.e. with perfect knowledge of θ_0) r-linearizing controllers. Thereafter, the adaptive case is investigated in section 6.3. Finally the case where assumption A2 is postulated instead of A2bis is discussed in section 6.4.

6.1. Preliminaries.

In this section we present two preliminary properties useful for the analysis. Lemma 3 is a stability property while lemma 4 is an algebraic property related to the existence of linearizing control laws.

Lemma 3 :

For the system
$Dx(k) = Ax(k) + v_1(k) + Dv_2(k)$

If $(I+\delta A)$ *is a stable matrix i.e. there exist :*

K_1 (≥ 1) *and a* $(0<a\leq 1)$ *such that :* $\|(I+\delta A)^k\| \leq K_1 a^k \ \forall k$

$v_1(k)$ *and* $v_2(k)$ *are uniformly bounded :*

$\|v_1(k)\| \leq C_1$ *and* $\|v_2(k)\| \leq C_2 \ \forall k$

Then :

a) $\|x(k)\|$ *is uniformly bounded as follows :*

$$\|x(k)\| \leq K_1 \|x(0)\| + \delta\frac{K_1}{1-a}(C_1 + \|A\|.C_2) + (K_1+1)C_2$$

If, in addition ,

$\lim\sup_{k\to\infty} \|v_1(k)\| = c_1$ *and* $\lim\sup_{k\to\infty} \|v_2(k)\| = c_2$

b) $\lim\sup_{k\to\infty} \|x(k)\| \leq \frac{K_1\delta}{1-a}(c_1 + \|A\|.c_2) + c_2$

We now present an algebraic property which is critical to guarantee the existence of both non adaptive r-linearizing control laws.

We define the infinite sequence of vectors $b_j (j=1,\dots,\infty)$ which satisfy the following conditions :

C1. $b_1 \in B_\theta$

C2. The dimension of b_j is equal to the dimension of $\bar{\beta}_j(\theta_0)$

C3. there exists a positive constant d_0 such that

$$\|b_j\| < d_0^j \ \forall j \tag{6.1}$$

These vectors will serve as true parameters $\bar{\beta}_j(\theta_0)$ as well as estimates of them in theorems 2 and 4 respectively .

Lemma 4 :

Under assumptions A1-A2bis-A3 and conditions C1-C2-C3, for every δ *and every* $\xi \in B_x$,
there exists an infinite sequence of functions
$v_i(\xi, b_1, \dots, b_{i+1})$
such that, for every r, the function defined as :

$$v(\xi,\delta,b_1,..b_r) \overset{\Delta}{=} \sum_{i=0}^{r-1} \frac{\delta^i}{(i+1)!} v_i(\xi,b_1,\dots b_{i+1}) \tag{6.2}$$

(i) is uniformly bounded i.e. there exists $\bar{v}(\delta)$ *such that :* $\|v(\xi,\delta,b_1,..b_r)\| \leq \bar{v}(\delta)$ *(6.3)*

$$(ii) \ \sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \varphi_j^T(\xi,v)\, b_j = \bar{A}_r(\delta)\,\xi + R_2(\delta^r) \tag{6.4}$$

Notice in particular that

$v_0(\xi,b_1) \equiv u_0(\xi,b_1)$ where $u_0(\xi,b_1)$ is defined in (2.4)

With appropriate choice of $x, b_1, \dots, b_r$, v will give the r-linearizing control law in 6.2 and the adaptive control law in 6.3

6.2. Perfectly known systems.

It can be checked that $\bar{\beta}_i(\theta_0)$ satisfies (6.1) with $d_0 = p\|\theta_0\|$. The r-linearizing control law is then obtained replacing in (6.2) ξ by $x(k)$ the state of the system and the arbitrary values b_k by the actual values of the corresponding $\bar{\beta}_k(\theta_0)$:

$$u(k) = v(x(k),\delta,\bar{\beta}_1(\theta_0),\dots\bar{\beta}_r(\theta_0)) \overset{\Delta}{=} \sum_{i=0}^{r-1}\frac{\delta^i}{(i+1)!} v_i(x(k),\bar{\beta}_1(\theta_0),\dots,\bar{\beta}_{i+1}(\theta_0)) \tag{6.5}$$

The following theorem then follows :

Theorem 2:

Under assumptions A1-A2bis-A3. For every δ, *for every x in Bx , for every r, the control law u(k) defined in (6.5) is uniformly bounded and :*

((i)realizes the r-linearization of the system

i.e. $Dx(k) = \bar{A}_r\delta x(k) + R_3(\delta^r)$ *(6.6)*

(ii) ensures that $R_3(\delta^r)$ *is bounded as follows :*

$\|R_3(\delta^r)\| \leq M_3(\delta)\frac{\delta^r}{(r+1)!}$ *with* $M_3(\delta) < \infty$

Now we look for the domain of admissible initial values x(0) ensuring that x(k) remains continuously in B_x and derive the asymptotic convergence properties of (6.5).

For every r and every $\delta \leq \delta_r$ defined in lemma 1, the application of lemma 3 to (6.6) with $v_1(k) = R_3(\delta^r)$ and $v_2(k)=0$ give rise to the following bounds :

$$\|x(k)\| \leq K_1\|x(0)\| + K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} \leq K_1\|x(0)\| + \bar{K}_2\frac{\delta^{r+1}}{(r+1)!}$$

$$\lim\sup_{k\to\infty}\|x(k)\| \leq K_2(\delta)\frac{\delta^{r+1}}{(r+1)!}$$

where K_1 follows from lemma 1

$$K_2(\delta) = \frac{K_1}{1-a}M_3(\delta)$$

$$\bar{K}_2 = \max\{K_2(\delta) : \delta \in [0,\delta_r]\}$$

We define $\delta_2 = \min\left(\delta_r, \left((r+1)!\frac{k_1}{\bar{K}_2}\right)^{1/r+1}\right)$

Then, for every $\delta<\delta_2$, $k_1 - K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} \overset{\Delta}{=} \eta_1(\delta) > 0$

Then the following set :

$B_0 = \left\{x : K_1\|x\| + K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} \leq k_1\right\}$ is a closed ball of radius $\frac{\eta_1(\delta)}{K_1} > 0$ centered on $x=0$: $B\left(0,\frac{\eta_1(\delta)}{K_1}\right)$

Theorem 3 :

Under assumptions A1-A2bis-A3, for every r, for every $\delta < \delta_2$, *the control law (6.5) ensures that*

if $x(0) \in B_0$,

(i) $x(k) \in B_x$ *and theorem (2) holds* $\forall k$

(ii) $\lim\sup_{k\to\infty}\|x(k)\| \leq K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} \leq \bar{K}_2\frac{\delta^{r+1}}{(r+1)!}$

In Grizzle (1986), linearizability conditions are given for discrete systems and it is shown that sampling of continuous systems can destroy their feedback linearizability. In the present paper we focus on approximate linearization and give conditions (on the sampling period and the initial error) for r-linearizability (theorem 3).

6.3. Systems with parameter uncertainty.

For fixed values of δ, r and q, using the certainty equivalence principle, an adaptive control law u(k) is obtained, from expression 6.2, by replacing

- $.\xi$ by x(k)
- $.b_i$ by the estimate β_i of $\bar{\beta}_i(\theta_0)$ provided by the parameter adaptation algorithm (5.6) $\forall$ i = 1,..q
- $.b_i$ by 0 $\forall$ i > q

The adaptation algorithm (5.6) is implemented with $\rho(\delta)$ and $M_1(\delta)$ defined according to lemma 2 where the upper bound U is chosen as the maximum of $\|v(\xi,\delta,b_1,\dots,b_r)\|$ for $\xi \in B_x$ and $(b_1,\dots,b_r)$ belonging to the following compact set :

$\{ (b_1,..b_r) : \exists\theta \in B_\theta$ such that $\|(b_1-\bar{\beta}_1(\theta)), \dots,(b_r-\bar{\beta}_r(\theta))\| \le k_2\}$

The existence of this bound is ensured by lemma 4.

The closed loop dynamics is then written as follows :

$$Dx(k) = \bar{\Lambda}_r\delta\, x(k) + R_4(\delta^r) + \sum_{j=1}^{q} \frac{\delta^{j-1}}{j!} \varphi_j^T(k)\, \tilde{\beta}_j(k) + \sum_{j=q+1}^{\infty} \frac{\delta^{j-1}}{j!} \varphi_j^T(k)\, \bar{\beta}_j(\theta_0)$$

From the properties of the parameter adaptation, we have the following theorem :

Theorem 4 :

Under assumptions A1-A2bis-A3, for every δ,r,q, if $x(k) \in B_x$ and $\|\tilde{\beta}(k)\| < k_2$, $\forall k$, the control law u(k) is uniformly bounded and

(i) leads to the following closed loop dynamics

$$Dx(k) = \bar{\Lambda}_r(\delta)\, x(k) + (\omega I_n + \Gamma(e))\, e(k) + De(k) + R_4(\delta^r)$$

$$\text{where } \Gamma(e) = \alpha\psi^T(k+1)P(k)\psi(k) \ \text{ if } \|e\| \le d(\delta)\frac{\delta^q}{(q+1)!}$$

$$= 0 \ \text{ otherwise} \qquad (6.7)$$

(ii) ensures that $R_4(\delta^r)$ is bounded as follows

$$\|R_4(\delta)^r\| \le M_4(\delta)\frac{\delta^r}{(r+1)!} \text{ with } M_4(\delta) < \infty$$

Using an argumentation parallel to that of section 6.2, we analyze the convergence conditions of this adaptive algorithm.

For every r and $\delta \le \delta_r$, lemma 3 can be applied to (6.7) with the following definitions and bounds :

- $.v_2(k) = e(k)$ with C_2 and c_2 defined in (5.9) and (5.10)
- $.\ v_1(k) = (\omega I_n + \Gamma(e))\, e(k)$
- $.\ C_1 = (\omega + \alpha\gamma\rho^2(\delta))\, C_2$
- $.\ c_1 = \omega\, C_2$

Hence :

$$\|x(k)\| \le K_1\|x(0)\| + K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} + K_3(\delta)\|\tilde{\beta}(0)\| + K_4(\delta)\frac{\delta^q}{(q+1)!}$$

$$\text{and } \lim_{k\to\infty}\sup\|x(k)\| \le K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} + K_5(\delta)\frac{\delta^q}{(q+1)!}$$

where K_1 and $K_2(\delta)$ are defined in section 6.1. and 6.2

$$K_3(\delta) = \frac{\delta K_1}{1-a}\left[(\omega + \alpha\gamma\rho^2(\delta)) + \|\bar{\Lambda}_r(\delta)\| + K_1 + 1\right] M_2(\delta)$$

$$K_4(\delta) = \frac{\delta K_1}{1-a}\left[(\omega + \alpha\gamma\rho^2(\delta)) + \|\bar{\Lambda}_r(\delta)\| + K_1 + 1\right] M_1(\delta)$$

$$K_5(\delta) = \frac{\delta K_1}{1-a}\left[(\omega + \|\Lambda_r(\delta)\| + 1\right] d(\delta)$$

$K_2(\delta)$, $K_3(\delta)$, $K_4(\delta)$, $K_5(\delta)$ can be respectively bounded on $[0,\delta_r]$ by $\bar{K}_2$, $\bar{K}_3$, $\bar{K}_4$, $\bar{K}_5$

We define δ_* such that $\bar{K}_2\frac{\delta_*^{r+1}}{(r+1)!} + \bar{K}_4\frac{\delta_*^q}{(q+1)!} = k_1$

and $\delta_3 = \min(\delta_r, \delta_*)$

Then, for every $\delta < \delta_3$:

$$k_1 - K_2(\delta)\frac{\delta^q}{(q+1)!} - K_4(\delta)\frac{\delta^{r+1}}{(r+1)!} = \eta_2(\delta) > 0$$

This guarantees the existence of the closed set

$$D_0 = \{ (x,\tilde{\beta}) : K_1\|x\| + K_3(\delta)\|\tilde{\beta}\| \le \eta_2(\delta) \text{ and } \|\tilde{\beta}\| \le k_2\}$$

Theorem 5 :

Under assumptions A1-A2bis-A3, for every r, for every $\delta < \delta_3$, if $(x(0), \beta(0)) \in D_0$,

(i) $x(k) \in B_x$, $\|\tilde{\beta}(k)\| \le k_2$ and theorem 4 holds $\forall$ k

(ii) $$\lim_{k\to\infty}\sup\|x(k)\| \le K_2(\delta)\frac{\delta^{r+1}}{(r+1)!} + K_5(\delta)\frac{\delta^q}{(q+1)!}$$

$$\le \bar{K}_2\frac{\delta^{r+1}}{(r+1)!} + \bar{K}_5\frac{\delta^q}{(q+1)!}$$

As in theorem 3 for the non-adaptive case, this theorem provides the convergence properties and the conditions (on the sampling period and on the initial errors of the state and parameters) for approximate linearizability.

Remarks :

R.1 Up to now, the values of the linearization degree r and of the estimation degree q have been left to the choice of the user and are independent. However from the results of theorem 5, it clearly appears that a good and coherent choice is : q = r + 1.

R.2 It must be pointed out that regardless of the choice of r and q, the two control laws of section 6.2. and 6.3. coïncide respectively with the continuous time linearizing control $u_0(x,\theta_0)$ and with the continuous linearizing adaptive control law of (Bastin and Campion 1989), when $\delta \to 0$.

6.4. Adaptive r-linearizing control (m < n).

For systems with $G(x,\theta)$ not square, it can be easily shown that the results of lemma 4 hold only for $r \le 2$. Theorem 2 to 5 do not explicitely use that $G(x,\theta)$ is square and may thus be rewritten replacing " $\forall$ r" by "for $r \le 2$ ".

7. CONCLUSIONS.

An indirect adaptive approximate state feedback linearization scheme has been derived for a class of sampled-data non linear models obtained from sampling of state-feedback linearizable and linearly parametrized continuous time systems. The applicability conditions and convergence properties have been discussed in details.

APPENDIX.

Proof of lemma 2 :

$\forall j$, φ_j is a continuous function non linear in x and polynomial of degree j in the components of u.
For x(k) and u(k) belonging to compact sets ($x(k) \in B_x$ and $\|u(k)\| \le U$), φ_j is uniformly bounded as follows :

$\exists \mu < \infty$ such that $\| \varphi_j(k) \| < \mu^j \ \forall k$

From (5.3.b), with $\varphi_{jF}(0) = 0$, we obtain

$$\| \varphi_{jF}(k) \| \le \frac{\mu^j}{\omega} \quad \forall k \qquad (a.1)$$

a. Bound on $\psi(k)$:

Using (5.5.b) we obtain :

$$\| \psi(k) \| \le \sum_{j=1}^{q} \frac{\delta^{j-1}}{j!} \| \varphi_{jF}(k) \| \le \frac{1}{\omega} \sum_{j=1}^{q} \frac{\delta^{j-1}}{j!} \mu^j \overset{\Delta}{=} \rho(\delta)$$

b. Bound on $R_1(\delta^q)$:

$$R_1(\delta^q) = \sum_{j=q+1}^{\infty} \frac{\delta^{j-1}}{j!} \varphi_{jF}(k)^T \bar{\beta}_j(\theta_0)$$

Since $\|\bar{\beta}_j(\theta_0)\| \le p^j \|\theta_0\|^j$ we obtain from (a.1) :

$$\| R_1(\delta^q) \|$$

$$\le \frac{\delta^q}{(q+1)!} \frac{1}{\omega} (\mu p \|\theta_0\|)^q \left[1 + \frac{\delta \mu p \|\theta_0\|}{q+2} + \frac{\delta^2 (\mu p \|\theta_0\|)^2}{(q+2)(q+3)} + \ldots \right]$$

$$\le \frac{\delta^q}{(q+1)!} \frac{1}{\omega} (\mu p \|\theta_0\|)^q \exp[\delta \mu p \|\theta_0\|] \overset{\Delta}{=} \frac{\delta^q}{(q+1)!} M_1(\delta)$$

Proof of theorem 1 :

a. We first show by induction that $P(k) > 0$, $\forall k$.

Let P(k) be positive definite.
Since $\alpha \delta P(k) \psi(k) \psi^T(k) P(k)$ is positive semidefinite, P(k+1) is positive definite iff the spectral radius of $\alpha \delta \psi(k) \psi^T(k) P(k)$ is ≤ 1 which is verified since α satisfies (5.7.b).
Then, since $\gamma > 0$, obviously : $0 < P(k+1) \le P(k) \le P(0)$ and therefore, since P(k) is invertible,

$$P(0)^{-1} \le P(k)^{-1} \le P(k+1)^{-1} \quad \forall k \qquad (a.2)$$

b. Let $V(k) = \beta(k)^T P(k)^{-1} \beta(k)$
Then $V(k+1) - V(k) = -\alpha\delta \, [\, e(k)^T e(k) - R_1(\delta^q)^T H(k) R_1(\delta^q) \,]$
where $H(k) \overset{\Delta}{=} (I - \alpha \delta \psi^T(k) P(k) \Psi(k))^{-1}$

and $\| H(k) \| \le \dfrac{1}{1 - \alpha\delta\gamma\rho^2(\delta)}$

Hence, the dead zone (5.7.a) ensures that $\exists \varepsilon > 0$ such that

$$V(k+1) - V(k) < -\varepsilon < 0 \quad \text{if } \|e\| > d(\delta) \frac{\delta^q}{(q+1)!}$$
$$= 0 \quad \text{otherwise} \qquad (a.3)$$

(5.8) and (5.9) follows then directly from (a.2), (a.3) and lemma 2.

c. It follows from (a.3) that after a limited number of adaptation steps ($\le V(0)/\varepsilon$), the prediction error e(k) lies definitively within the deadzone and that ensures (5.10).

Proof of Theorem 2 :

The functions φ_j defined in (5.1) are non linear in x and polynomial in u. Evaluating them in $x = \xi$ and $u = \sum_{i=0}^{\infty} \frac{\delta^i}{(i+1)!} v_i$, the expression $\sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \varphi_j^T(x,u)\, b_j$ can be rewritten using (5.1) and (3.3) as a power series in δ of the following structure :

$$\sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} [\, F_j(\xi, v_0, v_1, \ldots, v_{j-2}, b_1, \ldots b_p) + G(\xi, b_1)\, v_{p-1} \,]$$

where . the F_j are analytic in its variables and $F_1(\xi, b_1) = f(\xi, b_1)$

. $G(\xi, b_1)$ is analytic and invertible since $b_1 \in B_\theta$.

Identifying this power series term by term with

$$\bar{\Lambda}_r(\delta)\, \xi = \sum_{j=1}^{\infty} \frac{\delta^{j-1}}{j!} \Lambda^j \xi \,,$$

we obtain a solvable system of equations in v_i (i=0,...) which ensured, by induction, the existence of the sequence of functions v_i :

$$v_0(\xi, b_1) = G(\xi, b_1)^{-1} [\, \Lambda\xi - f(\xi, b_1) \,] = u_0(\xi, b_1)$$

$$v_{j-1}(\xi, b_1, \ldots, b_j) = G(\xi, b_1)^{-1} [\Lambda^j \xi - F_j(\xi, v_0, \ldots, v_{j-2}, b_1, \ldots b_j) \,]$$

For every r, it follows then directly from the construction of the function v_i that $v(\xi, \delta, b_1, \ldots b_r)$ defined in (6.2) ensures (6.4).

Since B_x is a compact set, since C1-C2-C3 imply that b_i (i=1,..r) belongs to compact sets and since the functions v_i are continuous, the function $v(\xi, \delta, b_1, \ldots b_r)$ is uniformly bounded in $\xi, b_1, \ldots b_r$.

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